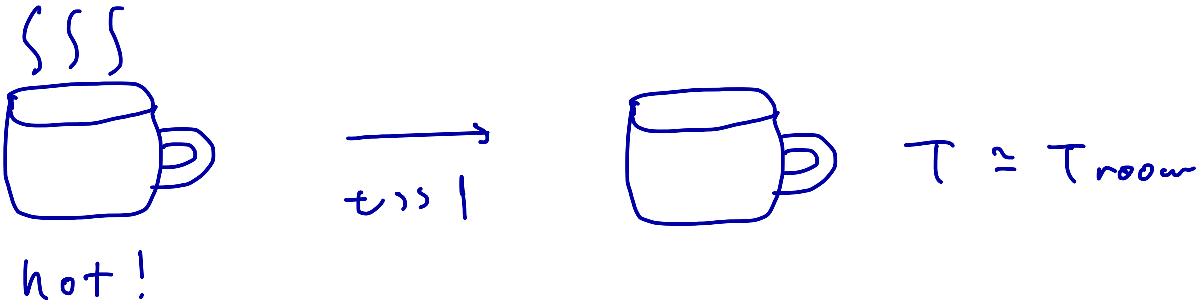
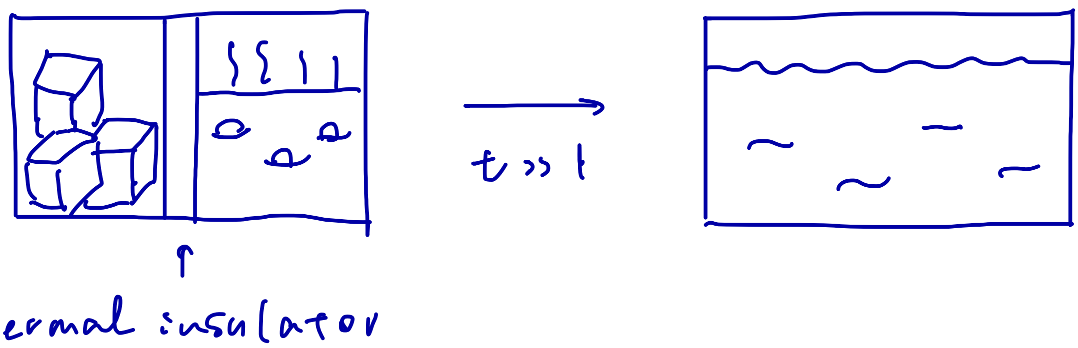
Thermalization & Relaxation of isolated quantum systems

"Thermalization"



Hot coffee cools down due to heat exchange with external system.



An isolated system is also thermalized.

- How about quantum systems?
- If not thermalized, why?
- Microscopic description of thermalization?

We use the knowledge of

- Quantum mechanics

  ← quantum systems

- Statistical mechanics

  ← in order to understand the mechanism of thermalization

  (connecting micro world to macro world)

- Integrable systems

  ← representative examples in which thermalization does not occur.

[CM arXiv. 2002.01069]

Quantum mechanics ↘ ↙ Statistical mechanics

- <u>Relaxation of quantum integrable systems</u>

Outline

§1. Postulates of quantum mechanics

§2. Postulates of statistical mechanics

§3. Thermalization vs. Integrable systems

§4. Summary

§ 1. Postulates of quantum mechanics

Quantum systems are mathematically formulated by using "linear algebra".

Notations

$\mathcal{H}$: Hilbert space ← where q-systems live in.

z^* : complex conjugate of the complex number z

$\mathcal{H} \ni |\psi\rangle$: vector (called as "ket").

$\langle\psi|$: vector dual to $|\psi\rangle$ (called as "bra").

$\langle\varphi|\psi\rangle$: inner product between the vectors $|\varphi\rangle$ and $\langle\psi|$.

$|\varphi\rangle\otimes|\psi\rangle = |\varphi\rangle|\psi\rangle$: tensor product of $|\varphi\rangle$ and $|\psi\rangle$.

A^* : complex conjugate of the matrix A.

$A^\dagger = (A^T)^*$: Hermitian conjugate or adjoint of the matrix A.

$\langle\varphi|A|\psi\rangle$: inner product between $|\varphi\rangle$ and $A|\psi\rangle$
equivalently, inner product between $A^\dagger|\varphi\rangle$ and $|\psi\rangle$.

Hilbert space (finite-dim. case)

Complex vector space equipped with inner products.

Inner products

$(\ , \ ) = \langle \ | \ \rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ s.t.

for $|v\rangle, |w\rangle \in \mathcal{H}$, $\lambda_j \in \mathbb{C}$,

(i) Bilinear

$$\langle v | \left(\sum_j \lambda_j | w_j \rangle \right) = \sum_j \lambda_j \langle v | w_j \rangle$$

$$\left(\sum_j \lambda_j^* \langle v_j | \right) | w \rangle = \sum_j \lambda_j^* \langle v_j | w \rangle$$

(ii) Hermitian

$$\langle v | w \rangle = \langle w | v \rangle^*$$

(iii) Positive definite

$$\langle v | v \rangle \geq 0 \quad (= 0 \text{ iff } |v\rangle = 0).$$

※ A closed bracket produces a scalar.

Outer product

Linear operator

$$|w\rangle\langle v| : \underset{\mathcal{H}}{\underset{\cap}{|v'\rangle}} \longmapsto \underbrace{\langle v|v'\rangle}_{\text{inner prod.}} \underset{\mathcal{H}}{\underset{\cap}{|w\rangle}}$$

※ An open bracket acts as a linear operator on a Hilbert space.

Adjoint & Hermitian operators

$\mathcal{H}$: Hilbert space

A : Linear operator on $\mathcal{H}$

$A^\dagger$: Adjoint operator of A

if $A^\dagger$ is a unique linear operator s.t.

$$\forall\, |v\rangle, |w\rangle \in \mathcal{H},\ \big(|v\rangle, A|w\rangle\big) = \big(A^\dagger|v\rangle, |w\rangle\big).$$

Same as Hermitian conjugate when $\dim \mathcal{H}$ is finite.

$\langle v|$: Dual vector of $|v\rangle \in \mathcal{H}$

$$|v\rangle^\dagger = \langle v|$$

$$\Rightarrow \big(A|v\rangle\big)^\dagger = \langle v|A^\dagger.$$

※ An adjoint operator is anti-linear.

$$\left(\sum_{i=1}^{n} a_i A_i\right)^\dagger = \sum_{i=1}^{n} a_i^* A_i^\dagger$$

Tensor products

- Tensor product of vectors

$\mathcal{H}_1, \mathcal{H}_2$: Hilbert space

$$\begin{cases} \cdot \ \dim \mathcal{H}_1 = m \\ \cdot \ \dim \mathcal{H}_2 = n \end{cases}$$

tensor product

$\mathcal{H}_1 \otimes \mathcal{H}_2$: mn-dimensional vector space.

$$\begin{array}{ccc} \mathcal{H}_1 & & \mathcal{H}_2 \\ \cup\!\!\!\!\!\: \ni & & \cup\!\!\!\!\!\: \ni \end{array}$$

$|v\rangle \otimes |w\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2$.

$\quad \shortparallel$

$|v\rangle |w\rangle$

$\quad \shortparallel$

$|v, w\rangle$.

Especially,

$$\begin{cases} \langle |i\rangle \rangle_{i=1\cdots m} & : \text{orthonormal basis for } \mathcal{H}_1 \\ \langle |j\rangle \rangle_{j=1\cdots n} & : \qquad \text{"} \qquad\qquad \mathcal{H}_2 \end{cases}$$

$\Rightarrow \langle |i\rangle \otimes |j\rangle \rangle_{\substack{i=1\cdots m \\ j=1\cdots n}}$: basis for $\mathcal{H}_1 \otimes \mathcal{H}_2$.

Properties)

(i) $\forall\, z \in \mathbb{C},\ |v\rangle \in \mathcal{H}_1,\ |w\rangle \in \mathcal{H}_2,$

$$z\left(|v\rangle \otimes |w\rangle\right) = \left(z|v\rangle\right) \otimes |w\rangle$$

$$= |v\rangle \otimes \left(z|w\rangle\right)$$

(ii) $\forall\, |v_1\rangle, |v_2\rangle \in \mathcal{H}_1,\ |w\rangle \in \mathcal{H}_2,$

$$\left(|v_1\rangle + |v_2\rangle\right) \otimes |w\rangle = |v_1\rangle \otimes |w\rangle + |v_2\rangle \otimes |w\rangle$$

(iii) $\forall\, |v\rangle \in \mathcal{H}_1,\ |w_1\rangle, |w_2\rangle \in \mathcal{H}_2,$

$$|v\rangle \otimes \left(|w_1\rangle + |w_2\rangle\right) = |v\rangle \otimes |w_1\rangle + |v\rangle \otimes |w_2\rangle$$

- Tensor product of operators

$$
\begin{cases}
A : \mathcal{H}_1 \to \mathcal{H}_1 \\
B : \mathcal{H}_2 \to \mathcal{H}_2
\end{cases}
$$

$$
\Rightarrow \underbrace{(A \otimes B)}_{\substack{\mathcal{H}_1 \otimes \mathcal{H}_2 \\ \to \mathcal{H}_1 \otimes \mathcal{H}_2}} \big(\underset{\in \mathcal{H}_1}{|v\rangle} \otimes \underset{\in \mathcal{H}_2}{|w\rangle} \big) := A|v\rangle \otimes B|w\rangle .
$$

Inner product on $\mathcal{H}_1 \otimes \mathcal{H}_2$

$$
\Big(\sum_i a_i \overset{\mathcal{H}_1}{|v_i\rangle} \otimes \overset{\mathcal{H}_2}{|w_i\rangle} ,\ \sum_j b_j \overset{\mathcal{H}_1}{|v'_j\rangle} \otimes \overset{\mathcal{H}_2}{|w'_j\rangle} \Big)
$$

$$
:= \sum_{i,j} a_i^* b_j \langle v_i | v'_j \rangle \langle w_i | w'_j \rangle .
$$

$$
\Rightarrow \left.\begin{array}{l} \text{Bilinearity} \\ \text{Hermiticity} \\ \text{Positive definiteness} \end{array}\right\} \text{ satisfied.}
$$

Matrix representations (Kronecker product)

For $A \in M(m,n;\mathbb{C})$ and $B \in M(p,q;\mathbb{C})$,

$$A \otimes B = \left.\overbrace{\begin{pmatrix} A_{11}B & \cdots & A_{1n}B \\ \vdots & & \vdots \\ A_{m1}B & \cdots & A_{mn}B \end{pmatrix}}^{nq}\right\} mp \;.$$

matrix

Ex.)

