On the equivalence of two non-Riemannian curvatures in Warped product Finsler metrics

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Motivation

- The notion of warped products plays an important roles not only in geometry but also in mathematical physics, especially in general relativity.
- Specifically, many basic solutions of the Einstein field equations, including the Schwarzschild and the Robertson-Walker models are warped product metrics.

History and introduction

- In the renowned lecture of 1854 entitled, "On the Hypothesis, which lie at the foundations of Geometry" B. Riemann discussed the possibility of a metric, more general than the square root of a quadratic differential form (what we call a Riemannian metric).
- In particular, a metric, which is the fourth root of a quartic differential form. The origin of Finsler geometry seems to appear in this Habilitation lecture.

History and introduction

- Formally introduced by P. Finsler in 1918 in his thesis under supervision of C. Carathoédory and developed by several geometers including J. Taylor, L. Berwald, E. Cartan, H. Rund, M. Hashiguchi, M. Matsumoto etc.
- In the last decade of previous century S. S. Chern had also taken interest in development of global Finsler geometry.
- Finsler geometry is considered as a generalization of Riemannian geometry. In fact, every Riemannian metric is Finsler metric.

Finsler Manifold

Let M be a connected n-dimensional smooth manifold and $TM = \bigsqcup_{u \in M} T_u M$ be its tangent bundle.

Finsler metric

The Finsler metric on M is a continuous function $F : TM \longrightarrow [0, \infty)$ satisfies the following conditions:

(1) F is smooth on TM_0,

(2) F is a positively 1-homogeneous on the fibers of tangent bundle TM,

(3) The Hessian of $\frac{F^2}{2}$ with element $g_{ij} = \frac{1}{2}\frac{\partial^2 F^2}{\partial v^i \partial v^j}$ is positive definite on TM_0.

The pair (M, F) is called a Finsler manifold and g_{ij} is called the fundamental metric tensor.

Remark

- Condition (2) means, in general $F(u, -v) \neq F(u, v)$.
- Condition (3) implies that, $F(v_1 + v_2) \leq F(v_1) + F(v_2) \forall v_1, v_2 \in TM \backslash \{0\}$ (Triangle inequality)

- Every Riemannian metric is an example of a Finsler metric.

Riemann vs Finsler metric

- A Riemannian metric on a smooth manifold is a smoothly varying family of inner products one on each tangent space.
- A Finsler metric on a smooth manifold is a smoothly varying family of Minkowski norm one on each tangent space.

Some non-Riemannian examples of Finsler metric

Randers metric

The Randers metric is of the form $F = \alpha + \beta$, where α is a Riemannian metric and β is a one form, which is one of the most simplest non-Riemannian Finsler metric.

Funk metric is an example of Randers metric on $\mathbb{B}^n$ given by

$$F_\xi(x, \xi) = \frac{\langle x, \xi \rangle}{(1 - ||x||^2)} + \frac{\sqrt{\langle x, \xi \rangle^2 + (1 - ||x||^2)||\xi||^2}}{(1 - ||x||^2)}.$$

- Berwald metric is another important class of non-Riemannian Finsler metric in which all the tangent spaces are linearly isometric to each other.
- Various generalizations of Berwald metrics have been introduced and studied by several geometers. Landsberg metric, Douglas metric, weakly Berwald metric are few of them.

Warped product

- The warped product Riemannian manifolds, first introduced by Bishop and O'Neill are the natural and fruitful generalization of Riemannian products of two manifolds (R. L. Bishop, B. O'Neill. *Manifolds of negative curvature*,Trans Amer Math Soc 145: 1-49 (1969)).
- Later on, the warped product metric was extended to the case of Finsler manifolds in the works of Chen et al. and Kozma et al.
  - B. Chen, Z. Shen, L. Zhao, *Constructions of Einstein Finsler Metrics by Warped Product.* Internat J Math, 29(11): 1850081, (2018).
  - L. Kozma, R. Peter, C.Varga, *Warped product of Finsler manifolds.* Ann Univ Sci Budapest 44: 157–170 (2001).

