An Overview of the E_{11} Program

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23/01/24

Presented to Okinawa Institute of Science and Technology

Introduction: Motivation

The method of spontaneous symmetry breaking had an enormous impact on theoretical physics in the 1960's and 1970's, contributing to the development of the standard model of particle physics which unifies three of the four known forces of nature: the strong, weak, and electromagnetic forces.

Spontaneous symmetry breaking was abstracted to a general abstract symmetry group in the 1960's, allowing one to construct connections and covariant derivatives transforming under these groups, and is referred to as the 'Method of Nonlinear Realizations' (also called the CCWZ Formalism) [1-2, 3 Sec. 19.6].

It was natural to ask whether one could apply the method of spontaneous symmetry breaking to the fourth force of nature: the gravitational field.

In the 1973, Ogievetskii [4] showed that the general covariance group of general relativity can be generated as the closure of the affine group $A(4) = \mathrm{GL}(4, \mathbb{R}) \otimes_s P_4$ with the conformal group $\mathrm{C}(1,3) \simeq \mathrm{SO}(4,2)$.

Then, using the previous result, Borisov and Ogievetskii [5] showed that general relativity arises from the nonlinear realization of the closure of these two groups, each 'spontaneously breaking' down to the Lorentz group, the Nambu-Goldstone particles being Gravitons.

It was natural to ask whether one could apply this result to supergravity, in particular eleven-dimensional supergravity.

Introduction: Motivation

In 2000, West [6] constructed a partial analogue of the result of Borisov and Ogievetskii for the bosonic sector of eleven-dimensional supergravity, based on an algebra $G_{11} := \{K^a{}_b, R^{a_1 a_2 a_3}, R^{a_1 \cdots a_6}; P_a\}$, which is an extension of $A(4)$ by the R^3 and R^6, again 'breaking' to the Lorentz group.

Without simultaneous use of the conformal group or its supersymmetric extension, the dynamics is only fixed up to constant factors (which must be chosen by hand when using G_{11}), they would presumably be uniquely fixed by simultaneous (super)-conformal symmetry but this has not been derived explicitly.

However there is evidence this would not be the right approach.

Introduction: Motivation

It is known that when 11D supergravity is dimensionally reduced on a torus down to $11 - n$ dimensions ($n = 1, \ldots, 8$), the resulting Kaluza-Klein scalars can be written in terms of a non-linear realisations [7].

Dimension	Exceptional Symmetry Group	Coset Space
10 (IIA)	$\mathrm{O}(1,1)$	-
10 (IIB)	$\mathrm{SL}(2)$	$\mathrm{SL}(2)/\mathrm{SO}(2)$
9	$\mathrm{GL}(2)$	$\mathrm{GL}(2)/\mathrm{SO}(2)$
8	$E_3 \sim \mathrm{SL}(3) \times \mathrm{SL}(2)$	$\mathrm{SL}(3) \times \mathrm{SL}(2)/\mathrm{SO}(3) \times \mathrm{SO}(2)$
7	$E_4 \sim \mathrm{SL}(5)$	$\mathrm{SL}(5)/\mathrm{SO}(5)$
6	$E_5 \sim \mathrm{SO}(5,5)$	$\mathrm{SO}(5,5)/\mathrm{SO}(5) \times \mathrm{SO}(5)$
5	E_6	$E_6/\mathrm{Sp}(8)$
4	E_7	$E_7/\mathrm{SU}(8)$
3	E_8	$E_8/\mathrm{SO}(16)$

Each local symmetry group is the maximal compact subgroup of the exceptional symmetry group.

It was conjectured that in dimensions 2 and 1 we find an E_9 and E_{10} Exceptional Symmetry Group respectively [8].

The G_{11} symmetry group used in the non-linear realization for 11D supergravity contains none of this structure: G_{11} is not a Kac-Moody algebra (although $E_3, .., E_9, E_{10}$ are), and the local Lorentz subgroup chosen for G_{11} is not the maximal compact subgroup of G_{11}. However, G_{11} gets gravity.

In the highly-cited paper [9] that formed the beginning of the E_{11} program, it was shown that $G_{11}/\{P_a\}$ can be extended to the Kac-Moody algebra E_{11} without affecting the dynamics arising from the non-linear realization, and that the local Lorentz subgroup can be extended to the maximal compact subgroup of E_{11}, denoted $I_c(E_{11})$.

E_{11} tells us to accept an infinite collection of fields $g_{ab}, A_{a_1 a_2 a_3}, A_{a_1,\dots,a_6}, h_{a_1\dots,a_8,b}, \dots$ associated to the generators $K^a{}_b, R^{a_1 a_2 a_3}, R^{a_1\cdots a_6}, R^{a_1\cdots a_8,b}, \dots$ of the Kac-Moody algebra.

However generalizing from G_{11} to E_{11} still does not appear to fix the dynamics of 11D supergravity uniquely as things stand, there is still the 'conformal group' question.