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \otimes \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \left(\begin{array}{c} v_1 w_1 \\ v_1 w_2 \\ \hdashline v_2 w_1 \\ v_2 w_2 \end{array}\right)$$

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} = \left(\begin{array}{cc:cc} A_{11}B_{11} & A_{11}B_{12} & A_{12}B_{11} & A_{12}B_{12} \\ A_{11}B_{21} & A_{11}B_{22} & A_{12}B_{21} & A_{12}B_{22} \\ \hdashline A_{21}B_{11} & A_{21}B_{12} & A_{22}B_{11} & A_{22}B_{12} \\ A_{21}B_{21} & A_{21}B_{22} & A_{22}B_{21} & A_{22}B_{22} \end{array}\right)$$

Operator functions

$f : \mathbb{C} \to \mathbb{C}$ (normal function)

$\Downarrow$

"Matrix function"

uniquely defined

$$f(A) := \sum_a f(a) |a\rangle\langle a|$$

for a normal matrix $A = \underline{\sum_a a |a\rangle\langle a|}$.

spectral decomposition

Commutator and anti-commutator

- Commutator

$$[A, B] := AB - BA = 0$$

A & B are simultaneously diagonalizable if Hermitian.

- Anti-commutator

$$\{A, B\} := AB + BA$$

Postulate 1) State space

Isolated physical system $\leftrightarrow$ Hilbert space $\mathcal{H}$

Quantum state $\leftrightarrow$ Unit vector $|\psi\rangle$

"state vector"

Remark)

Globally phase shifted states $e^{i\theta}|\psi\rangle$ ($\theta \in \mathbb{R}$) are identified with $|\psi\rangle$.

Postulate 2) Physical quantities (Observables)

Physical quantity $\Leftrightarrow$ Self-adjoint operator (Hermitian) in $\mathcal{H}$.

Postulate 3) Born's probability rule.

$A : \mathbb{C}^N \to \mathbb{C}^N$ Hermite

Observable

Physical quantity

$$\Rightarrow A = \sum_{n=1}^{N} \alpha_n |\alpha_n\rangle\langle\alpha_n| \quad : \text{spectral decomp.}$$

eigenvalue eigenvector

$$= \sum_{n=1}^{N} \alpha_n P(\alpha_n)$$

projector onto $\{|\alpha_n\rangle\}$

In quantum measurement on $|\psi\rangle \in \mathcal{H}$,

- Eigenvalues of A $\Leftrightarrow$ Measurement outcomes
- Probability to obtain the outcome "a"

$$: p(\alpha) = \langle\psi| P(\alpha)^\dagger P(\alpha) |\psi\rangle$$

$$= \| P(\alpha)|\psi\rangle \|^2$$

$$= |\langle\alpha|\psi\rangle|^2 \quad \text{square of the amplitude}$$

$$\geq 0 \quad (\text{non-negative}).$$

※ $\sum_\alpha |\langle\alpha|\psi\rangle|^2 = \sum_\alpha \langle\psi|\alpha\rangle\langle\alpha|\psi\rangle$

$= \langle\psi|\psi\rangle$

$= 1$ Sum rule.

Expectation value of A on $|\psi\rangle$.

$$\langle A \rangle = \sum_{n=1}^{N} \alpha_n P(\alpha_n)$$

$$= \sum_{n=1}^{N} \alpha_n \underbrace{|\langle \alpha_n | \psi \rangle|^2}_{\langle \psi | \alpha_n \rangle \langle \alpha_n | \psi \rangle}$$

$$= \langle \psi | \underbrace{\sum_{n=1}^{N} \alpha_n | \alpha_n \rangle \langle \alpha_n |}_{A} \psi \rangle$$

$$= \langle \psi | A | \psi \rangle .$$

Postulate 4) (Time) Evolution

Closed quantum system

$$\boxed{|\psi\rangle} \longrightarrow \boxed{|\psi'\rangle} = \underbrace{U(t_1, t_2)}_{\text{unitary operator}} |\psi\rangle$$

$t = t_1$ $\qquad\qquad$ t_2

$$\simeq e^{-iH(t_2 - t_1)}$$

$$\Leftrightarrow \quad i \frac{d}{dt} |\psi\rangle = \underline{H} |\psi\rangle \quad : \text{Schrödinger eq.}$$

Hermitian operator

(Hamiltonian ; energy operator)

Density operator

- The system is in one of the state out of $\{ \underset{\substack{\cap \\ \mathcal{H}}}{|\psi_i\rangle} \}$.

- We only know the prob. for the system in $|\psi_i\rangle$ is p_i.

$$\boxed{\rho} = \sum_i p_i |\psi_i\rangle\langle\psi_i| , \quad \sum_i p_i = 1$$

density matrix

$\{p_i, |\psi_i\rangle\}$: ensemble of pure states.

Pure state and mixed state

$$\boxed{|\psi\rangle} = \boxed{\rho = |\psi\rangle\langle\psi|}$$: pure state ($\mathrm{tr}\,\rho^2 = 1$)

$$\boxed{\begin{matrix} |\psi_i\rangle \\ \text{w.p. } p_i \end{matrix}} = \boxed{\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|}$$: mixed state ($\mathrm{tr}\,\rho^2 < 1$)

dist. $\{p_i\}$

mixture of pure state

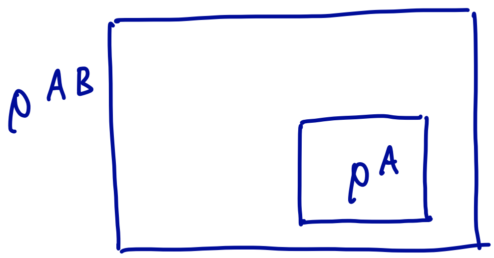
Properties of density operator

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i| \quad : \text{density operator}$$

$\Leftrightarrow$ $\left(\begin{array}{l} \cdot \ \text{Trace condition} \quad \mathrm{tr}\,\rho = 1 \\ \cdot \ \text{Positivity condition} \quad \rho \geq 0 . \end{array}\right.$

Reduced density matrix

↳ Description of the subsystem.



- Density matrix of the whole system ρ^{AB}
- Density matrix of the subsystem

$$\rho^{A} := \underbrace{\mathrm{tr}_B}_{\text{partial trace}} \left(\rho^{AB} \right)$$

mixed state if ρ^{AB} is an entangled pure state.

$$\mathrm{tr}_B \big(\underbrace{|a_1\rangle\langle a_2|}_{\in \mathcal{H}_A} \otimes \underbrace{|b_1\rangle\langle b_2|}_{\in \mathcal{H}_B} \big)$$

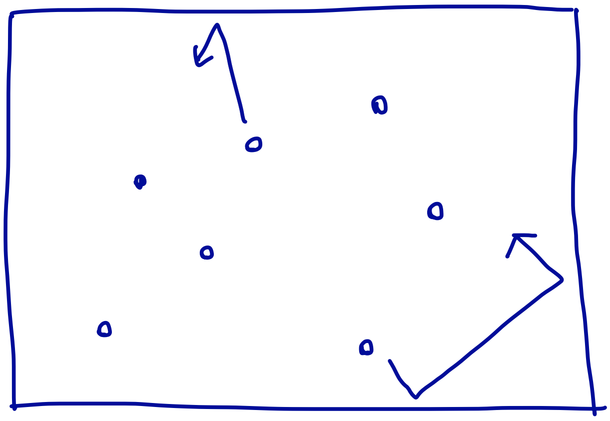
$$:= |a_1\rangle\langle a_2| \ \mathrm{tr}\left(|b_1\rangle\langle b_2| \right)$$

§ 2. Postulates of statistical mechanics

Aim of statistical mechanics :

to understand macroscopic properties

from microscopic description.



Classical Newton's eq. of motion

$$
\begin{cases}
m\ddot{x}_1 = F(x_1, x_2, \cdots) \\
m\ddot{x}_2 = F(x_1, x_2, \cdots) \\
\quad \vdots \\
m\ddot{x}_{10^{24}} = F(x_1, x_2, \cdots)
\end{cases}
$$

How do we solve it ?

⇒ Use statistics instead of

solving Newton eq.

Equilibrium

Macroscopic thermodynamics

$\Updownarrow$

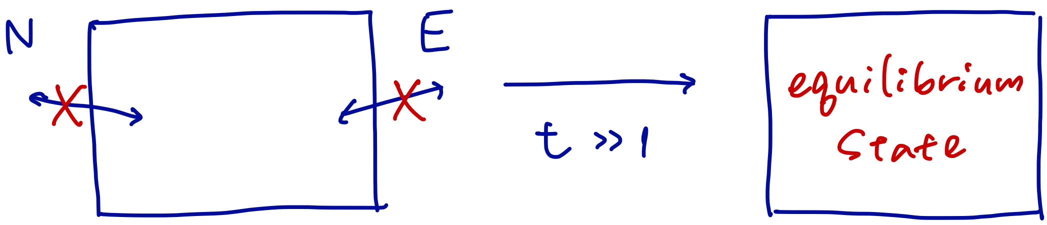
Microscopic mechanics + statistics

Postulate 1)

$\boxed{\text{equilibrium state}}$ $\Leftrightarrow$ A few thermodynamic var.

