Kite and Triangle diagrams through Symmetries of Feynman Integrals

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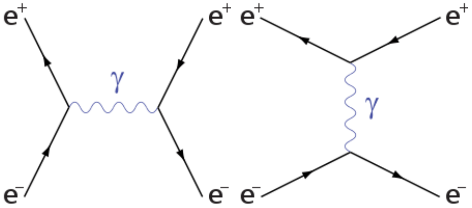
References

- Work motivated from:
  1. arXiv:1507.01359:Symmetries of Feynman integrals and the Integration By Parts method.Barak Kol
  2. arXiv:1604.07827:The algebraic locus of Feynman integrals.Barak Kol
  3. arXiv:1606.09257:Bubble diagram through the Symmetries of Feynman Integrals method.Barak Kol
  4. arXiv:1704.02187:Vacuum seagull: Evaluating a three-loop Feynman diagram with three mass scales.Barak Kol et. al
  5. arXiv:1804.01175:Algebraic aspects of when and how a Feynman diagram reduces to simpler ones. Barak Kol
  6. arXiv:1807.07471:Two-loop vacuum diagram through the Symmetries of Feynman Integrals method.Barak Kol
- Talk based on : PhysRevD.99.045018(1808.02494 [hep-th]) and JHEP 03 (2020) 156(1909.04055 [hep-th]) Barak Kol, Subhajit Mazumdar

Motivation

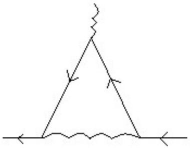
- Feynman diagrams and the associated integrals are at the computational core of quantum field theory and their evaluation attracts considerable attention.
- For example it is necessary for experimental design and data analysis including at the LHC.
- Bhabha Scattering

$$e^+ + e^- \to e^+ + e^-$$

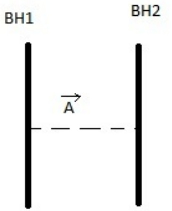


Motivation

- Loops represent virtual quantum process and involve integrals. For example, Quantum Correction to electron gyro-magnetic ratio.



- Growing usage throughout physics. Current-Current effective potential between two black holes.



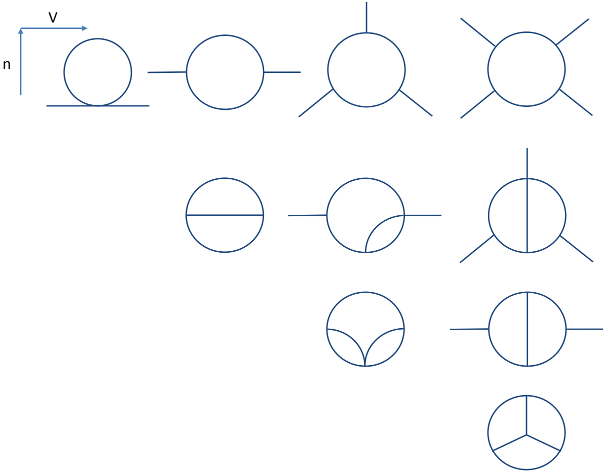
Motivation

- Several important and well-known methods for their computation are Integration By Parts (IBP), Differential Equations (DE) method, Dimensional Recurrence, Canonical Basis for DE, Intersection Theory etc.
- Is there a general theory for computing Feynman diagrams? Despite 70 years of work the answer is NO.

Introduction to SFI (Symmetries of Feynman Integral)

- SFI is a step in that direction. It considers a Feynman diagram of fixed topology (fixed graph), but varying kinematical invariants, masses and spacetime dimension.
- Each diagram is associated with a system of differential equations in this parameter space.
- The equation system defines a Lie group G which acts on parameter space and foliates it into orbits.
- This geometry allows to reduce the diagram to its value at some convenient base point within the same orbit plus a line integral over simpler diagrams, namely with one edge contracted.
- The SFI method is related to both the Integration By Parts method as well as to the Differential Equations method etc.. SFI novelties include the definitions of the group and its orbits, as well as the reduction to a line integral.

Hierarchy of diagrams according to edge contraction



Brief Review of SFI method: Basic Set up

- An associated Feynman Integral with a diagram with L loops, P propagators and N external legs is given by

$$I(m_1^2, \ldots, m_P^2, p_i \cdot p_j) = \int \frac{d^d l_1 \ldots d^d l_L}{(k_1^2 - m_1^2) \ldots (k_P^2 - m_P^2)} = \int d^d l_1 \ldots d^d l_L \tilde{I}$$

- where $\tilde{I}$ is the integrand.
- where

$$k_i = A_{ia} l_a + B_{ij} p_j$$

- Integral is function of mass squares and kinematical invarinats, which is known as "Parameter Space X" of the Diagram.