In 2016 [10] it was shown that the non-linear realization of $E_{11} \otimes_s l_1$, where we take the semi-direct product of E_{11} and its vector representation, denoted l_1, such that it spontaneously breaks to the maximal compact subgroup $I_c(E_{11})$, results uniquely in the equations of motion of the bosonic sector of 11D supergravity.

This is due to the infinite-dimensional vector representation l_1, containing an infinite collection of generators $P_a, Z^{a_1 a_2}, Z^{a_1 \ldots a_5}, \ldots$, [11] the presence of which constrains the E_{11} symmetry to fix the equations of motion uniquely [10].

Kac-Moody Algebra

A Kac-Moody algebra (KMA) of rank r is defined [17] in terms of an $r \times r$ (symmetric, indecomposable) generalized Cartan matrix (GCM) A_{ab}, which possesses the properties:

- $A_{aa} = 2$;
- A_{ab} is a negative integer or zero for $a \neq b$

The KMA associated to a GCM is the Lie algebra generated by elements $H_a, E_a, F_a, a = 1, ..., r$ satisfying the 'Chevalley-Serre' relations

$$[H_a, H_b] = 0 \ , \qquad [H_a, E_b] = A_{ab} E_b,$$
$$[E_a, F_b] = \delta_{ab} H_b \ , \qquad [H_a, F_b] = -A_{ab} F_b,$$
$$\mathrm{ad}_{E_a}^{1-A_{ab}}(E_b) = 0 \ , \qquad \mathrm{ad}_{F_a}^{1-A_{ab}}(F_b) = 0,$$

where $\mathrm{ad}_{E_a}(E_b) = [E_a, E_b]$. We can define an inner product and the notion of a simple root α_a to find $A_{ab} = 2\frac{(\alpha_a, \alpha_b)}{(\alpha_a, \alpha_a)}$ which can be represented via a Dynkin diagram consisting of r nodes and $-A_{ab}$ lines between nodes a and b. This encompasses the symmetric finite and affine simple Lie Algebras (LAs).

Representations of Kac-Moody Algebra

As for finite simple LAs, an irrep can be specified by a highest weight $\mu = \sum_{a=1}^{r} p_a l_a$ where the l_a are the fundamental weights $(\alpha_a, l_b) = \delta_{ij}$ }.

Recall the fundamental weights of the classical LAs can be associated to tensors and spinors, e.g. for $\mathrm{SL}(r)$ a rank $1 \leq k \leq r$ anti-symmetric tensor $T^{a_1 .. a_k}$ is associated to λ_{r-k}. A root α can be expressed in terms of the fundamental weights via $\alpha = \sum_a A_{ab} l_b$.

The program SimpLie has automated the process of constructing the generators of many Kac-Moody algebras at low 'levels'.

Kac-Moody Algebras: Example

The Lie algebra of $A_2 = \mathrm{SL}(3)$ generated by ${K^a}_b$, $a, b = 1, 2, 3$, satisfy $\sum_a {K^a}_a = 0$ and $[{K^a}_b, {K^c}_d] = {\delta^c}_b {K^a}_d - {\delta^a}_d {K^c}_b$. Setting $H_a = {K^a}_a - {K^{a+1}}_{a+1}$; $E^a = {K^a}_{a+1}$; $F_a = {K^{a+1}}_a$; $a = 1, 2$. We then have for example $E^1 = {K^1}_2$, $E^2 = {K^2}_3$ and so e.g. $[E^1, E^2] = {K^1}_3$. The Lie algebra can be written as a GKM Algebra with $A_{ab} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$, where the Serre relations hold. The Dynkin diagram is thus •—•. A_2 can be fixed by first working with A_1, •, i.e. $\{H_1, E^1, F_1\}$ and then adding H_2, E^2, F_2 s.t. Serre holds.

The l_1 vector representation arises by adding a third node to form •—•—•, meaning we consider the generators ${K^A}_B$, $A, B = 0, 1, 2, 3 = 0, a$. If $A \neq 0, B \neq 0$ we have ${K^a}_b$ and •—•—•, however if $A = 0$ and $B \neq 0$ we have ${K^0}_c := P_c$. This transforms as a vector under ${K^a}_b$: $[{K^a}_b, P_c] = -{\delta^a}_c P_b$.

The construction of E_{11} and its l_1 representation is roughly an infinite-dimensional generalization of this example for $A_{10} = \mathrm{SL}(11)$.

Non-Linear Realisations

Recall in SSB of a group G down to a subgroup H [3, Sec. 19.6], a wave function can be parametrized as $\Psi(x) = g(x)\tilde{\psi}(x)$, where $g(x) \in G/H$ defines the Nambu-Goldstone bosons, and $\tilde{\psi}(x)$ lives in a linear irrep of H.

Non-linear realizations simply work with the $g(x) \in G/H$ directly, without using wave functions.

The non-linear realisation of a group of the form $G \otimes_s l_1$ with respect to a subgroup $I_c(G)$ is, by definition, a set of equations of motion which are invariant under the transformations [18]

$$g \to g_0 g, \quad g_0 \in G \otimes_s l_1, \quad \text{as well as} \quad g \to gh, \quad h \in I_c(G).$$

Dynamics that are invariant under these transformations are naturally constructed in terms of the Cartan forms.