$\underbrace{E,\ V,\ N,\ \cdots}_{\text{finite \# in thermodynamic limit.}}$

Postulate 2)

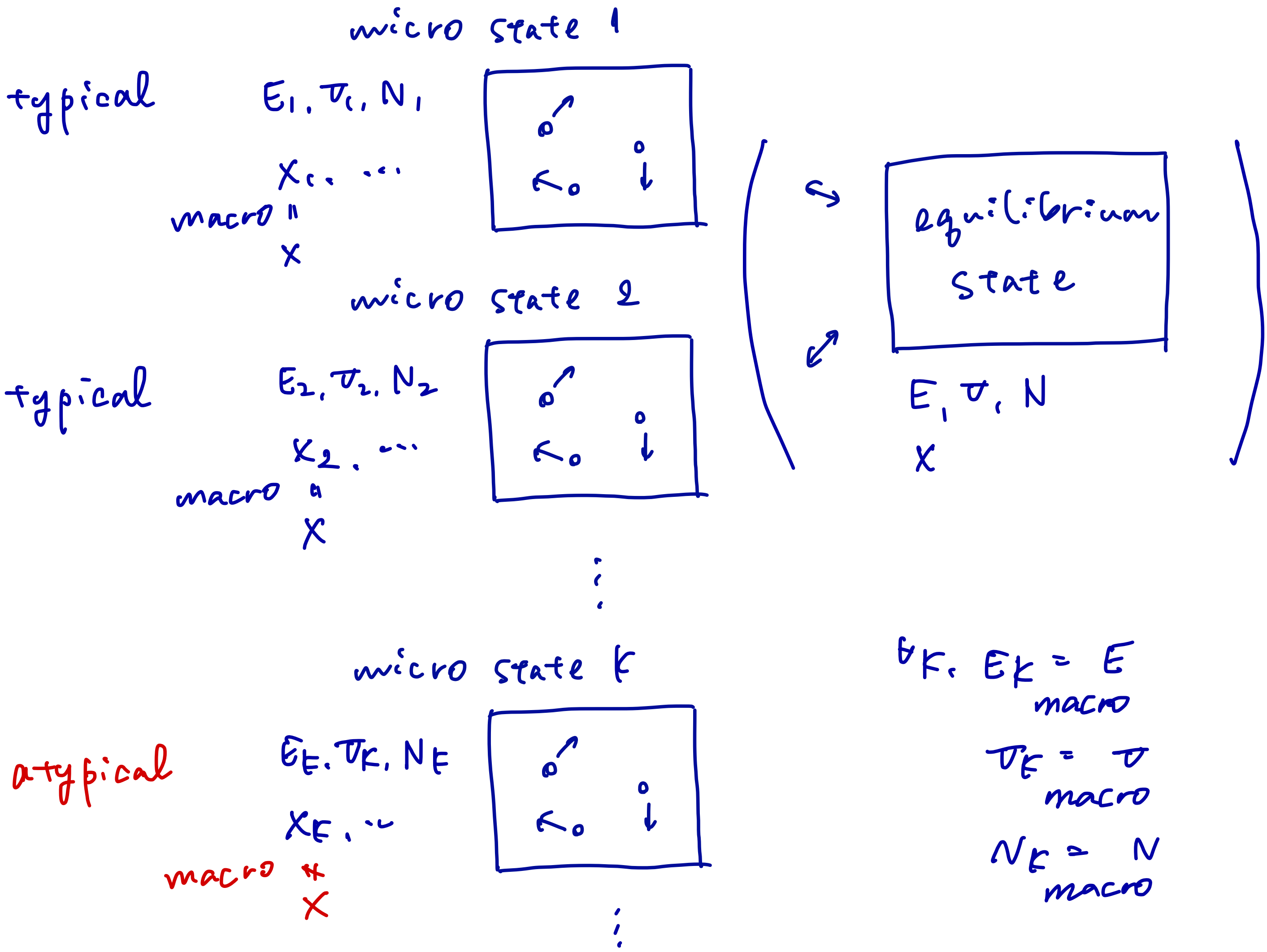


Macroscopic system

- Any macro variable = const.
- Any macro current = 0.

Postulate 3)

Almost all microstates are indistinguishable by macro variables.



Postulate 4)

Properties of equilibrium state

= Properties of corresponding typical micro states.

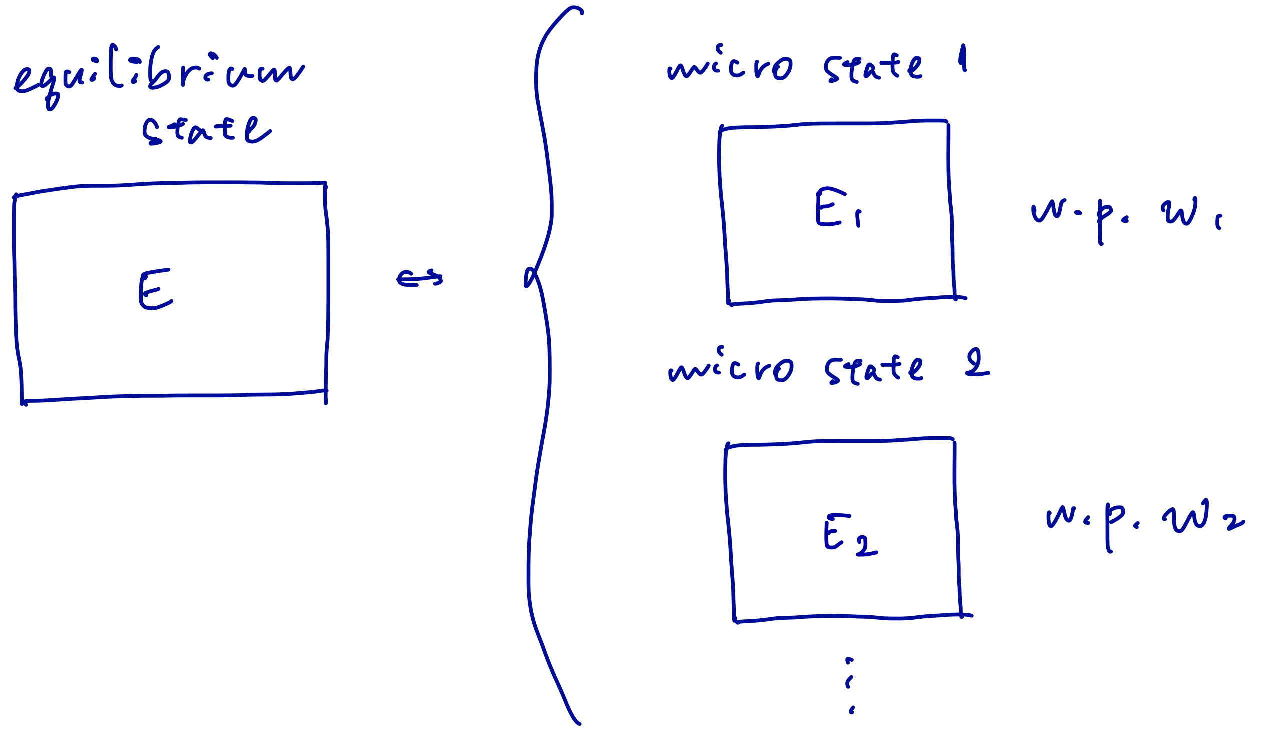
Microcanonical ens.

Postulate 5) Principle of equal prob.

In an isolated system, microstates with energy $E_i \in [E-\delta E, E]$ are realized with equal prob.

$$w_1 = w_2 = \cdots = \underbrace{\frac{1}{W(E)}}$$

# of micro states with energy in $[E-\delta E, E]$.



$$w_j = \begin{cases} \dfrac{1}{W(E)} & E_j \in [E - \underbrace{\delta E}_{\text{macroscopically small}}, E] \\ 0 & \text{otherwise} \end{cases}$$

(E_j are macroscopically undistinguishable.)

Expectation values of a macro var. X :

$$\langle X \rangle = \sum_i \underbrace{w_i}_{= \frac{1}{W(E)}} \underbrace{X(x_i, p_i)}$$

the value of X computed in the "i"th micro state.

Thermodynamics uses a macro var. which is not a function of x_i and p_i.

$\downarrow$

Entropy

Postulate 5) Boltzmann's relation

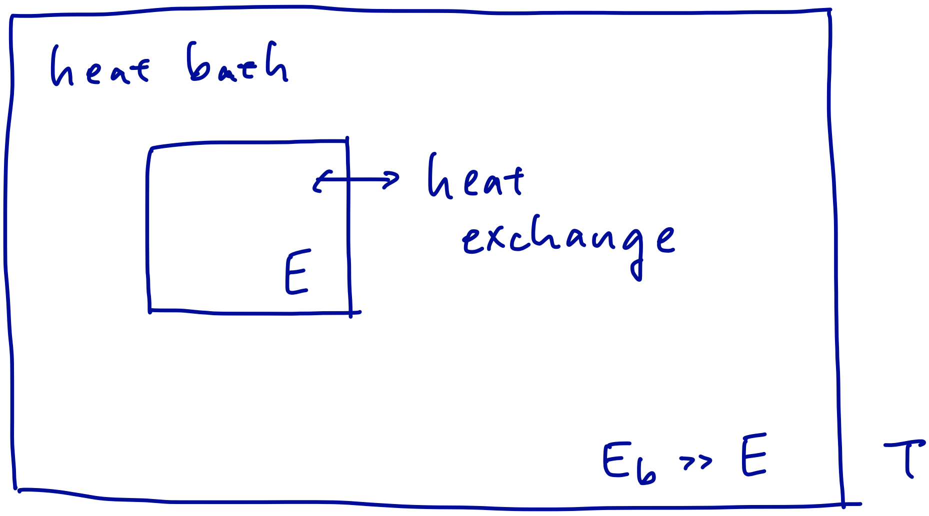
$$S(E) = k_B \ln W(E) .$$

SI

1.38×10^{-23} J/K

(Boltzmann const.)

Canonical ens.



Q.) Prob. to obtain the microstate of the subsystem with energy E ?