General idea to obtain SFI Equation Set

- The change of dummy integration variables $l'_a = l_a + \epsilon_{ab} k_b$ shouldn't affect the answer of the integral. Where q_a are linear combination of loop momenta and external momenta.
- This allows us to show that we have the identity

$$0 = \int dl \frac{\partial}{\partial l^\mu} k^\nu \tilde{I}$$

- This gives us a set of partial differential equations, namely,

$$c^a\, I + Tx^a_j\, \partial^j\, I + J^a = 0$$

where c^a are constants depending on the dimensions, Tx^a_j is a matrix linear in parameters J^a depends on simpler diagrams

- Solution for the Integral: Line integral over sources.

$$\hat{I}(x) = \hat{I}(x_0) + \int_{x_0}^{x} J^\alpha(\xi) d\xi_\alpha, \text{ where } \hat{I} = \frac{I}{I_0}, \; I_0 = \text{homogeneous soln.}$$

The identity towards SFI equation set

- Each variation as shown earlier yields,

$$0 = \int d^d l_1 ... d^d l_L \frac{\partial}{\partial_{l_a}} (l_b \tilde{I}) = dI\delta_{ab} - \Sigma_{k_i \in loop\ a} \int_l \frac{2k_i \cdot l_b}{(k_i^2 - x_i)^2} \tilde{I}$$

- We can rewrite the numerators like,

$$2k_i \cdot l_b = \Sigma(k_j^2 - x_j) + x_j$$

-

$$\begin{aligned} 0 &= dI\delta_{ab} - \Sigma_{k_i \in loop\ a} \int_l \frac{\Sigma(k_j^2 - x_j) + x_j}{(k_i^2 - x_i)^2} \tilde{I} \\ &= dI\delta_{ab} - \int \frac{k_j^2 - x_j}{...(k_i^2 - x_i)^2 ... (k_j^2 - x_j)} - x_j \int \frac{1}{...(k_i^2 - x_i)^2 ..} + ... \\ &= dI - J_{ij} - x_j \frac{\partial}{\partial x_i} I + ... \end{aligned}$$

-

$$c^a I + T x_j^a \partial^j I + J^a = 0$$

G Group, Algebraic Locus and Algebraic Solution

- We have seen that the Feynman integral satifies a set of partial differential equation and one can show that the differential operators in the equation set satisfies a Lie Algebra (Lie Group G $\subseteq$ of upper block upper Triangular matrices) which acts on the parameter space and foliates it into orbits.
- If one can find a left null vector for Tx matrix ,at a particular locus in parameter space, of the SFI equation set, it reduces it to an Algebraic Equation.
- Then the original Feynman Integral in study is given by linear combinations of simpler feynman integrals.
- This particular locus where this happens in parameter space is known as Algebraic Locus and the expression of original Feynman Integral in terms of linear combination of simpler diagrams is known as Algebraic Solution.

The Kite Diagram

- The Kite Feynman Diagram

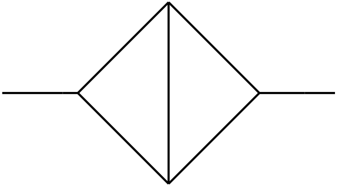
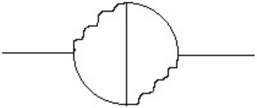


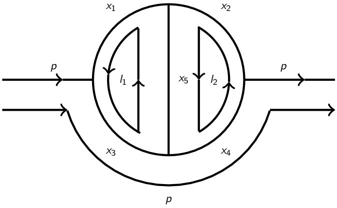
Figure: The kite diagram drawn in a way which explains its name.

- Applications: e.g. e-Field Strength Renormalization in QED.



The Kite Integral

- The kite diagram with its parameters and a choice of currents.



- The associated integral

$$
\begin{aligned}
I & \left(p^2; x_1, x_2, x_3, x_4, x_5\right) = \\
= & \int \frac{d^d l_1\, d^d l_2}{(l_1^2 - x_1)(l_2^2 - x_2)((l_1 + p)^2 - x_3)((p + l_2)^2 - x_4)((l_1 - l_2)^2 - x_5)}
\end{aligned}
$$

G group for Kite

- $$G = T_{2,1} \equiv \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix}$$

  and the number of equations is

  $$\dim(T_{2,1}) = 7 \ .$$

- More precisely the Lie algebra is $T_{2,1}$ and the group G consists of invertible upper triangular matrices.