Group elements of $G \otimes_s l_1$ can be parameterized as $g = g_G g_l$, with $g_G \in G$ and $g_l \in l_1$.

Non-Linear Realizations

Partial derivatives of $\Psi(x) = g(x)\tilde{\psi}(x)$ will produce covariant derivatives $\tilde{\psi}(x)$, with $D_\mu = \partial_\mu + g^{-1}\partial_\mu g$ [3].

We construct the dynamics of the $G \otimes_s l_1$ non-linear realisation in terms of Cartan forms directly, which are given by

$$\mathcal{V} \equiv g^{-1}dg = \mathcal{V}_G + \mathcal{V}_l,$$

where

$$\mathcal{V}_G = g_G^{-1}dg_G = dz^\Pi G_{\Pi,\underline{\alpha}} R^{\underline{\alpha}} \quad \text{and} \quad \mathcal{V}_l = g_G^{-1}(g_l^{-1}dg_l)g_G = dz^\Pi E_\Pi{}^A l_A \,.$$

Here $\mathcal{V}_G$ belongs to G, while $\mathcal{V}_l$ belongs to the l_1 representation.

The forms $\mathcal{V}_A$ and $\mathcal{V}_l$ are invariant under rigid transformations, and under local $I_c(G)$ transformations they change as

$$\mathcal{V}_A \to h^{-1}\mathcal{V}_A h + h^{-1}dh \quad \text{and} \quad \mathcal{V}_l \to h^{-1}\mathcal{V}_l h$$

From these transformations, $E_\Pi{}^A$ can be interpreted as a vielbein,

$$\mathcal{V} = dz^\Pi E_\Pi{}^A (G_{A,\underline{\alpha}} R^{\underline{\alpha}} + l_A)$$

where $G_{A,\underline{\alpha}} = E_A{}^\Pi G_{\Pi,\underline{\alpha}}$.

E_{11} and it's l_1 Representation

The E_{11} Kac-Moody algebra is defined by the following Dynkin diagram

Taking node 11 as the central node, decomposed with respect to it's A_{10} subalgebra, this results in the generators

$$K^a{}_b \ , \ R^{a_1 a_2 a_3} \ , \ R_{a_1 a_2 a_3} \ , \ R^{a_1..a_6} \ , \ R_{a_1..a_6} \ , \ R^{a_1..a_8,b} \ , \ \ldots$$

$$R^{[a_1 a_2 a_3]} = R^{a_1 a_2 a_3} \ , \ R^{[a_1 \ldots a_6]} = R^{a_1 \ldots a_6} \ , \ R^{[a_1 \ldots a_8],b} = R^{a_1 \ldots a_8,b} \ , \ R^{[a_1 \ldots a_8,b]} = 0 \ , \ \ldots$$

which satisfy an algebra that is fixed by implementing the Serre relations step by step

$$[K^a{}_b, K^c{}_d] = \delta^c{}_b K^a{}_d - \delta^a{}_d K^c{}_b \ , \ [K^a{}_b, R^{c_1 \cdots c_3}] = 3\delta^{[c_1}{}_b R^{|a| c_2 c_3]} \ ,$$

$$[R^{a_1 a_2 a_3}, R^{a_4 a_5 a_6}] = 2R^{a_1..a_6} \quad , \quad [R^{a_1..a_6}, R^{b_1 b_2 b_3}] = 3R^{a_1..a_6[b_1 b_2, b_3]} \ , \ \ldots$$

E_{11} possesses an l_1 vector representation containing the generators

$$P_a \ , \ Z^{a_1 a_2} \ , \ Z^{a_1 \ldots a_5} \ , \ Z^{a_1 \ldots a_8} \ , \ Z^{a_1 \ldots a_7,b} \ , \ \ldots$$

which transform under E_{11} as

$$[K^a{}_b, P_c] = -\delta^a_c P_b + \tfrac{1}{2}\delta^a{}_b P_c \quad , \quad [R^{a_1 a_2 a_3}, P_b] = 3\delta^{[a_1}_b Z^{a_2 a_3]} \quad , \quad \ldots$$

The following map defines an involution on E_{11}

$$K^a{}_b \rightarrow -\eta^{ac} K^d{}_c \eta_{db} \quad , \quad R^{a_1 a_2 a_3} \rightarrow -\eta^{a_1 b_1} .. \eta^{a_3 b_3} R_{b_1 b_2 b_3} \ ,$$
$$R^{a_1 .. a_6} \rightarrow \eta^{a_1 b_1} .. \eta^{a_6 b_6} R_{b_1 .. b_6} \quad , \quad R^{a_1 .. a_8 , b} \rightarrow -\eta^{a_1 c_1} .. \eta^{a_8 c_8} \eta^{bd} R_{c_1 .. c_8 , d} \ , \ \ldots$$