Prob. to obtain the subsystem in the microstate n:

$$p(E_n) = \frac{W_b(E_t - E_n)}{\sum_{E'} W_b(E_t - E')\, w(E')}$$

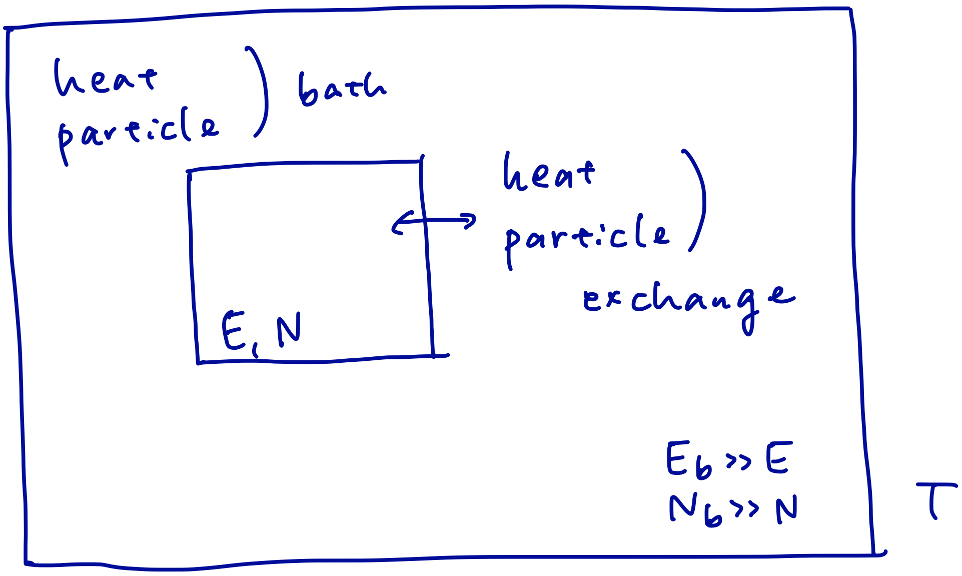
∵ principle of equal prob.

$$= \underbrace{\frac{1}{Z} e^{-\frac{E_n}{k_B T}}}$$ ∵ Boltzmann's relation

canonical (Gibbs) distribution

$$Z := \sum_n e^{-\frac{E_n}{k_B T}}$$ "partition func."

Grand canonical ensemble



Prob. to obtain the microstate n:

$$p_n = \frac{1}{\Xi} e^{-\frac{1}{k_B T}(E_n - \mu N_n)} \quad \left(\sum_n p_n = 1 \right).$$

grandcanonical dist.

(dist. of subsystem attached to the heat & particle bath with E & N).

$$\Xi = \sum_n e^{-\frac{1}{k_B T}(E_n - \mu N_n)}$$

Combining quantum mech. & stat. mech.

Postulate 1')

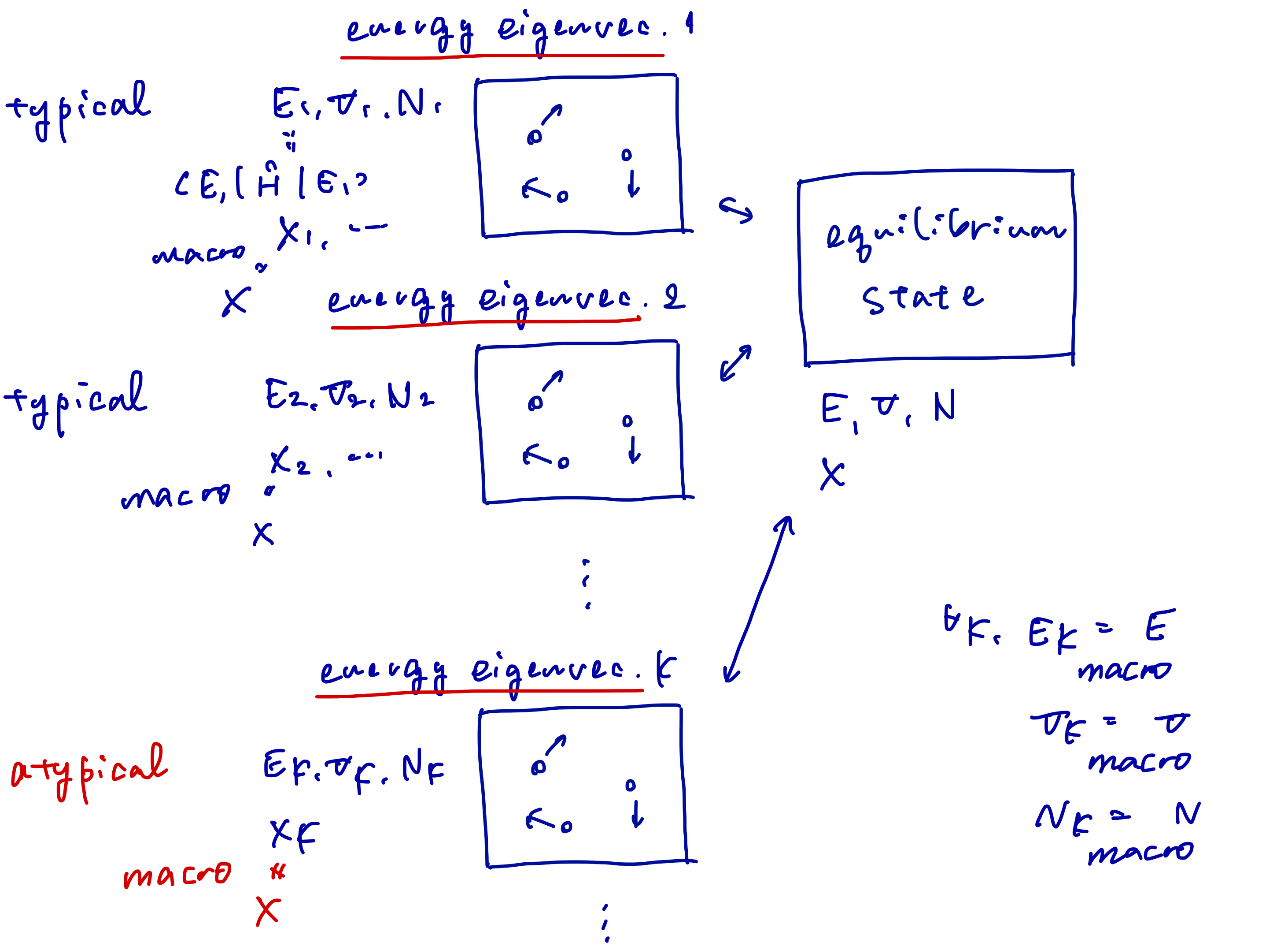
A generic quantum system reaches equilibrium after enough time.

Postulate 2')

An equilibrium state is characterized by a few thermodynamic operators.

Postulate 3') : Eigenstate thermalization hypothesis (ETH)

Almost all energy eigenvec. are indistinguishable by macro variables.



Postulate 4')

Properties of equilibrium state

= Properties of corresponding typical energy eigenvec.

Postulate 5')

Expectation values of a macro operator X :

$$\langle X \rangle = \frac{1}{W(E)} \sum_i \underbrace{\langle E_i | X | E_i \rangle}$$

the expectation value of X measured on $|E_i\rangle$.

$$= \mathrm{tr}\left(\underbrace{\rho_{MC}}_{\parallel} X \right)$$

$$\sum_{E_i \in [E-\delta E, E]} \frac{1}{W(E)} |E_i\rangle\langle E_i| .$$

: microcanonical density matrix.

$W(E)$: # of energy eigenvec. with $E_i \in [E-\delta E, E]$.

$\Leftrightarrow$ Principle of equal prob.

cf.)

Density matrix for canonical ens.

$$\rho_{GE} = \frac{1}{Z} e^{-\beta H}$$

H : Hamiltonian

$\beta \in \mathbb{R}$: inverse temperature

$$Z = \mathrm{tr}\, \rho_{GE}$$

$$= \sum_n \langle E_n | e^{-\beta H} | E_n \rangle$$

$$= \sum_n e^{-\beta E_n}$$

Density matrix for grandcanonical ens.

$$\rho_{GCE} = \frac{1}{\Xi} e^{-\beta(H - \mu N)}$$

N : particle number

$\mu \in \mathbb{R}$: chemical potential

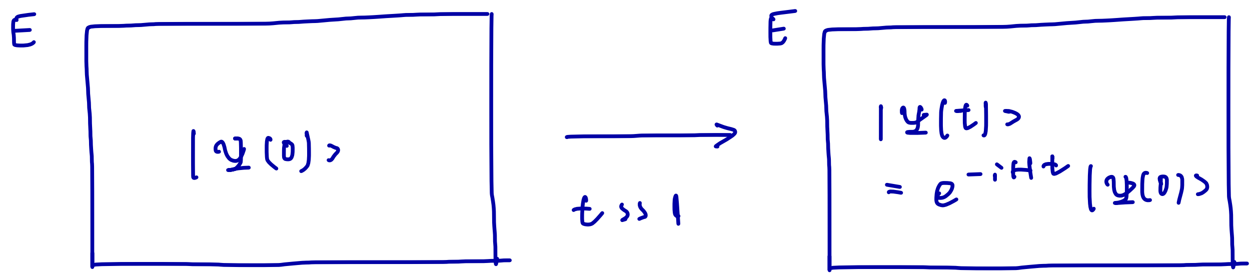
$$\Xi = \mathrm{tr}\, \rho_{GCE}$$

$$= \sum_n \langle E_n, N_n | e^{-\beta(H-\mu N)} | E_n, N_n \rangle$$

$$= \sum_n e^{-\beta(E_n - \mu N_n)}$$

§ 3. Thermalization vs. integrable systems

Thermalization of generic quantum systems



Thought experiment)

Let $|\Psi(0)\rangle = \sum_n c_n |E_n\rangle$, $E_n \in (E-\delta E, E]$

$$\sum_n |c_n|^2 = 1.$$

$c_n \in \mathbb{C}$

$$\lim_{t\to\infty} |\Psi(t)\rangle\langle\Psi(t)| = \lim_{t\to\infty} e^{-iHt}|\Psi(0)\rangle\langle\Psi(0)|e^{iHt}$$

$$= \lim_{t\to\infty} \sum_{n,m} e^{-i(E_n-E_m)t} c_n c_m^* |E_n\rangle\langle E_m|$$

$$\neq \frac{1}{W(E)} \sum_n |E_n\rangle\langle E_n|$$

never reaches the MC density matrix.

mixed state.