SFI Equation Set

- The equations are given by the usual SFI form

$$c^a\, I + Tx^a_j\, \partial^j\, I + J^a = 0$$

- where c^a, Tx^a_j and J^a shall be defined immediately within the above-mentioned basis. The vector of constants, c^a, is given by

$$c^a = \begin{pmatrix} d-4 \\ d-4 \\ d-4 \\ d-4 \\ d-4 \\ d-4 \\ 2\,d-10 \end{pmatrix} .$$

SFI Equation Set

- The generator matrix $Tx^a_j\partial^j$ is given by

$$Tx^a_j\partial^j = -2\begin{pmatrix} x_1 & s^6_L & 0 & 0 & s^2 & 0 \\ s^6_L & x_3 & 0 & 0 & s^4 & 0 \\ s^2 & s^4 & 0 & 0 & x_5 & 0 \\ 0 & 0 & x_2 & s^6_R & s^1 & 0 \\ 0 & 0 & s^6_R & x_4 & s^3 & 0 \\ 0 & 0 & s^1 & s^3 & x_5 & 0 \\ x_1 & x_3 & x_2 & x_4 & x_5 & x_6 \end{pmatrix}\begin{pmatrix} \partial^1 \\ \partial^3 \\ \partial^2 \\ \partial^4 \\ \partial^5 \\ \partial^6 \end{pmatrix}.$$

- The s variables are defined as follows

$$s^1 := (x_5 + x_2 - x_1)/2 \qquad s^2 := (x_5 + x_1 - x_2)/2$$
$$s^3 := (x_5 + x_4 - x_3)/2 \qquad s^4 := (x_5 + x_3 - x_4)/2$$
$$s^6_L := (x_1 + x_3 - x_6)/2 \qquad s^6_R := (x_2 + x_4 - x_6)/2$$

Sources: Two Kinds

- The source vector J^a is given by

$$J^a = \begin{pmatrix} \partial^5 O_2 - (\partial^3 + \partial^5) O_1 \\ \partial^5 O_4 - (\partial^1 + \partial^5) O_3 \\ \partial^1 O^2 + \partial^3 O^4 - (\partial^1 + \partial^3) O_5 \\ \partial^5 O_4 - (\partial^4 + \partial^5) O_2 \\ \partial^5 O_3 - (\partial^2 + \partial^5) O_4 \\ \partial^2 O^1 + \partial^4 O^3 - (\partial^2 + \partial^4) O_5 \\ 0 \end{pmatrix}$$

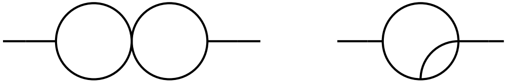
- O_i denote the diagram gotten by omitting, or contracting, the i'th propagator. Two possible topologies appear.
- 

Figure: The two source topologies (a) figure 8 (b) propagator seagull

Geometry of parameter space

- By the method of maximal minor M_a is found to be

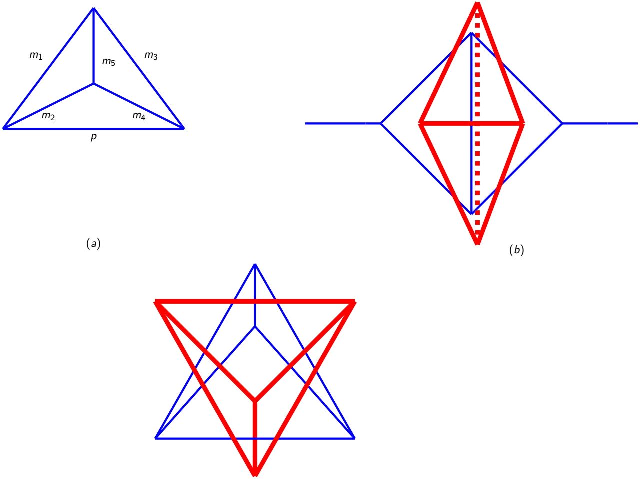
$$M_a = 4\,p^2\,B_3(x)\,K_a(x)$$

where the notation $B_3(x), K_a(x)$ will be defined now.
$B_3(x)$ is a cubic polynomial defined by

$$\begin{aligned} B_3 &= x_1\,x_4(x_1+x_4)+x_2\,x_3(x_2+x_3)+x_5\,x_6(x_5+x_6)+ \\ &+ x_1\,x_2\,x_5+x_1\,x_3\,x_6+x_2\,x_4\,x_6+x_3\,x_4\,x_5+ \\ &- (x_1\,x_4(x_2+x_3+x_5+x_6)+x_2\,x_3(x_1+x_4+x_5+x_6) \\ &+ x_5\,x_6(x_1+x_2+x_3+x_4)) \end{aligned}$$