leading to an involution invariant subalgebra $I_c(E_{11})$ generated by

$$J_{ab} = \eta_{ac} K^c{}_b - \eta_{bc} K^c{}_a \qquad S_{a_1 a_2 a_3} = R^{b_1 b_2 b_3} \eta_{b_1 a_1} \eta_{b_2 a_2} \eta_{b_3 a_3} - R_{a_1 a_2 a_3},$$
$$S_{a_1 .. a_6} = R^{b_1 .. b_6} \eta_{b_1 a_1} .. \eta_{b_6 a_6} - R_{a_1 .. a_6} \quad , \quad S_{a_1 .. a_8 , b} = R^{c_1 .. c_8 , d} \eta_{c_1 a_1} .. \eta_{c_8 a_8} \eta_{db} - R_{a_1 .. a_8 , b}$$

where, using the above commutators, the generators can be shown to satisfy the commutation relations

$$[J_{ab}, J_{cd}] = \eta_{bc} J_{ad} - \eta_{bd} J_{ac} + \eta_{ad} J_{bc} - \eta_{ac} J_{bd} \ ,$$
$$[S^{a_1 a_2 a_3}, S_{b_1 b_2 b_3}] = -18 \delta^{[a_1 a_2}_{[b_1 b_2} J^{a_3]}{}_{b_3]} + 2 S^{a_1 a_2 a_3}{}_{b_1 b_2 b_3} \ ,$$
$$[S^{a_1 a_2 a_3}, P_b] = 3 \delta^{[a_1}{}_b Z^{a_1 a_3]}, \qquad [S_{a_1 a_2 a_3}, Z^{b_1 b_2}] = Z_{a_1 a_2 a_3}{}^{b_1 b_2} - 6 \delta^{b_1 b_2}_{[a_1 a_2} P_{a_3]} \ , \ \ldots$$

The Nonlinear Realisation of $E_{11} \otimes_s l_1 / I_c(E_{11})$ [18]

We can now consider the nonlinear realization of $E_{11} \otimes_s l_1$ with respect to the $I_c(E_{11})$ subgroup. Elements of the coset $E_{11} \otimes_s l_1 / I_c(E_{11})$ can be parametrized as

$$g = g_E g_l \quad , \quad g_E = \dots e^{h_{a_1 \dots a_8, b} R^{a_1 \dots a_8, b}} e^{A_{a_1 \dots a_6} R^{a_1 \dots a_6}} e^{A_{a_1 a_2 a_3} R^{a_1 a_2 a_3}} e^{h_a{}^b K^a{}_b},$$
$$g_l = e^{x^a P_a} e^{x_{ab} Z^{ab}} e^{x_{a_1 \dots a_5} Z^{a_1 \dots a_5}} \dots$$

The dynamics of E_{11} can be constructed using the following Cartan forms

$$\mathcal{V} \equiv g^{-1} dg = \mathcal{V}_E + \mathcal{V}_l,$$

for

$$\mathcal{V}_E = G_a{}^b K^a{}_b + G_{a_1 \dots a_3} R^{a_1 \dots a_3} + G_{a_1 \dots a_6} R^{a_1 \dots a_6} + G_{a_1 \dots a_8, b} R^{a_1 \dots a_8, b} + \dots$$

where

$$G_a{}^b = (e^{-1} de)_a{}^b, \qquad G_{a_1 \dots a_3} = e_{a_1}{}^{\mu_1} \dots e_{a_3}{}^{\mu_3} dA_{\mu_1 \dots \mu_3}$$
$$G_{a_1 \dots a_6} = e_{a_1}{}^{\mu_1} \dots e_{a_6}{}^{\mu_6} (dA_{\mu_1 \dots \mu_6} - A_{[\mu_1 \dots \mu_3} dA_{\mu_4 \dots \mu_6]})$$
$$G_{a_1 \dots a_8, b} = e_{a_1}{}^{\mu_1} \dots e_{a_8}{}^{\mu_8} e_b{}^{\nu} (dh_{\mu_1 \dots \mu_8, \nu} - A_{[\mu_1 \dots \mu_3} dA_{\mu_4 \mu_5 \mu_6} A_{\mu_7 \mu_8]\nu} + 3A_{[\mu_1 \dots \mu_6} dA_{\mu_7 \mu_8]\nu}$$
$$+ A_{[\mu_1 \dots \mu_3} dA_{\mu_4 \mu_5 \mu_6} A_{\mu_7 \mu_8 \nu]} - 3A_{[\mu_1 \dots \mu_6} dA_{\mu_7 \mu_8 \nu]}).$$

The vielbein is given to low levels by

$$E_{\Pi}{}^{A} = (\det e)^{-\frac{1}{2}} \begin{bmatrix} e_{\mu}{}^{a} & -3e_{\mu}{}^{c}A_{cb_1b_2} \\ 0 & (e^{-1})_{[b_1}{}^{\mu_1}(e^{-1})_{b_2]}{}^{\mu_2} \end{bmatrix}.$$