★ Do isolated quantum systems thermalize?

Definitions)

The system shows relaxation if

$$\delta X^2 := \lim_{T\to\infty} \frac{1}{T}\int_0^T \langle X\rangle^2(t)\, dt - \left(\lim_{T\to\infty}\frac{1}{T}\int_0^T \langle X\rangle(t)\, dt\right)^2$$

$$\ll \underbrace{\|X\|^2}_{\text{operator norm}}$$

The system thermalizes if

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T \langle X\rangle(t)\, dt \underset{\text{T.L.}}{=} \langle X\rangle_{\text{eq.}}$$

Remarks)

Relaxation occurs for most initial states of non-reasonance systems.

$\because$ $|\Psi(0)\rangle = \sum_n c_n |E_n\rangle$, $E_n \in [E-\delta E, E]$.

$$\Rightarrow \delta X^2 = \sum_{n\neq m}\sum_{k\neq l} c_n^* c_m c_k c_l^* e^{i(E_n - E_m - E_k + E_l)t} \times \langle E_n|X|E_m\rangle (\langle E_k|X|E_l\rangle)^*$$

$$= \sum_{n\neq m}\sum_{k\neq l} c_n^* c_m c_k c_l^* \delta_{E_n - E_m, E_k - E_l} \times \langle E_n|X|E_m\rangle (\langle E_k|X|E_l\rangle)^*$$

"Non-resonance condition"

$$E_n - E_m = E_k - E_l \neq 0 \Rightarrow n = k,\ m = l.$$

$$\Rightarrow \delta X^2 = \sum_{n,m} |c_n|^2 |c_m|^2 \left| \langle E_n | X | E_m \rangle \right|^2$$

$$\leq \sum_n |c_n|^4 \langle E_n | X X^\dagger | E_n \rangle$$

$$\leq \underbrace{\sum_n |c_n|^4}_{\simeq\ \#\text{ of eigenvectors included in } |\Psi(0)\rangle} \| X \|^2$$

$\simeq$ # of eigenvectors included in $|\Psi(0)\rangle$.

Sufficient condition for thermalization :

"Strong eigenstate thermalization hypothesis"

(ETH)

$$\langle \Psi(t) | X | \Psi(t) \rangle$$

$$= \langle \Psi(0) | e^{iHt} X e^{-iHt} \underbrace{| \Psi(0) \rangle}_{\parallel}$$

$$\sum_{E_a \in [E-\delta E, E]} c_a | E_a \rangle, \quad \sum_{E_a \in [E-\delta E, E]} |c_a|^2 = 1$$

$$= \sum_{\substack{E_a, E_b \\ \in [E-\delta E, E]}} c_a^* c_b e^{i(E_a - E_b)t} \langle E_a | X | E_b \rangle$$

$$= \sum_{E_a \in [E-\delta E, E]} |c_a|^2 \langle E_a | X | E_a \rangle \ : \text{diag.}$$

$$+ \sum_{E_a \neq E_b} c_a^* c_b e^{i(E_a - E_b)t} \langle E_a | X | E_b \rangle \ : \text{off-diag.}$$

- Diag. part

$$\sum_{E_a \in [E-\delta E, E]} |c_a|^2 \langle E_a | X | E_a \rangle$$

$$\parallel \quad \text{if} \quad \forall a, \ \langle E_a | X | E_a \rangle \underset{\text{T.L.}}{=} \langle X \rangle_{eq}.$$

$$\langle X \rangle_{eq} := \frac{1}{W(E)} \sum_{E_a \in [E-\delta E, E]} \langle E_a | X | E_a \rangle$$

- Off-diag. part (fluctuation term)

$$\langle E_a | X | E_{b \neq a} \rangle \underset{\text{T.L.}}{=} 0$$

ETH guarantees thermalization of the system in any initial state $|\Psi(0)\rangle$.

Questions)

Does strong ETH true for any quantum system?

If not, when it is not true?

Quantum systems with more conserved macro op.

$$a^{(1)} = E,\ a^{(2)} \qquad \boxed{|\psi(0)\rangle} \quad \xrightarrow[t \gg 1]{} \quad a^{(1)} = E,\ a^{(2)} \qquad \boxed{\begin{aligned} &|\psi(t)\rangle \\ &= e^{-iHt}|\psi(0)\rangle \end{aligned}}$$

$$[\hat{a}^{(2)}, \hat{H}] = 0$$

∵ $\langle \psi(t) | \hat{a}^{(2)} | \psi(t) \rangle$

$= \langle \psi(0) | e^{i\hat{H}t} \hat{a}^{(2)} e^{-i\hat{H}t} | \psi(0) \rangle$

$= \langle \psi(0) | \hat{a}^{(2)} | \psi(0) \rangle .$

Principle of equal prob. must be stated for

equilibrium state

$$\boxed{E,\ a^{(2)}} \quad \Leftrightarrow \quad \left\{ \begin{array}{ll} \text{energy eigenvec 1} & \\ \boxed{\begin{array}{cc} E_1, & a_1^{(2)} \\ \text{macro} \ \| & \| \\ E & a^{(2)} \end{array}} & \text{w.p. } \dfrac{1}{W(E, a^{(2)})} \\ \text{energy eigenvec 2} & \\ \boxed{\begin{array}{cc} E_2, & a_2^{(2)} \\ \text{macro} \ \| & \| \\ E & a^{(2)} \end{array}} & \text{w.p. } \dfrac{1}{W(E, a^{(2)})} \\ \vdots & \end{array} \right.$$

Unless $\hat{O}_2 = f(H)$, macroscopic # of $|E_i\rangle$ violate strong ETH :

$$\langle E_i | \hat{O}_2 | E_i \rangle \underset{\text{macro}}{\neq} \langle \hat{O}_2 \rangle_{MC(E)}$$

$$:= \frac{1}{W(E)} \sum_{E_a \in [E-\delta E, E]} \langle E_a | \hat{O}_2 | E_a \rangle .$$

Example)

$\hat{O}_2 = N$ (particle number).

$$\langle N \rangle_{MC(E)} := \frac{1}{W(E)} \sum_{E_a \in [E-\delta E, E]} \langle E_a | \hat{N} | E_a \rangle$$

$$= \frac{1}{W(E)} \sum_{N} \sum_{\substack{E_a \in [E-\delta E, E] \\ N_a \in [N-\delta N, N]}} \langle E_a, N_a | N | E_a, N_a \rangle$$

$$= \sum_{N} \frac{W(E,N)}{W(E)} \langle N \rangle_{MC(E,N)} \underset{\text{macro}}{\neq} \langle N \rangle_{MC(E,N)}$$

$$\Rightarrow \langle E_i | N | E_i \rangle \underset{\text{macro}}{\neq} \langle N \rangle_{MC(E)}$$

for macroscopic # of $|E_i\rangle$.

Instead,

$$\langle E_i, N_i | N | E_i, N_i \rangle \underset{\text{macro}}{=} \langle N \rangle_{MC(E,N)}$$

$$:= \frac{1}{W(E,N)} \sum_{\substack{E_a \in [E-\delta E, E] \\ N_a \in [N-\delta N, N]}} \langle E_a, N_a | N | E_a, N_a \rangle$$

Integrable systems have many conserved quantities.

$Q_1 = E, Q_2, Q_3, \cdots$ $\qquad\qquad$ $Q_1 = E, Q_2, Q_3, \cdots$

$$\boxed{|\Psi(0)\rangle} \xrightarrow[t \gg 1]{} \boxed{\begin{aligned}&|\Psi(t)\rangle \\ &= e^{-iHt}|\Psi(0)\rangle\end{aligned}}$$

$\forall k, \ [Q_k, H] = 0$

We cannot define the equilibrium state since

(i) $MC(E, Q_2, Q_3, \cdots)$ ensemble can't be characterized by a few thermodynamic var.
($E, Q_2, Q_3, \cdots$ must be thermodynamic var.)

(ii) $MC(E, Q_2, Q_3, \cdots)$ average of a certain current becomes non-zero.

$\Downarrow$

contradicts to the def. of equilibrium state.

Questions)

- Does relaxation phenomena occur in integrable systems?

- If so, which conserved quantities are enough to describe the relaxation state?

Integrable systems with many conserved quantities

"Integrable systems"

- The systems with many conserved quantities.
- No common understanding of quantum integrability.
- Sufficient condition : Yang-Baxter eq.

$R : V \otimes V \to V \otimes V$

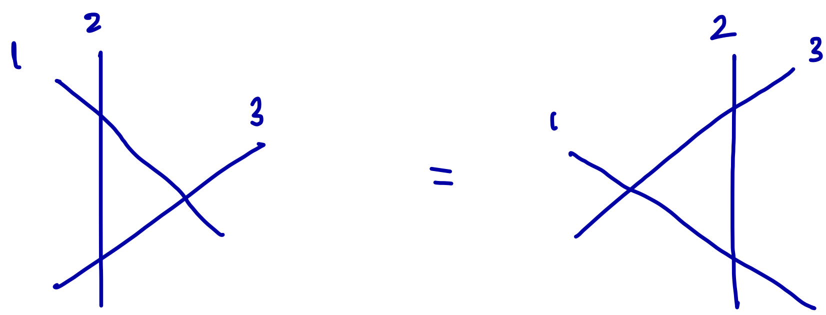
$R_{12}, R_{13}, R_{23} : V \otimes V \otimes V \to V \otimes V \otimes V$

$R_{12} = R \otimes I$, $\quad R_{23} = I \otimes R$

$$R_{12}(\lambda_1-\lambda_2)\, R_{13}(\lambda_1-\lambda_3)\, R_{23}(\lambda_2-\lambda_3)$$

$$= R_{23}(\lambda_2-\lambda_3)\, R_{13}(\lambda_1-\lambda_3)\, R_{12}(\lambda_1-\lambda_2)$$

$$(\lambda_1, \lambda_2, \lambda_3 \in \mathbb{C})$$



Sufficient condition that decomposition of many-body scattering does not depend on the way to decompose.