Warped product

- Let (M, ds_1^2) and (N, ds_2^2) are the Riemannian manifolds. A warped product of M and N is the manifold $M \times N$ endowed with a Riemannian metric of the form

$$ds^2 = ds_1^2 + f^2 ds_2^2$$

where f is a smooth function depending on the coordinates of M only, known as the warping function.

- In non-Riemannian setting, Kozma, Peter and Varga have studied product manifolds $M \times N$ endowed with a Finsler metric called warped product

$$F^2 = F_1^2 + f^2 F_2^2$$

where (M, F_1) and (N, F_2) are Finsler manifolds and f is a smooth function on M.

- In the present work, we consider the product manifold

$$M = I \times \bar{M}$$

where I is an interval of $\mathbb{R}$ and $\left(\bar{M}, \bar{\alpha}\right)$ is an $(n-1)$ dimensional Riemannian manifold $(\bar{\alpha}^2 = \bar{a}_{ab}\, v^a\, v^b)$. A warped product Riemannian metric on M can be written as

$$\alpha = dr^2 + w(r)\bar{\alpha}^2 \tag{1}$$

where r is the standard coordinate on I and $w(r)$ is a positive function. It is well-known that if

$$w(r) = \begin{cases} \frac{1}{K}\sin^2\sqrt{K}r;\ K > 0 \\ r^2;\ K = 0 \\ \frac{1}{-K}\sinh^2\sqrt{-K}r;\ K < 0 \end{cases}$$

$\bar{\alpha}$ has constant Ricci curvature $(n-2)$, then α has constant Ricci curvature $(n-1)K$.

- Now, we express the Riemannian warped product in Finslerian language.
- Let $\{\theta^a\}_{a=2}^n$ be a local coordinate system on $\bar{M}$. Then $\{u^i\}_{i=1}^n$ gives us a local coordinate on M by setting $u^1 = r$ and $u^a = \theta^a$.
- A vector v on M can be written as $v = v^i\partial/\partial u^i$ and its projection on $\bar{M}$ is denoted by

$$\bar{v} = v^a\partial/\partial u^a = v^a\partial/\partial\theta^a.$$

- The Finsler version of (1) is given below

$$\alpha(u,v) = \sqrt{v^1v^1 + w(r)\bar{\alpha}^2(\theta,\bar{v})}, \tag{2}$$

where $\bar{\alpha}(\theta,\bar{v}) = \sqrt{\bar{a}_{ij}(\theta)v^iv^j}$. We generalize it to the form

$$F = \sqrt{w(v^1, r, \bar{\alpha})}$$

- Since F is positive homogeneous of degree one, we have

$$F = \bar{\alpha}\sqrt{w\left(\frac{v^1}{\bar{\alpha}}, r, 1\right)}$$

for $\bar{\alpha} \neq 0$

- Let $s = \frac{v^1}{\bar{\alpha}}$, then a warped product Finsler metric can be written in the form

$$F = \bar{\alpha}\sqrt{w(s,r)}, \tag{3}$$

where w is a suitable function defined on an open subset of $\mathbb{R}^2$. It can be rewritten as

$$F = \bar{\alpha}\phi(s,r), \qquad \text{where} \quad \phi(s,r) = \sqrt{w(s,r)}. \tag{4}$$