Under $I_c(E_{11})$, the Cartan forms transform as

$$\delta G_a{}^b = 18\Lambda^{c_1c_2b}G_{c_1c_2a} - 2\delta_a{}^b\Lambda^{c_1c_2c_3}G_{c_1c_2c_3}$$

$$\delta G_{a_1a_2a_3} = -\frac{5!}{2}G_{b_1b_2b_3a_1a_2a_3}\Lambda^{b_1b_2b_3} - 6G_{(c[a_1)}\Lambda^{c}{}_{a_2a_3]} \quad , \quad \ldots$$

and $I_c(E_{11})$ acts on the derivative indices via

$$\delta G_{a,\bullet} = -3G^{b_1b_2}{}_{,\bullet}\Lambda_{b_1b_2a}, \qquad \delta G^{a_1a_2}{}_{,\bullet} = 6\Lambda^{a_1a_2b}G_{b,\bullet}$$

for example

$$\begin{aligned}\delta G_{a_1,a_2a_3a_4} &= \delta[(E^{-1})_{a_1}{}^{\Pi}G_{\Pi,a_2a_3a_4}] \\ &= -3\Lambda_{b_1b_2a_1}G^{b_1b_2}{}_{,b_1b_2a_2a_3a_4} - \frac{5!}{2}G_{a_1,b_1b_2b_3a_2a_3a_4}\Lambda^{b_1b_2b_3} - 6G_{a_1,(d[a_2)}\Lambda^{d}{}_{a_3a_4]}.\end{aligned}$$

E_{11} Graviton Dual-Graviton Duality Equations

E_{11} fixes the following first order equation of motion involving ∂_a derivatives uniquely

$$D_{a,b_1b_2} = (\det e)^{1/2}\omega_{a,b_1b_2} - \frac{1}{4}\varepsilon_{b_1b_2}{}^{e_1..e_9}G_{e_1,e_2..e_9,a}$$

relating the graviton and dual graviton. In order to do this, E_{11} actually fixes a generalization of this equation involving derivatives involving all generalized coordinates $\partial_{a_1a_2}, \partial_{a_1...a_5}, \dots$

$$\begin{aligned}\mathcal{D}_{a,b_1b_2} =&(\det e)^{1/2}\omega_{a,b_1b_2} - 3G^{c_2}{}_{a,c_2b_1b_2} + 6G^{c}{}_{[b_1,b_2]ac} \\ &+ 2\eta_{a[b_1}G^{c_2c_3,}{}_{|c_2c_3|b_2]} - \frac{1}{4}\varepsilon_{b_1b_2}{}^{e_1..e_9}G_{e_1,e_2..e_9,a} + \dots\end{aligned} \tag{1}$$

which varies under $I_c(E_{11})$ into

$$\begin{aligned}\delta\mathcal{D}_{a,b_1b_2} =& -36\Lambda^{c_8c_9}{}_{a}D_{b_1b_2c_8c_9} - 8\eta_{a[b_1}D_{b_2]c_1c_2c_3}\Lambda^{c_1c_2c_3} \\ &- \frac{55}{2}\varepsilon^{d_1..d_{10}}{}_{[b_1}\Lambda_{b_2]}{}^{c_1c_2}D_{d_1..d_{10},ac_1c_2} - \frac{55}{18}\eta_{a[b_1}\varepsilon_{b_2]}{}^{d_1..d_{10}}\Lambda^{c_1c_2c_3}D_{d_1..d_{10},c_1c_2c_3} \\ &+ \partial_a\tilde{\Lambda}_{b_1b_2} + \dots\end{aligned}$$

E_{11} Graviton Dual-Graviton Duality Equations

In this result we defined the following first order duality equation between the three form and six form

$$D_{a_1 a_1 a_3 a_4} = G_{[a_1,a_2 a_3 a_4]} - \frac{1}{2 \cdot 4!} \varepsilon_{a_1..a_4}{}^{b_1..b_7} G_{b_1,b_2..b_7}.$$

and the term known to be part of a duality relation with terms above level three:

$$D_{d_1..d_{10},ab_1 b_2} = -\frac{144}{5 \cdot 11!} \varepsilon_{d_1..d_{10}}{}^{e} G_{[e,ab_1 b_2]}.$$

The variation only transforms up to additional terms (interpreted as a 'generalized gauge transformation' [10])

$$\partial_a \tilde{\Lambda}_{b_1 b_2} = -\varepsilon_{b_1 b_2}{}^{c_1..c_9} [\frac{1}{12} G_{a,c_1..c_6} \Lambda_{c_7 c_8 c_9}].$$

E_{11} $3-6$ Duality Relations

We can also I_1 extend the $D_{a_1 \ldots a_4} = 0$ equation by the requirement that the variation fully anti-symmetrized Cartan forms, resulting in

$$\mathcal{D}_{a_1 a_2 a_3 a_4} = \mathcal{G}_{a_1, a_2 a_3 a_4} - \frac{1}{2 \cdot 4!} \varepsilon_{a_1 a_2 a_3 a_4}{}^{b_1 .. b_7} \mathcal{G}_{b_1, b_2 .. b_7} + \frac{1}{2} G_{[a_1 a_2, a_3 a_4]}$$

where

$$\mathcal{G}_{a_1 a_2 a_3 a_4} = G_{[a_1, a_2 a_3 a_4]} + \frac{15}{2} G^{b_1 b_2}{}_{, b_1 b_2 a_1 .. a_4} \; , \; \mathcal{G}_{b_1, b_2 .. b_7} = G_{[b_1, b_2 .. b_7]} + 28 G^{c_1 c_2}{}_{, c_1 c_2 [b_1 .. b_6, b_7]} .$$