Many conserved quantities

$\Leftarrow$ ② Commuting transfer matrices

$\Leftarrow$ ① Yang-Baxter eq. (YBE)

①

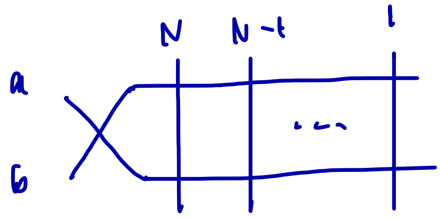
$M_a(\lambda) \in \mathrm{End}(V_a \otimes V^{\otimes N})$: monodromy matrix

$\quad \shortparallel$

$R_{aN}(\lambda) \cdots R_{a2}(\lambda) R_{a1}(\lambda)$.

$$R_{ab}(\lambda-\mu) M_a(\lambda) M_b(\mu)$$

$$\underset{\uparrow \text{YBE}}{=} M_b(\mu) M_a(\lambda) R_{ab}(\lambda-\mu)$$



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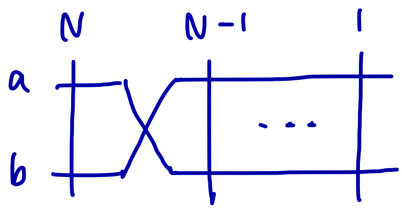

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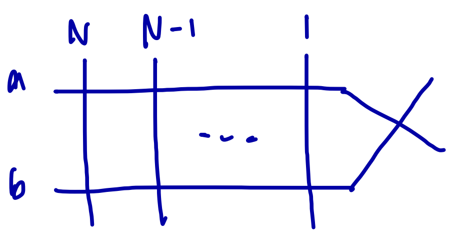


$$T(\lambda) \in \mathrm{End}(V^{\otimes N}) \quad : \text{transfer matrix}$$

$$\shortparallel$$

$$\mathrm{tr}_a M_a(\lambda)$$

$$\mathrm{tr}_a \mathrm{tr}_b \left(R_{ab}(\lambda-\mu) M_a(\lambda) M_b(\mu) R_{ab}^{-1}(\lambda-\mu) \right)$$

$$\underset{\uparrow}{=} \mathrm{tr}_a \mathrm{tr}_b \left(M_b(\mu) M_a(\lambda) \right)$$

R is invertible

$$\Rightarrow \quad T(\lambda) T(\mu) = T(\mu) T(\lambda).$$

2 Expanding $T(\lambda), T(\mu)$ around $\lambda = 0$, $\mu = 0$

$$T(\lambda) = \sum_r \lambda^r X_r$$

$$T(\mu) = \sum_{r'} \mu^{r'} X_{r'}.$$

$$[T(\lambda), T(\mu)] = 0 \Rightarrow [X_r, X_{r'}] = 0$$

Conventionally, we choose

$$\log T(\lambda) = \sum_r \lambda^r \underbrace{Q_r}, \quad Q_1 =: H$$

extensive concerved quantities

-1 $R(0) = P : \underset{\substack{\cap \\ V}}{v_1} \otimes \underset{\substack{\cap \\ V}}{v_2} \mapsto v_2 \otimes v_1$

XXZ model

$$H_{XXZ} := \sum_{n=1}^{N} \left(\sigma_n^x \sigma_{n+1}^x + \sigma_n^y \sigma_{n+1}^y + \Delta \sigma_n^z \sigma_{n+1}^z \right)$$

$$\sigma_n^\alpha = \mathbb{1} \otimes \cdots \otimes \mathbb{1} \otimes \sigma^\alpha \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1}$$

$$\mathbb{1}, \sigma^\alpha : \mathbb{C}^2 \to \mathbb{C}^2 \qquad (\alpha = x, y, z)$$

$$\mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- Periodic b.c.

$$\sigma_{N+1}^\alpha := \sigma_1^\alpha$$

- Anisotropy parameter Δ

$$\begin{cases} \Delta = \cos\gamma \ (\gamma \in \mathbb{R}) \Rightarrow \text{gapless excitation} \\ \Delta = \cosh\eta \ (\eta \in \mathbb{R}) \Rightarrow \text{gapped excitation} \end{cases}$$

Alternatively,

$$\Delta = \frac{1}{2}(q + q^{-1}) \leftarrow \text{"}q\text{" of } U_q(sl_2).$$

$$\text{"} \quad e^{i\gamma} \text{ for gapless regime}$$
$$\quad e^{\eta} \text{ for gapped regime.}$$

- "Ladder" operators

$$\sigma^\pm := \sigma^x \pm i\sigma^y$$

$$\sigma^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

R-matrix

The solution of the Yang-Baxter eq.

Ex.)

$$R(\lambda) = \begin{pmatrix} \frac{1}{2}(qe^{\lambda}-q^{-1}e^{-\lambda}) & & & \\ & \frac{1}{2}(e^{\lambda}-e^{-\lambda}) & \frac{1}{2}(q-q^{-1}) & \\ & \frac{1}{2}(q-q^{-1}) & \frac{1}{2}(e^{\lambda}-e^{-\lambda}) & \\ & & & \frac{1}{2}(qe^{\lambda}-q^{-1}e^{-\lambda}) \end{pmatrix}$$

The solution associated with $\underline{U_q(sl_2)}$:

$$\{ S^+, S^-, K^{\pm 1} \}.$$

$$[R(\lambda), X] = 0, \quad X = S^+, S^-, K^{\pm 1}.$$

$$[S^+, S^-] = \frac{K^2 - K^{-2}}{q - q^{-1}}, \quad K S^{\pm} K^{-1} = q^{\pm 1} S^{\pm}.$$

$$\Delta(S^+) = S^+ \otimes K^{-2} + I \otimes S^+$$

Equivalently,

$$\Delta(S^-) = S^- \otimes I + K^2 \otimes S^-$$

$$\Delta(K^{\pm 1}) = K^{\pm 1} \otimes K^{\pm 1}$$

$$\begin{aligned} R_{an}(\lambda) = \; & 1_a \cdot \mathrm{sh}\left(\lambda + \frac{i\gamma}{2}\sigma_n^z\right)\mathrm{ch}\left(\frac{i\gamma}{2}\right) \\ & + \sigma_a^z \cdot \mathrm{ch}\left(\lambda + \frac{i\gamma}{2}\sigma_n^z\right)\mathrm{sh}\left(\frac{i\gamma}{2}\right) \\ & + \sigma_a^+ \cdot \frac{1}{2}(q - q^{-1}) S_n^- \\ & + \sigma_a^- \cdot \frac{1}{2}(q - q^{-1}) S_n^+ \end{aligned}$$

$$R_{an}(0) = \frac{1}{2}(q + q^{-1}) \underbrace{P_{an}}_{\text{permutation}}.$$

$$\log \underbrace{T(\lambda)}_{\shortparallel} = \sum_r \lambda^r Q_r$$

$$\mathrm{tr}_a\left(R_{aN}(\lambda) \cdots R_{a1}(\lambda) \right)$$

$$Q_1 = \frac{d}{d\lambda} \log T(\lambda) \Big|_{\lambda=0} = H_{XXZ}.$$

Conserved quantities

- $\log T(\lambda) = \sum_r \lambda^r Q_r$

$$\Rightarrow \forall r, \quad [H_{XXZ}, \underbrace{Q_r}_{\text{conserved quantities}}] = 0$$

- $T^{(s)}(\lambda) := \mathrm{tr}_a M_a^{(s)}(\lambda)$

$$M_a^{(s)}(\lambda) = R^{(s)}_{aN}(\lambda) \cdots R^{(s)}_{a1}(\lambda) \in \mathrm{End}\left(V_a \otimes (\mathbb{C}^2)^{\otimes N} \right)$$

$$V_a = \mathbb{C}^{2s+1} \quad s: \text{(half-) integer}$$

$$\mathbb{C}^{\infty} \quad s \in \mathbb{C} \text{ for generic}$$

$$\log T^{(s)}(\lambda) = \sum_r \lambda^r Q_r^{(s)}$$

$$\underset{\text{YBE}}{\overset{\uparrow}{\Rightarrow}} \forall r, s, \quad [H_{XXZ}, \underbrace{Q_r^{(s)}}_{\text{conserved quantities}}] = 0$$

Spin-s (2s+1 -dim.) rep. of $U_q(sl_2)$

$$K^{\pm 1} = \sum_{r=0}^{2s} q^{\pm(s-r)} |r\rangle\langle r|$$

$$S^{+} = \sum_{r=0}^{2s-1} \frac{q^{r+1} - q^{-r-1}}{q-q^{-1}} |r\rangle\langle r+1|$$

$$S^{-} = \sum_{r=0}^{2s-1} \frac{q^{2s-r} - q^{-2s+r}}{q-q^{-1}} |r+1\rangle\langle r|$$

for (half-) integer s.

$$K^{\pm 1} = \sum_{r=0}^{\infty} q^{\pm(s-r)} |r\rangle\langle r|$$

$$S^{+} = \sum_{r=0}^{\infty} \frac{q^{r+1} - q^{-r-1}}{q-q^{-1}} |r\rangle\langle r+1|$$

$$S^{-} = \sum_{r=0}^{\infty} \frac{q^{2s-r} - q^{-2s+r}}{q-q^{-1}} |r+1\rangle\langle r|$$

for $s \in \mathbb{C}$
q = generic.

$\langle r | r' \rangle = \delta_{r,r'}$

$\{|r\rangle\}_{r=0}^{2s}$ spans $\mathbb{C}^{2s+1}$.

: orthonormal basis.

Eigenvalues & eigenvectors of H

determines the dynamics (time evolution)

Transfer matrix is the generating func. of conserved quantities (including H).

$\Rightarrow$ Diagonalize $T(\lambda)$ instead of H.