Fundamental metric tensor of warped product Finsler metric

The coefficients of fundamental metric tensor of the warped product Finsler metrics are given by

$$\begin{pmatrix} g_{11} & g_{1j} \\ g_{i1} & g_{ij} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}w_{ss} & \frac{1}{2}\chi_s \bar{\alpha}_{v^j} \\ \frac{1}{2}\chi_s \bar{\alpha}_{v^i} & \frac{1}{2}\chi \bar{a}_{ij} - \frac{1}{2}s\chi_s \bar{\alpha}_{v^i}\bar{\alpha}_{v^j} \end{pmatrix},$$

where $\chi := 2w - sw_s$ and $\chi_s := w_s - sw_{ss}$.
By some simple calculation we have, $\det(g_{ij}) = \frac{1}{2^{n-1}}\chi^{n-2}\Lambda$, where $\Lambda = 2ww_{ss} - w_s^2$.
Which can be rewritten as

$$\det(g_{ij}) = \phi^{n+1}\phi_{ss}(\phi - s\phi_s)^{n-2} \tag{5}$$

Geodesic and spray coefficients

- A smooth curve in a Finsler manifold is a geodesic if it has constant speed and is locally length minimizing.
- Thus, a geodesic in a Finsler manifold (M, F) is a curve $\gamma : I = (a, b) \to M$ with $F(\gamma(t), \dot{\gamma}(t)) =$ constant and for any $t_0 \in I$, there is a small number $\epsilon > 0$, such that γ is length minimizing on $[t_0 - \epsilon, t_0 + \epsilon] \cap I$.

Geodesic and spray coefficients

- A smooth curve γ in a Finsler manifold (M, F) is a geodesic if and only if $\gamma(t) = (u^i(t))$ satisfies the following second order non linear differential equation:

$$\frac{d^2u^i(t)}{dt^2} + G^i\left(u^i, \frac{du^i}{dt}\right) = 0, \qquad 1 \leq i \leq n,$$

where $G^i = G^i(u, v)$ are local functions on TM defined by

$$G^i = \frac{1}{4} g^{i\ell}\left\{\left[F^2\right]_{u^k v^\ell} v^k - \left[F^2\right]_{u^\ell}\right\}.$$

The coefficients G^i are called the spray coefficients and the quantity

$$G := v^i \frac{\partial}{\partial u^i} + G^i \frac{\partial}{\partial v^i}$$

is called spray on M.

Lemma

The spray coefficients of warped product Finsler metrics F are given by

$$G^1 = \Phi\bar{\alpha}^2, \quad G^k = \bar{G}^k + \Psi\bar{\alpha}v^k \tag{6}$$

where

$$\Phi = \frac{1}{4}\left\{\left(W_r - \chi_r\right)U + s\chi_r V\right\}, \tag{7}$$

$$\Psi = \frac{1}{4}\left\{\left(W_r - \chi_r\right)V + s\chi_r\left(W + X\right)\right\} \tag{8}$$

and

$$\chi = 2w - sw_s, \quad \Lambda = 2ww_{ss} - w_s^2, \quad U = \frac{2\chi - 2s\chi_s}{\Lambda},$$

$$V = -\frac{2\chi_s}{\Lambda}, \quad W = \frac{2}{\Lambda} \quad X = \frac{2w_s\chi_s}{\chi\Lambda}.$$

Volume forms in Finsler geometry

- A volume form $d\mu$ on Finsler manifold (M, F) is nothing but a global nondegenerate n-form on M.
- In local coordinates we can express $d\mu$ as

$$d\mu = \sigma(u) du^1 \wedge \dots \wedge du^n$$

- For a vector $v \in T_uM \setminus \{0\}$, define

$$\tau_F = \log \frac{\sqrt{\det(g_{ij}(u,v))}}{\sigma(u)}.$$

where τ_F is called distortion of $(M, F, d\mu)$.

S-curvature

- let $\gamma = \gamma(t)$ be the geodesic with $\gamma(0) = u$ and $\dot{\gamma}(0) = v$. To measure the rate of distortion along geodesics, we define

$$S(u, v) = \frac{d}{dt}\left[\tau_F(\gamma(t), \dot{\gamma}(t))\right]\Big|_{t=0},$$

S is called the S-curvature.