The variation of $\mathcal{D}_{a_1 .. a_4}$ under $I_c(E_{11})$ is then

$$\delta \mathcal{D}_{a_1 .. a_4} = \frac{1}{4!} \varepsilon_{a_1 a_2 a_3 a_4}{}^{b_1 .. b_7} D_{b_1 b_2 b_3 b_4} \Lambda_{b_5 b_6 b_7} + 3 D_{c, [a_1 a_2} \Lambda^{c}{}_{a_3 a_4]} .$$

E_{11} Equations of Motion

We can project $D_{a_1 \ldots a_4}$ into $E^{\mu_1\mu_2\mu_3} = \partial_\nu[(\det e)^{1/2} D^{\nu\mu_1\mu_2\mu_3}]$ giving

$$\partial_\nu[(\det e)^{1/2} G^{[\nu,\mu_1\mu_2\mu_3]}) + \frac{1}{2 \cdot 4!}(\det e)^{-1} \varepsilon^{\mu_1\mu_2\mu_3\tau_1\ldots\tau_8} G_{[\tau_1,\tau_2\tau_3\tau_4]} G_{[\tau_5,\tau_6\tau_7\tau_8]} = 0,$$

familiar as the second order supergravity equation of motion for $A_{\mu_1\mu_2\mu_3}$.
This can be put back into tangent indices $E^{a_1a_2a_3}$, and the variation of a suitable l_1 extension can be shown to give

$$\delta \mathcal{E}^{a_1a_2a_3} = \frac{3}{2} E_b{}^{[a_1} \Lambda^{|b|a_2a_3]} + \ldots$$

where $\ldots$ indicates more contributions depending on $D_{a_1..a_4}$, and

$$E_a{}^b = (\det e) R_a{}^b - 48 G_{[a,c_1c_2c_3]} G^{[b,c_1c_2c_3]} + 4\delta_a{}^b G_{[c_1,c_2c_3c_4]} G^{[c_1,c_2c_3c_4]}.$$

is the supergravity Einstein equation of motion, reproducing the correct energy-momentum tensor.

E_{11} Equations of Motion

This can also be l_1 extended

$$
\begin{aligned}
\mathcal{E}_a{}^b = (\det e)\mathcal{R}_a{}^b - 48G_{[a,c_1c_2c_3]}G^{[b,c_1c_2c_3]} + 4\delta_a{}^b G_{[c_1,c_2c_3c_4]}G^{[c_1,c_2c_3c_4]} \\
- 360G^{d_1d_2,}{}_{d_!d_2ac_1c_2c_3}G^{[b,c_1c_2c_3]} - 360G^{d_1d_2,}{}_{d_1d_2}{}^{bc_1c_2c_3}G_{[a,c_1c_2c_3]} \\
+ 60\delta_a{}^b G^{d_1d_2,}{}_{d_1d_2c_1c_2c_3c_4}G^{[c_1,c_2c_3c_4]} - 12G_{c_1c_2,ac_3}G^{[b,c_1c_2c_3]} + 3G_{c_1c_2,d}{}^d G_{[a,}{}^{bc_1c_2]} \\
- 6(\det e)e_a{}^\lambda e^{b\mu}\partial_{[\mu}\{(\det e)^{-1/2}G^{\tau_1\tau_2,}{}_{\tau_1\tau_2\lambda]}\} \\
- (\det e)^{1/2}\omega_{c,b}{}^c G^{d_1d_2,}{}_{d_1d_2a} - 3(\det e)^{1/2}\omega_{a,b}{}^c G^{d_1d_2,}{}_{d_1d_2c},
\end{aligned}
$$

and it's $I_c(E_{11})$ variation can be shown to be

$$
\begin{aligned}
\delta\mathcal{E}_a{}^b = & -36E_{ac_1c_2}\Lambda^{bc_1c_2} - 36E^{bc_1c_2}\Lambda_{ac_1c_2} + 8\delta_a{}^b\Lambda^{c_1c_2c_3}E_{c_1c_2c_3} \\
& - 2\varepsilon_{ac_1\ldots c_7d_1d_2d_3}G^{[b,c_1c_2c_3]}D^{c_4\ldots c_7}\Lambda^{d_1d_2d_3} - 2\varepsilon^{bc_1\ldots c_7d_1d_2d_3}G_{[a,c_1c_2c_3]}D_{c_4\ldots c_7}\Lambda_{d_1d_2d_3} \\
& + \frac{1}{3}\delta_a{}^b\varepsilon^{c_1\ldots c_8d_1d_2d_3}G_{c_1,c_2c_3c_4}D_{c_5\ldots c_8}\Lambda_{d_1d_2d_3}.
\end{aligned}
$$

Thus E_{11} exactly reproduces the equations of motion of the bosonic sector of eleven-dimensional supergravity when we neglect the effects of the higher level derivatives in these equations.