Baikov Polynomial

- According to the Cayley-Menger formula B_3 describes the squared volume of the dual tetrahedron.



(a)

(b)

(c)

Global Stabilizer

- B_3 appeared in the physics literature in the work of Baikov on the 3-loop vacuum diagram (tetrahedron)
  The vector K_a is given by

$$K = \begin{pmatrix} -\partial^2 B_3 \\ -\partial^4 B_3 \\ \lambda_L \\ \partial^1 B_3 \\ \partial^3 B_3 \\ -\lambda_R \\ 0 \end{pmatrix}^T$$

Definition of the Heron / Källén invariant

$$\lambda := x^2 + y^2 + z^2 - 2\,x\,y - 2\,x\,z - 2\,y\,z$$

$$\lambda_L = \lambda(p^2, x_1, x_3)$$
$$\lambda_R = \lambda(p^2, x_2, x_4)$$

Algebraic Constraint

- K_a is a global stabilizer, namely it satisfies

$$K_a \, T x_j^a = 0$$

- Since $K_a \, c^a = 0$ multiplying the equation set by K_a generates a global constraint among the sources, namely,

$$K_a \, J^a = 0$$

- This is defined as Algebraic Constraint.

SFI maximally effective in Kite

- The dimension of the G-orbit through any point $x \in X$ is given by the rank of Tx at that point.
- The dimension of the G-orbit is generically 6. Since $\dim(X) = 6$

$$\mathrm{codim}(G - \mathrm{orbit}) = 0 \ .$$

- This means that SFI is maximally effective for the kite diagram.

Algebraic locus and solution

- At the singular locus, namely when $B_3(x) = 0$ or $p^2 = 0$ the dimension of the G orbit is reduced and accordingly an additional stabilizer appears.
- Given a stabilizer Stb_a, if the associated constant is non-zero, namely $Stb_a\, c^a \neq 0$ one can reduce the diagram to a linear combination of simpler ones by multiplying the equation set on the left by the stabilizer.

Algebraic locus and solution

- B_3 **locus**. At $B_3 = 0$ the global stabilizer K splits into a pair of stabilizers K^L, K^R as follows

$$K^L = \begin{pmatrix} -\partial^2 B_3 \\ -\partial^4 B_3 \\ \lambda_L \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}^T \qquad K^R = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -\partial^1 B_3 \\ -\partial^3 B_3 \\ \lambda_R \\ 0 \end{pmatrix}^T$$

Algebraic Solution of the Kite

- **Algebraic solution**. The algebraic solution is now gotten by multiplying the equation set on the left by an arbitrary linear combination $\alpha_L K^L + \alpha_R K^R$. We notice that

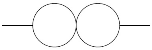
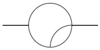
$$\left(\alpha_L K_a^L + \alpha_R K_a^R\right) c^a = (\alpha_L + \alpha_R)(d-4)\,\partial^5 B_3$$

- So the Algebraic Solution is

$$(4-d)\,I|_{B3=0} = \frac{\left(\alpha_L K_a^L + \alpha_R K_a^R\right) J^a}{(\alpha_L + \alpha_R)\,\partial^5 B_3}$$

Tests Special Cases

- In the massless case $m_1 = \cdots = m_5 = 0$ it was shown already in[Chetyrkin,Tkachov 1981] that the diagram can be reduced as follows

$$I_{\text{massless}} = \frac{2}{d-4} \Bigg($$  $$-$$  $$\Bigg)$$

- The algebraic solution generalizes the reduction of the massless case to the most general parameters, namely $B_3(m_1^2, \dots, m_5^2, p^2) = 0$.
  The case $m_3 = m_4 = m_5 = 0$ is of special interest. In this case $B_3 = 0$ simplifies to the following two alternative forms

$$(4-d)I_{x_3=x_4=x_5=0} = \frac{(x_2 - x_1)J^2 + (x_1 - x_6)J^3}{x_2 - x_6} = \frac{(x_1 - x_2)J^5 + (x_2 - x_6)J^6}{x_1 - x_6}$$

This case falls into the applicability regime of the "diamond rule" (with $L = S = 1$)[Ruijl, Ueda and Vermaseren 2015].