Let $V_a = \mathbb{C}^2 = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} =: |\uparrow\rangle, \begin{pmatrix} 0 \\ 1 \end{pmatrix} =: |\downarrow\rangle \right\}$.

$$M_a(\lambda) = \begin{array}{cc} \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}_a & \begin{array}{c} |\uparrow\rangle \\ |\downarrow\rangle \end{array} \end{array}$$

$\mathbb{C}^{\otimes N}$ with weight M_z.

$$\left(\begin{array}{lll} A, D : & \mathcal{H}_{M_z} \to \mathcal{H}_{M_z} \\ B : & \mathcal{H}_{M_z} \to \mathcal{H}_{M_z+1} \\ C : & \mathcal{H}_{M_z} \to \mathcal{H}_{M_z-1} \end{array} \right.$$

☺ For $S^z := \frac{1}{2} \sum_n \sigma_n^z$, $[H, S^z] = 0$

$\Rightarrow \mathcal{H} = \bigoplus_{M_z} \mathcal{H}_{M_z}$, $S^z |\psi_{M_z}\rangle = M_z |\psi_{M_z}\rangle$

$|\psi_{M_z}\rangle \in \mathcal{H}_{M_z}$.

YBE

$$\Rightarrow \quad R_{ab}(\lambda-\mu)\, M_a(\lambda)\, M_b(\mu) = M_b(\mu)\, M_a(\lambda)\, R_{ab}(\lambda-\mu)$$

tells the algebraic (commutation) relations among A, B, C, and D.

Examples)

$$A(\lambda)\, B(\mu) = \frac{\operatorname{sh}(\mu-\lambda+i\gamma)}{\operatorname{sh}(\mu-\lambda)}\, B(\mu)\, A(\lambda) - \frac{\operatorname{sh}(i\gamma)}{\operatorname{sh}(\mu-\lambda)}\, B(\lambda)\, A(\mu)$$

$$D(\lambda)\, B(\mu) = \frac{\operatorname{sh}(\lambda-\mu+i\gamma)}{\operatorname{sh}(\lambda-\mu)}\, B(\mu)\, D(\lambda) - \frac{\operatorname{sh}(i\gamma)}{\operatorname{sh}(\lambda-\mu)}\, B(\lambda)\, D(\mu)$$

$$[B(\lambda), B(\mu)] = 0.$$

Diagonalization of T

Remind $T(\lambda) := \mathrm{tr}_a M_a(\lambda) = A(\lambda) + D(\lambda)$.

Trivial eigenvector of $T(\lambda)$

$$|\psi_{M_z=\frac{N}{2}}\rangle = |\uparrow\uparrow\cdots\uparrow\rangle = \bigotimes_{n=1}^{N} |\uparrow\rangle_n .$$

: the highest weight state.

One of the vector $|\psi_{M_z<\frac{N}{2}}\rangle \in \mathcal{H}_{M_z}$ is obtained as

$$|\psi_{M_z<\frac{N}{2}}\rangle = \prod_{n=1}^{\frac{N}{2}-M_z} B(\lambda_n)\,|\psi_{M_z=\frac{N}{2}}\rangle \quad — \quad (*)$$

Remark)

$\langle\psi_{M_z}|\psi_{M_z'}\rangle = 0$ unless $M_z = M_z'$.

The vector ★ is the eigenvec. of T if

$$T(\lambda)\prod_{n=1}^{M} B(\lambda_n)\,|\psi_{M_z=\frac{N}{2}}\rangle$$

$$= \big(A(\lambda)+D(\lambda)\big)\prod_{n=1}^{M} B(\lambda_n)\,|\psi_{M_z=\frac{N}{2}}\rangle$$

$$= \left(a(\lambda)\prod_{n=1}^{M}\frac{\mathrm{sh}(\lambda_n-\lambda+i\gamma)}{\mathrm{sh}(\lambda_n-\lambda)} + d(\lambda)\prod_{n=1}^{M}\frac{\mathrm{sh}(\lambda-\lambda_n+i\gamma)}{\mathrm{sh}(\lambda-\lambda_n)} \right)$$

$$\times\, B(\lambda_1)\cdots B(\lambda_M)\,|\psi_{M_z=\frac{N}{2}}\rangle$$

$$\underbrace{- \sum_{k=1}^{M}\left(a(\lambda_k)\frac{\mathrm{sh}(\lambda_k-\lambda)}{\mathrm{sh}(i\gamma)}\prod_{n\neq k}\frac{\mathrm{sh}(\lambda_n-\lambda_k+i\gamma)}{\mathrm{sh}(\lambda_n-\lambda_k)} + d(\lambda_k)\frac{\mathrm{sh}(\lambda-\lambda_k)}{\mathrm{sh}(i\gamma)}\prod_{n\neq k}\frac{\mathrm{sh}(\lambda_k-\lambda_n+i\gamma)}{\mathrm{sh}(\lambda_k-\lambda_n)} \right) \times B(\lambda_1)\cdots \underset{j}{B(\lambda)}\cdots B(\lambda_M)\,|\psi_{M_z=\frac{N}{2}}\rangle .}_{=\,0\,.}$$

— ★★

$a(\lambda)$ & $d(\lambda)$ are eigenvalues of $A(\lambda)$ & $D(\lambda)$ on $|\psi_{M_z=\frac{N}{2}}\rangle$:

$$A(\lambda)\,|\psi_{M_z=\frac{N}{2}}\rangle = \underbrace{a(\lambda)}_{=\left(\mathrm{sh}\left(\lambda+\frac{i\gamma}{2}\right)\right)^N}|\psi_{M_z=\frac{N}{2}}\rangle$$

$$D(\lambda)\,|\psi_{M_z=\frac{N}{2}}\rangle = \underbrace{d(\lambda)}_{=\left(\mathrm{sh}\left(\lambda-\frac{i\gamma}{2}\right)\right)^N}|\psi_{M_z=\frac{N}{2}}\rangle .$$

"Bethe equations"

$$\left(\frac{\mathrm{sh}(\lambda_k+\frac{i\gamma}{2})}{\mathrm{sh}(\lambda_k-\frac{i\gamma}{2})}\right)^N = \prod_{n\neq k}\frac{\mathrm{sh}(\lambda_k-\lambda_n+i\gamma)}{\mathrm{sh}(\lambda_k-\lambda_n-i\gamma)}, \quad k=1,2,\cdots,M.$$

Remarks)

- Bethe states $\prod_{n=1}^{M} B(\lambda_n)|\psi_{M_z=\frac{N}{2}}\rangle$ are the h.w.s.

  λ_n: sol. of the Bethe eq.

$$S^+\prod_{n=1}^{M} B(\lambda_n)|\psi_{M_z=\frac{N}{2}}\rangle = 0.$$

- The other eigenvector of T (not Bethe states) are constructed by applying S^-

$$\prod_{n=1}^{M} B(\lambda_n)|\psi_{M_z=\frac{N}{2}}\rangle$$

$$\downarrow S^-$$

$$S^-\prod_{n=1}^{M} B(\lambda_n)|\psi_{M_z=\frac{N}{2}}\rangle$$

$$\downarrow S^-$$

$$(S^-)^2\prod_{n=1}^{M} B(\lambda_n)|\psi_{M_z=\frac{N}{2}}\rangle$$

$$\vdots$$

$$\downarrow S^-$$

$$(S^-)^{N-2M}\prod_{n=1}^{M} B(\lambda_n)|\psi_{M_z=\frac{N}{2}}\rangle$$

lowest weight state.

Bethe roots

Bethe equations

$$\left(\frac{\mathrm{sh}\left(\lambda_j-\frac{i\gamma}{2}\right)}{\mathrm{sh}\left(\lambda_j+\frac{i\gamma}{2}\right)}\right)^N=\prod_{k\neq j}\frac{\mathrm{sh}(\lambda_j-\lambda_k-i\gamma)}{\mathrm{sh}(\lambda_j-\lambda_k+i\gamma)}$$

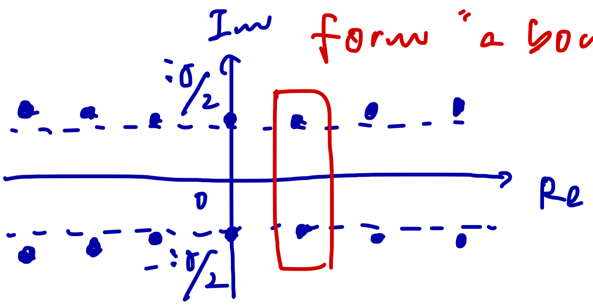
are known to form "string solutions" in large N :

$$\lambda_a^{j,r}\approx \underset{\substack{\uparrow\\ \mathbb{R}\,:\ \text{string center}}}{\lambda_a^{j}}+\frac{i\gamma}{2}(l-2r+1)+i\underbrace{\delta_a^{j,r}}_{O(e^{-N})},\quad r=1,2,\cdots,l$$

l-string solution.

Examples)

2-string solutions



- $q=e^{i\gamma}$, $\gamma=\frac{\pi}{l}$ (l : coprime)

$$\left(\begin{array}{l}1,2,\cdots,(l-1)\text{-string sol.}\\ \text{anti-string sol.}\end{array}\right.$$

- $b=e^{\eta}$

  1, 2, $\cdots$ string sol.

Bethe solutions in the $N \to \infty$ lim.

Bethe eq.

$$\Rightarrow \theta_{kin}(\lambda_j) + \frac{1}{N}\sum_{k=1}^{M} \theta_{scat}(\lambda_j - \lambda_k) = \frac{2\pi}{L} \underset{\sim}{I_j}$$

(int. for odd N
half-int for even N

$$\left(\begin{aligned} \theta_{kin}(\lambda) &:= \frac{i}{N}\sum_{n=1}^{N} \ln \frac{\mathrm{sh}(\lambda - \frac{i\gamma}{2})}{\mathrm{sh}(\lambda + \frac{i\gamma}{2})} \\ \theta_{scat}(\lambda) &:= i \ln \frac{\mathrm{sh}(i\gamma + \lambda)}{\mathrm{sh}(i\gamma - \lambda)} \end{aligned} \right.$$

when $\forall j,\ \lambda_j \in \mathbb{R}$.

$\Rightarrow$ A set of (half-)integers $\{I_j\}$ uniquely correspond to an eigenvector.

String center of each string solutions becomes dense in the <u>thermodynamic limit</u>.

$$N \to \infty,\ M/N \text{ fixed}.$$

$\Rightarrow$ "Density of Bethe roots"

$$\rho(\lambda) := \frac{d}{d\lambda} \underset{\sim}{x(\lambda)}$$

$$\uparrow$$

$$x_j := \frac{I_j}{N}$$

for the ground state ($\forall j,\ I_{j+1} - I_j = 1$).