Busemann-Hausdorff volume form

In a local coordinate system (u^i, v^i), the Busemann-Hausdorff volume form is defined as $dV_{BH} = \sigma_{BH}(u)du^i$, where

$$\sigma_{BH}(u) = \frac{\text{Vol}(\mathbb{B}^n(1))}{\text{Vol}\{(v^i) \in \mathbb{R}^n; F\left(u, v^i \frac{\partial}{\partial u^i}\right) < 1\}} \tag{9}$$

where $\mathbb{B}^n(1)$ denotes the Euclidean unit n-ball, and vol the standard Euclidean volume.

Holmes-Thompson volume form

In a local coordinate system (u^i, v^i), the Homes-Thompson volume form is defined as $dV_{HT} = \sigma_{HT}(u)du^i$, where

$$\sigma_{HT}(u) = \frac{1}{\mathsf{Vol}(\mathbb{B}^n(1))} \int_{F\left(u, v^i \frac{\partial}{\partial u^i}\right) < 1} \det\left(g_{ij}(u,v)\right) dV_{S_u M}, \tag{10}$$

where

$$dV_{S_u M} := \sqrt{\det(g_{ij}(u,v))} \sum_{i=1}^{n} (-1)^{i+1} \frac{v^i}{F} \frac{dv^1}{F} \wedge \ldots \wedge \frac{d\bar{v}^i}{F} \wedge \ldots \wedge \frac{dy^n}{F}$$

is the induced volume form of $S_u M := \left\{ v \in T_u M | F(u,v) = 1 \right\}$ from the Riemannian metric $\bar{g} = g_{ij}(u,v) dv^i \bigotimes dv^j$ on $TM \backslash \{0\}$ and

$$V_{S_u M} = \int_{S_u M} dV_{S_u M}$$

is the corresponding volume of $S_u M$.

E-curvature of the Finsler metric

The E-curvature of the Finsler metric F is defined as

$$E_{ij} := \frac{1}{2} S_{v^i v^j}(u, v) = \frac{1}{2} \frac{\partial^2}{\partial v^i \partial v^j} \left(\frac{\partial G^m}{\partial v^m} \right),$$

where G^i are the spray coefficients of the Finsler metric F.

Isotropic S-curvature

A Finsler metric F is said to be isotropic S-curvature if

$$S = (n+1)c(u)F,$$

where $c(u)$ is the scalar function on M.
F is said to be constant S-curvature if $c(u)$ is constant.

Isotropic E-curvature

The Finsler metric F is said to be of isotropic E-curvature if there exist a scalar function $c = c(u)$ on M such that

$$E_{ij} = \frac{n+1}{2} c(u) F_{v^i v^j}.$$

F is said to be constant E-curvature if $c(u)$ is a constant

Main Results

Theorem (Shriwastawa, Tiwari and Gangopadhyay (2023))

The warped product Finsler metric $F = \bar{\alpha}\phi(r, s)$ defined in (4), has isotropic S-curvature if and only if it has isotropic E-curvature.

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Theorem (Shriwastawa, Tiwari and Gangopadhyay (2023))

The warped product Finsler metric $F = \bar{\alpha}\phi(r, s)$ defined in (4), has isotropic S-curvature if and only if

$$(\phi_s - s\phi_{ss}) + (n-2)\psi_s + n\psi + s^2\psi_{ss} = k\,(\phi - s\phi_s)$$

where ϕ and ψ are given in (7) and (8) and $k \neq 0$ is a scalar function on M.

Theorem (Shriwastawa, Tiwari and Gangopadhyay (2023))

The warped product Finsler metric $F = \bar{\alpha}\phi(r, s)$ defined in (4), has isotropic E-curvature w. r. to volume form dV_{BH} or dV_{HT} if and only if

$$\phi_s + n\psi - s\psi_s + g(r)s = c(u)(n+1)\phi$$

where $c(u)$ is a scalar function on M and

$$g(r) = -r\frac{k'(r)}{k(r)}.$$

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Arigatougozaimashita !