Discussion of SFI on Kite

- The G-orbits were found to be 6-dimensional in our 6d parameter space X, namely the orbit co-dimension is zero. This means that for this diagram the SFI method would be maximally effective.
- On the surface $B_3 = 0$ the integral degenerates into a linear combination of simpler diagrams and is given by the algebraic soln., thereby providing a maximal generalization of the massless case. This is our central result.

Introduction to Triangle

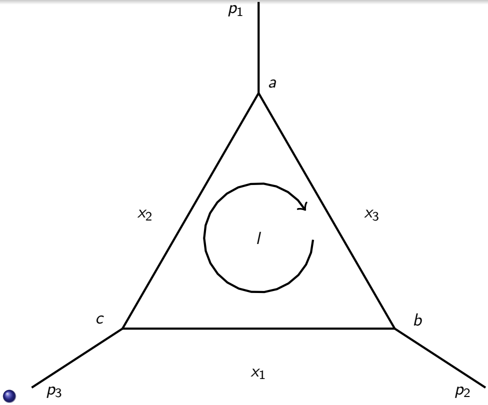


Figure: The triangle diagram. p_1, p_2, p_3 are the external currents of energy-momentum while x_1, x_2, x_3 are the squared masses of the respective propagators ($x_1 \equiv {m_1}^2$, etc.). The vertices are denoted by a, b, c.

SFI action on Triangle

- The associated Feynman integral defined by

$$I = \int \frac{d^d l}{\prod_{i=1}^{3} \left(k_i{}^2 - m_i{}^2\right)}$$

The propagator currents can be chosen as [1]

$$k_i = l + \left(p_{i+1} - p_{i-1}\right)/3 \quad , i = 1, 2, 3.$$

- Altogether, the parameter space X is given by

$$X = \left\{(x_1, x_2, x_3, x_4, x_5, x_6) = \left(m_1{}^2,\ m_2{}^2,\ m_3{}^2,\ p_1{}^2,\ p_2{}^2,\ p_3{}^2\right)\right\}$$

We consider a general spacetime dimension d where the mass dimension of the integral is $d - 6$.

[1] An alternative practical choice is given by $k_1 = l,\ k_2 = l + p_3,\ k_3 = l - p_2$.

SFI Equation System

- The equation system thus obtained can be summarized in matrix form by

$$0 = c^a\, I - 2\,(Tx)^a_j\, \partial^j I + J^a,\;\; a = 1, \ldots, 7$$

where the generator matrix is given by

$$(Tx)^a_j = \begin{pmatrix} 0 & s_c^1 & 0 & s_\infty^5 & 0 & x_6 \\ 0 & 0 & s_a^2 & x_4 & s_\infty^6 & 0 \\ s_b^3 & 0 & 0 & 0 & x_5 & s_\infty^4 \\ 0 & 0 & s_b^1 & s_\infty^6 & x_5 & 0 \\ s_c^2 & 0 & 0 & 0 & s_\infty^4 & x_6 \\ 0 & s_a^3 & 0 & x_4 & 0 & s_\infty^5 \\ x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \end{pmatrix};$$

- SFI differential equations for I belong to the SFI group (G) - the upper triangular group $T_{1,2}$.

SFI Equation System

- The x_i-independent constants are given by

$$c^a = \begin{pmatrix} -1 \\ -1 \\ -1 \\ -1 \\ -1 \\ -1 \\ d-6 \end{pmatrix} ;$$

and finally, the sources are given by

$$J^a = \begin{pmatrix} \frac{\partial}{\partial x_2} O_1 \, I \\ \frac{\partial}{\partial x_3} O_2 \, I \\ \frac{\partial}{\partial x_1} O_3 \, I \\ \frac{\partial}{\partial x_3} O_1 \, I \\ \frac{\partial}{\partial x_1} O_2 \, I \\ \frac{\partial}{\partial x_2} O_3 \, I \\ 0 \end{pmatrix} .$$

Geometry of parameter space

- The dimension of the G-orbit through any point $x \in X$ is given by the rank of Tx at that point.