In [17] similar techniques led to the derivation of the following second order dual gravity equation of motion

$$
\begin{aligned}
&E_{a_1\ldots a_8,b} \\
&= \frac{1}{9\cdot 2}(\det e)^{\frac{1}{2}} e^{c\mu}\partial_\mu G_{c,a_1\ldots a_8,b} + \frac{1}{2\cdot 9}(\det e)^{\frac{1}{2}} e^{c\mu}\partial_\mu (8G_{[a_1,a_2\ldots a_8]c,b} - G_{b,a_1\ldots a_8,c}) \\
&- \frac{4}{9\cdot 9}(\det e)^{\frac{1}{2}}(8e_b{}^\mu\partial_\mu G_{[a_1,a_2\ldots a_8]c,}{}^c + e_{[a_1|}{}^\mu\partial_\mu G_{b|,a_2\ldots a_8]c,}{}^{,c} - 7e_{[a_1}{}^\mu\partial_{|\mu|} G_{a_2,a_3\ldots a_8]bc,}{}^c) \\
&- \frac{2\cdot 7}{9\cdot 9}(G_{[a_1,|e|}{}^e G_{a_2,a_3\ldots a_8]bc,}{}^c - \frac{1}{2}G_{b,e}{}^e G_{[a_1,a_2\ldots a_8]c,}{}^c + \frac{1}{2}G_{[a_1,|e}{}^e G_{b|,a_2\ldots a_8]c,}{}^c) \\
&+ \frac{1}{9\cdot 4}G^{c,e}{}_e G_{c,a_1\ldots a_8,b} + \frac{1}{9\cdot 4}G^{c,e}{}_e(8G_{[a_1,a_2\ldots a_8]c,b} - G_{b,a_1\ldots a_8,c}) \\
&- \frac{1}{2\cdot 9}G^{e,c}{}_e(G_{c,a_1\ldots a_8,b} + 8G_{[a_1,a_2\ldots a_8]c,b} - G_{b,a_1\ldots a_8,c}) \\
&+ \frac{4\cdot 7}{9\cdot 9}\{(G_{a_1,}{}^{ce} + G_{a_1,}{}^{ec})G_{a_2,a_3\ldots a_8be,c} - \frac{1}{2}(G_{b,}{}^{ce} + G_{b,}{}^{ec})G_{a_1,a_2\ldots a_8e,c}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{2}(G_{a_1,}{}^{ce}+G_{a_1,}{}^{ec})G_{b,a_2\ldots a_8 e,c}\}\\
&-\frac{1}{9}(-2G_{[a_1,|b|a_2}{}^{c}G_{a_3,a_4\ldots a_8]}{}^{c}+5G_{[a_1,a_2a_3}{}^{c}G_{a_4,a_5\ldots a_8]bc}-8G_{b,[a_1a_2}{}^{c}G_{a_3,a_4\ldots a_8]c}\\
&-G_{[a_1,a_2a_3}{}^{c}G_{|b|,a_4\ldots a_8]c})\\
&+\frac{4}{9\cdot 9}(7G_{[a_1,a_2}{}^{e}G_{|e|,a_3\ldots a_8]bc,}{}^{c}-G_{[a_1,|b}{}^{e}G_{e|,a_2\ldots a_8]c,}{}^{c}-8G_{b,[a_1}{}^{e}G_{|e|,a_2\ldots a_8]c,}{}^{c})\\
&+\varepsilon^{c_1c_2e_1\ldots e_9}G_{e_1,e_2\ldots e_9,[a_1|}G_{[c_1,c_2|a_2\ldots a_8]],b}\\
&-\frac{4\cdot 7}{9\cdot 9}(6G_{[a_1,a_2}{}^{e}G_{a_3,a_4\ldots a_8]bec,}{}^{c}-G_{[a_1,|b|}{}^{e}G_{a_2,a_3\ldots a_8]ec,}{}^{c}-8G_{b,[a_1}{}^{e}G_{a_2,a_3\ldots a_8]ec,}{}^{c}\\
&+G_{[a_1,a_2}{}^{e}G_{|b,e|a_3\ldots a_8]c,}{}^{c})\\
&-\frac{4}{9}G_{c,[a_1}{}^{e}G_{|b,e|a_2\ldots a_8],}{}^{c}+\frac{4}{9}(G_{c,[a_1}{}^{e}G^{c,}{}_{|e|a_2\ldots a_8],b}-G_{e,[a_1}{}^{c}G_{|c|,}{}^{e}{}_{a_2\ldots a_8],b})\\
&+\frac{1}{9\cdot 2}(G_{c,b}{}^{e}G^{c,}{}_{[a_1\ldots a_8],e}-G_{e,b}{}^{c}G_{c,a_1\ldots a_8,}{}^{e})+\frac{7\cdot 4}{9}G^{[c,}{}_{[a_1}{}^{e]}G_{a_2,|ce|a_3\ldots a_8],b}\\
&+\frac{4}{9}G_{c,b}{}^{e}G_{[a_1,a_2\ldots a_8]}{}^{c}{}_{,e}
\end{aligned}
$$

Quantization Attempt: Current Algebras

In 2017 at OIST, a radical preliminary proposal for quantizing E_{11} via the method of 'current algebras' was proposed in [19] by Professor Sugawara, and used to study the M2 and M5 branes in [20], and cosmology in [21].