Remark) $W[\{\rho_i\}]$

Many microstates could be expressed by the same Bethe root density.

(We only discuss the macro quantities that are insensitive to the micro feature.)

$$S_{YY} := W[\{\rho_i\}] \quad : \text{Yang-Yang entropy.}$$

§ 4. Specific behavior of integrable systems

Due to existence of many conserved quantities, integrable systems show specific behaviors uncommon to normal thermalizing system.

Ralaxation state of the XXZ model

Proposition) [Rigol et al. 2008, CM 2020]

$$\rho_{GGE} = \frac{1}{Z} e^{-\sum_{r} \beta_r Q_r}$$

The XXZ model relaxes to the density matrix

$$\rho_{GGE} = \frac{1}{Z} e^{-\sum_{n=1}^{\infty}\left(\sum_{r=1}^{l-1} \beta_n^{r} Q_n^{r} + \beta_n^{s} Q_{n,p}(s)\right)}, \quad p \in \mathbb{Z}_{\geq 0}$$

$$\left(q = e^{i\frac{\pi}{l}} \times \overset{n}{\underset{\mathbb{Z}}{\cap}} \right)$$

in the thermodynamic Limit.

The relaxation state of the XXZ model in TL is completely described by $\underbrace{\{\rho^{(r)}(\lambda_{th})\}_{r=1,\cdots,l}}$.

$$\{Q_1^{(r)}(\lambda_{th})\}_{r=1,\cdots,l-1} \cap \{Q_{1,0}(\lambda_{th}, s)\}$$

Linearly-independent set of conserved quantities.

XXZ model

- eigenvectors for $T \to \infty$
  $\simeq \{\rho^r(\lambda)\}_r$
  $+$
  saddle point method

  $\Bigg\}$ $\{\rho^r(\lambda)\}_r$ for the steady state.

- $\{\rho^r(\lambda)\}_r \simeq \{\tau^r(\lambda)\}_r \simeq \{Q_n^r\}_{r,n}$

  $\{\tau^r(\lambda)\}_r$: transfer matrices

  $\{Q_n^r\}_{r,n}$: conserved quantities

Conserved quantities in terms of $\rho^{(r)}$

$$\log \underbrace{T^{(r)}(\lambda)}_{\shortparallel} = \sum_n (\lambda-\lambda_0)^n \mathcal{Q}_n^{(r)}(\lambda_0)$$

$$\mathrm{tr}_a M_a^{(r)}(\lambda), \quad M_a^{(r)}(\lambda) \in \mathrm{End}\Big(\underbrace{V_a}_{\in\ \mathbb{C}^{2r+1}} \otimes (\mathbb{C}^2)^{\otimes N}\Big)$$

$$\mathcal{Q}_1^{(r)}(\lambda_0) \underset{\text{T.L.}}{\propto} \sum_{j=1}^{l} \sum_{a=1}^{\min(n_j, r)} \theta^{(1,0)}_{|r-n_j|-1+2a,\, v_j} \underset{\uparrow}{*} \rho_j(\lambda_0)$$

convolution

$$\log \underbrace{T(\lambda, s)}_{\shortparallel} = \sum_{n,m} (\lambda-\lambda_0)^n (s-s_0)^m \mathcal{Q}_{n,m}(\lambda_0, s_0)$$

$$\mathrm{tr}_a M_a(\lambda, s), \quad M_a(\lambda, s) \in \mathrm{End}\Big(\underbrace{V_a}_{\in\ \mathbb{C}^{\infty}\ (\text{spin-}\underset{\in \mathbb{C}}{s}\ \text{rep.})} \otimes (\mathbb{C}^2)^{\otimes N}\Big)$$

$\mathcal{Q}_{n,m}(\lambda, s)$

$$\underset{\text{T.L.}}{\propto} \sum_{j=1}^{l} \sum_{a=1}^{n_j} \theta^{(n,m)}_{2s-\frac{l-1}{2}-n_j-1+2a,\, v_j} * \rho_j\Big(\lambda + i\big(\tfrac{l-1}{4}+\tfrac{1}{2}\big)\gamma\Big)$$

$$- \sum_{j=1}^{l} \sum_{a=1}^{\min(n_j, \frac{l-1}{2})} \theta^{(n,m)}_{|\frac{l-1}{2}-n_j|-1+2a,\, v_j} * \rho_j\Big(\lambda + i\big(\tfrac{l-1}{4}-s-\tfrac{1}{2}\big)\gamma\Big)$$

The kernels

$$\theta^{(1,0)}_{n,\nu}(\lambda) = \frac{\nu}{\pi} \frac{\sin(n\gamma)}{\mathrm{ch}(2\lambda) - \nu\cos(n\gamma)}$$

$$\theta^{(0,1)}_{n,\nu}(\lambda) = \frac{\nu\gamma}{\pi} \frac{\mathrm{sh}(2\lambda)}{\mathrm{ch}(2\lambda) - \nu\cos(n\gamma)}$$

$$\nu = \begin{cases} +1 & \text{for string} \\ -1 & \text{for anti-string} \end{cases}$$

satisfy

$$\sum_{a=1}^{\min(n_j, 2s+1)} F_k\left[\theta^{(1,0)}_{|2s+1-n_j|-1+2a,\, \nu_j}(\lambda)\right]$$

$$+ \sum_{a=1}^{\min(n_j, 2s-1)} F_k\left[\theta^{(1,0)}_{|2s-1-n_j|-1+2a,\, \nu_j}(\lambda)\right]$$

$$= 2\cosh\left(\frac{\pi k}{2\ell}\right) \sum_{a=1}^{\min(n_j, 2s)} F_k\left[\theta^{(1,0)}_{|2s-n_j|-1+2a,\, \nu_j}(\lambda)\right] - \delta_{j,2s}$$

Fourier transf.

$$F_k[\rho(\lambda)] := \int_{-\infty}^{\infty} d\lambda\, e^{-ik\lambda} \rho(\lambda)$$

Proposition)

Conserved quantities are expressed by the densities of string centers.

"String - charge duality"

- $$F_K[\rho^{(r)}(\lambda)] - \delta_{r,l-1} F_K[\rho^{(l)}(\lambda)] = 2\,\mathrm{ch}\left(\frac{\gamma K}{2}\right) F_K[\mathfrak{a}_1^{(r)}(\lambda)] - F_K[\mathfrak{a}^{(r+1)}(\lambda)] - F_K[\mathfrak{a}^{(r-1)}(\lambda)]$$
  $$(r = 1, 2, \cdots, l-1)$$

- $$F_K[\rho^{(l)}(\lambda)] = -\frac{\mathrm{sh}\left((l-2s)\frac{\gamma K}{2}\right)}{\mathrm{ch}\left((l-2s-1)\frac{\gamma K}{2}\right)} F_K[\mathfrak{a}_1^{(l-1)}(\lambda)] + \frac{\mathrm{sh}\left(\frac{\gamma K}{2}\right)}{\mathrm{sh}\left((l-2s-1)\frac{\gamma K}{2}\right)} F_K[\mathfrak{a}_{1,0}(\lambda, s)]$$

Lemma)

Spin-flip non-invariant charges are not linearly independent.

$$F_k[Q_{1,0}(\lambda,s)] = \frac{\mathrm{sh}\left((l-2s-1)\frac{\gamma k}{2}\right)}{\mathrm{sh}\left((l-2t-1)\frac{\gamma k}{2}\right)} F_k[Q_{1,0}(\lambda,t)] - \frac{\mathrm{sh}\left((2t-2s)\frac{\gamma k}{2}\right)}{\mathrm{sh}\left((l-2t-1)\frac{\gamma k}{2}\right)} F_k[Q_1^{(l-1)}(\lambda)]$$

$$F_k[Q_{r,2p}(\lambda,s)] = (-ik)^{r-1}(-\gamma k)^{2p} F_k[Q_{1,0}(\lambda,s)]$$

$$F_k[Q_{r,2p-1}(\lambda,s)] = (-ik)^{r-1}\frac{(-\gamma k)^{2p-1}}{\mathrm{sh}\left((l-2s-1)\frac{\gamma k}{2}\right)} F_k[Q_1^{(l-1)}(\lambda)] + (-ik)^{r-1}(-\gamma k)^{2p-1}\coth\left((l-2s-1)\frac{\gamma k}{2}\right) F_k[Q_{1,0}(\lambda,s)]$$

Non-zero (macroscopic) spin current

Drude weight (in linear response)

$$D\left(\beta=\frac{1}{k_BT}\right) = \lim_{t\to\infty}\lim_{N\to\infty}\frac{\beta}{2Nt}\int_0^t dt' \langle J(0), J(t')\rangle_\beta$$

$$\geq \lim_{N\to\infty}\frac{1}{2N}\frac{|\langle J, Q_k\rangle_\beta|^2}{\|Q_k\|_\beta^2}$$

$D > 0 \Rightarrow$ ballistic transport

(non-vanishing DC current)

Prop.) Drude weight for spin current is non-zero at high temperature.

$$\lim_{\beta\to 0} D_s(\beta) > 0$$

(c.f.) Conserved operator constructed from the monodromy matrix with the aux. space of complex-spin rep. has finite overlap with the spin current operator

$$\frac{i}{2}\sum_n\left(\sigma_n^+\sigma_{n+1}^- - \sigma_n^-\sigma_{n+1}^+\right)$$

$$\begin{cases} \langle J_s, Q_{r,2p-1}(\overset{\in\mathbb{C}}{\lambda}, \overset{\in\mathbb{C}}{s_0})\rangle \sim N^{r+p} \\ \langle J_s, Q_{r,2p}(\lambda, s_0)\rangle \sim 0 \\ \|Q_{r,p}(\lambda, s_0)\|^2 \sim N^{2r+p} \end{cases}$$

§ 4 Summary

- Integrable systems have many conserved quantities due to the Yang-Baxter eq.

- Equilibrium state can't be defined for integrable systems (equivalently, integrable systems do not thermalize).

  Instead, integrable systems relax to the steady state called "generalized Gibbs ensemble (GGE)".

- The XXZ model (an example of integrable systems) shows finite spin current even after relaxation.