$$M_a = S\,K_a \;;$$

- The singular factor $S(x)$ is given by

$$S(x) = 4\,\lambda_\infty\,B_3 \;;$$

- $K_a(x)$ is given by

$$K_a = (s_a^3,\, s_b^1,\, s_c^2,\, -s_a^2,\, -s_b^3,\, -s_c^1,\, 0)\;.$$

- The dimension of the G-orbit is generically 6 and $\dim(X) = 6$

$$\mathrm{codim}(G - \mathrm{orbit}) = 0$$

Definitions λ_∞ and B_3 and geometric significance

- The Heron/Kallen invariant

$$\lambda_\infty = x_4^2 + x_5^2 + x_6^2 - 2x_4x_5 - 2x_5x_6 - 2x_4x_6$$

- Significance: If x_4, x_5, x_6 denote the squared lengths of the sides of a triangle, then its squared area is given by $-\lambda_\infty/16$
- The Tartaglia/Baikov polynomial

$$\begin{aligned} B_3 &= {x_1}^2 x_4 + x_1 {x_4}^2 + {x_2}^2 x_5 + x_2 {x_5}^2 + {x_3}^2 x_6 + x_3 {x_6}^2 \\ &+ x_1 x_2 x_6 + x_1 x_3 x_5 + x_2 x_3 x_4 + x_4 x_5 x_6 \\ &- (x_2 x_5 (x_1 + x_3 + x_4 + x_6) + x_3 x_6 (x_1 + x_2 + x_4 + x_5) \\ &+ x_1 x_4 (x_2 + x_3 + x_5 + x_6)) \,. \end{aligned}$$

- Significance: $(-B_3)/144$ expresses the squared volume of a dual tetrahedron in terms of the squared lengths of its sides.

Solution at singular locus: $\lambda_\infty = 0$ & $B_3 = 0$

- We determined the solution at the λ_∞ locus to be

$$
\begin{aligned}
-I|_{\lambda_\infty=0} &= \frac{1}{2\,B_3}\left(\partial^1 B_3\, I_1 + \partial^2 B_3\, I_2 + \partial^3 B_3\, I_3\right) = \\
&= \frac{2\,x_4}{\partial^1 B_3} I_1 + \frac{2\,x_5}{\partial^2 B_3} I_2 + \frac{2\,x_6}{\partial^3 B_3} I_3
\end{aligned}
$$

where $I_i,\ i = 1, 2, 3$ denote bubble diagrams with propagator i contracted.

- We find that the solution at B_3 is given by

$$
\begin{aligned}
I|_{B_3=0} &= -\frac{2(d-3)}{(d-4)}\left(\frac{x_4}{\partial^1 B_3} I_1 + \frac{x_5}{\partial^2 B_3} I_2 + \frac{x_6}{\partial^3 B_3} I_3\right) \\
&+ \frac{2\lambda_\infty}{(d-4)}\left(\frac{x_1\, T_1}{(\partial_2 B_3)(\partial_3 B_3)} + \frac{x_2\, T_2}{(\partial_1 B_3)(\partial_3 B_3)} + \frac{x_3\, T_3}{(\partial_1 B_3)(\partial_2 B_3)}\right) .
\end{aligned}
$$

SFI general solution of Triangle

- $$I = \frac{c_\Delta}{\sqrt{|\lambda_\infty|/4}} \left[F(h^2, {c_1}^2, {a_2}^2) + F(h^2, {c_1}^2, {a_3}^2) + cyc. \right]$$

where $c_\Delta = -i\pi^{\frac{d}{2}} \Gamma(\frac{6-d}{2})$ and $h^2 = \frac{B_3}{\lambda_\infty}$.

$$F(h^2,\, c^2,\, a^2) \;\; := \;\; \int_{\Delta_{a,c}} d^2q \, \left(h^2 + q^2\right)^{\frac{d-6}{2}}$$

$$\int_{\Delta_{a,c}} d^2q \;\; := \;\; \int_0^{|a|} dq_y \int_0^{\frac{|b|}{|a|} q_y} dq_x$$