The essence of this approach is to implement the Dirac-Schwinger commutation relations

$$[\Theta_{00}(x), \Theta_{00}(y)] = -i\{\Theta_{0k}(x) + \Theta_{0k}(y)\}\eta^{kl}\partial_l\delta(\mathbf{x} - \mathbf{y})$$

to ensure covariance.

As it stands, the approach does not fully utilize the higher level coordinates, but suggests a vast generalization of Dirac-Schwinger may exist in E_{11}.

So E_{11} asks us to accept an infinite collection of fields $h_{ab}, A_{a_1a_2a_3}, A_{a_1\ldots a_6}, h_{a_1\ldots a_8,b}, \ldots$ and to generalize our notion of space-time to an infinite-dimensional 'generalized space-time' $P_a, Z^{a_1a_2}, Z^{a_1\cdots a_5}, \ldots$ implying the existence of a 'generalized geometry' on this space-time, and deep links with Higher Spin Theory [12].

What is the meaning of this? What have we bought into?

There is a famous saying in physics: "Spacetime is Doomed" [13], e.g., an infinitely precise measurement is thought to create a black hole, space-time intuition is based on sufficiently low energies.

E_{11} is attempting to discuss the low energy limit of M-theory. The higher fields and coordinates appear to be a low energy effect, and that space-time must be replaced by some more fundamental degrees of freedom in the final M-theoretical quantum theory of gravity. This has happened in the past:

Interpreting E_{11}

Kaluza-Klein dimensional reduction of 11D supergravity on a circle gives the unique (non-chiral) ten-dimensional IIA supergravity theory, the low energy limit of Type IIA superstring.

This was initially thought to illustrate a defect in the 11D theory [14], since it implies the existence of extra Kaluza-Klein states that were not initially seen in the 10D superstring theory.

However it was later recognized that the additional Kaluza-Klein states do arise in the 10D superstring theories in the form of 'soliton-like' solutions associated to (mem)branes of the theory.

In other words, the ten-dimensional supergravity theory with the additional soliton-like states appeared to be an 11D theory in disguise [14].

'*E* Theory' appears to be an extension of 11D supergravity containing new effects that may reflect the existence of a more fundamental set of d.o.f. than 'space-time' in the final quantum gravity [10], space-time is 'extended' in 'E Theory' from the get-go.

11D supergravity reduces to Einstein gravity when we (ignore spin and) set the 3-form $A_{a_1 a_2 a_3}$ to zero. Thus a subset of a Kac-Moody algebra, E_{11}, seems to describe Einstein's gravity in 11D as a special case.

In 2001 [15] it was conjectured that Einstein's gravity in D dimensions arises from a non-linear realization involving the Kac-Moody algebra A_{D-3}^{+++}, specifically $A_{D-3}^{+++} \otimes_s l_1 / I_C(A_{D-3}^{+++})$.

In 2020 [16] the dynamics of the nonlinear realization of $A_1^{+++} \otimes_s l_1 / I_C(A_1^{+++})$ were worked out, including a derivation of the dual graviton equation of motion, which inspired a derivation of the E_{11} dual graviton equation of motion [17].

In A_1^{+++} we are thus studying Einstein's gravity along with additional contributions presumably reflecting the breakdown of gravity at higher energy scales (shorter distances).

Future Research I

- $I_c(E_{11}) \otimes_s l_1$ is reminiscent of the 4D Poincaré group $\mathrm{SO}(1,3) \otimes_s P_4$, indeed it contains the Poincaré group. In Quantum Field Theory, particles are interpreted as irreducible representations of the Poincaré group via the so-called 'Wigner method'. In [22] it was conjectured that different branes arise as irreducible representations of $I_c(E_{11}) \otimes_s l_1$ for different choices of isotropy subgroups. In [22], [23], and [24], partial results on the isotropy groups for the M2, M5 branes, and IIA string, were presented. Much more research is needed on this very difficult problem, no other approach to string theory suggests a 'Wigner method' to branes in string theory.
- The low energy effective action of the 26D bosonic string was conjectured in [9] to arise from a nonlinear realization involving the Kac-Moody algebra K_{27}. The full power of E theory has yet to be applied to this difficult algebra.

Future Research II

- The current algebra methods of [19 — 21] require further study, they may apply to Einstein's gravity in 4D via A_1^{+++} and may shed light on quantum gravity.

- E_9 recently appeared in the swampland program [25], E_{11}?

- How do the above results manifest in the E_{10} program [26]?

- The E_{11} dual graviton was recently studied in the context of 'generalized symmetries' [27], what about the rest of E theory?

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