where $q^2 = -{q_x}^2 - {q_y}^2$,

$${c_1}^2 = x_1 - \frac{B_3}{\lambda_\infty}$$

$${a_1}^2 = -\frac{\lambda_a}{4x_4} - \frac{B_3}{\lambda_\infty}$$

Magic Connection of Triangle and Diameter diagrams

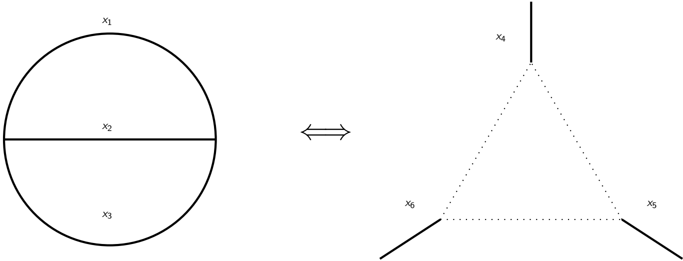


Figure: Magic Connection between Diameter and massless Triangle

Magic Connection

- We compare the integral normalized by its leading singularities, defined by

$$\hat{I} = I/I_0$$

- For the diameter we find

$$\partial_1 \hat{I}_D = -\frac{d-2}{x_1 \lambda} \frac{1}{I_{0D}} \left(-x_1 \, j_2 j_3 + s^3 j_1 j_3 + s^2 j_1 j_2 \right)$$

and similarly for $\partial_2 \hat{I}_D$ and $\partial_3 \hat{I}_D$. For the massless triangle we find

$$\partial_4 \hat{I}_\Delta = \frac{2(d-3)}{x_4 \, \lambda_\infty} \frac{1}{I_{0T}} \left(-x_4 \, I_1 + s^6 I_2 + s^5 I_3 \right)$$

and similarly for $\partial_5 \hat{I}_\Delta$ and $\partial_6 \hat{I}_\Delta$.

Magic Connection

- The respective homogenous solutions are given by

$$I_{0D}(d) = \lambda^{\frac{d-3}{2}}$$
$$I_{0\Delta}(d) = \lambda_{\infty}^{\frac{3-d}{2}}(x_4x_5x_6)^{\frac{d-4}{2}}$$

while the tadpole and bubble sources are given by

$$j_{\mu}(\mu; d) = i\pi^{\frac{d}{2}}\Gamma(2-\frac{d}{2})\mu^{(\frac{d}{2}-1)}$$
$$I_{Bi}(\mu_i; d) = \frac{i^{1-d}\pi^{\frac{d}{2}}\Gamma(2-\frac{d}{2})\Gamma^2(\frac{d}{2}-1)}{\Gamma(d-2)}\mu_i^{(\frac{d}{2}-2)}$$

Magic Connection

- After substituting for these we get

$$\partial_1 \hat{I}_D = -\frac{(d-2)\,c_T^2}{x_1\,\lambda}\,\frac{1}{\lambda^{\frac{d-3}{2}}}\left(-x_1\,(x_2\,x_3)^{\frac{d-2}{2}} + s^2\,(x_1\,x_2)^{\frac{d-2}{2}} + s^3\,(x_1\,x_3)^{\frac{d-2}{2}}\right)$$

$$\partial_4 \hat{I}_\Delta = \frac{2\,(d-3)c_B}{x_4\,\lambda_\infty}\,\lambda^{\frac{d-3}{2}}\left(-x_4\,(x_5\,x_6)^{-\frac{d-4}{2}} + s^5(x_4\,x_6)^{-\frac{d-4}{2}} + s^6(x_4\,x_6)^{-\frac{d-4}{2}}\right)$$

  where c_T, c_B are the tadpole and bubble constants

- The magic connection

$$I_D(x_1,\,x_2,\,x_3;\,d) = i^{1-d}\pi^{\frac{3d}{2}-3}\frac{\Gamma(3-d)}{\Gamma(\frac{d}{2})}(x_1x_2x_3)^{\frac{d}{2}-1} I_\Delta(\{p_i^2 = x_i\}_{i=1,2,3};\,6-d)\ .$$

- This result matches exactly with the relation discovered by A.I. Davydychev and J.B. Tausk (Phys. Rev. D 53 (1996) 7381 [hep-ph/9504431])

Discusson of SFI on Triangle Diagram

- The SFI equation system was determined and presented in a simple basis.
- We studied the geometry of parameter space and found that the SFI method is maximally effective here as the co-dimension of the G-orbit is 0.
- The singular locus was found to consist of two components where either λ_∞ or the Tartaglia/Baikov polynomial B_3 vanish. At these components the triangle was evaluated as a linear combination of descendant bubble diagrams.
- The general solution was derived.
- Magic connection was revisited.

THANK YOU