Celestial Amplitudes and Asymptotic Symmetries

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based on:

A. Fotopoulos, St.St., T.R. Taylor, Bin Zhu:

- **BMS Algebra from Soft and Collinear Limits**
  arXiv:1912.10973, JHEP 2003 (2020) 130

- **Extended Super BMS Algebra of Celestial CFT**
  arXiv:2007.03785, JHEP 2009 (2020) 198

- Wei Fan, A. Fotopoulos, St.St., T.R. Taylor:
  On Sugawara construction on Celestial Sphere
  arXiv:2005.10666, JHEP 2009 (2020) 139

- Wei Fan, A. Fotopoulos, St.St., T.R. Taylor, Bin Zhu:
  Conformal Blocks from Celestial Gluon Amplitudes
  arXiv:2103.0442

Recap: Amplitudes

Traditional momentum space

$$p_k^\mu, \quad k = 1, \ldots, n$$

$$p_k^2 = -m_k^2$$

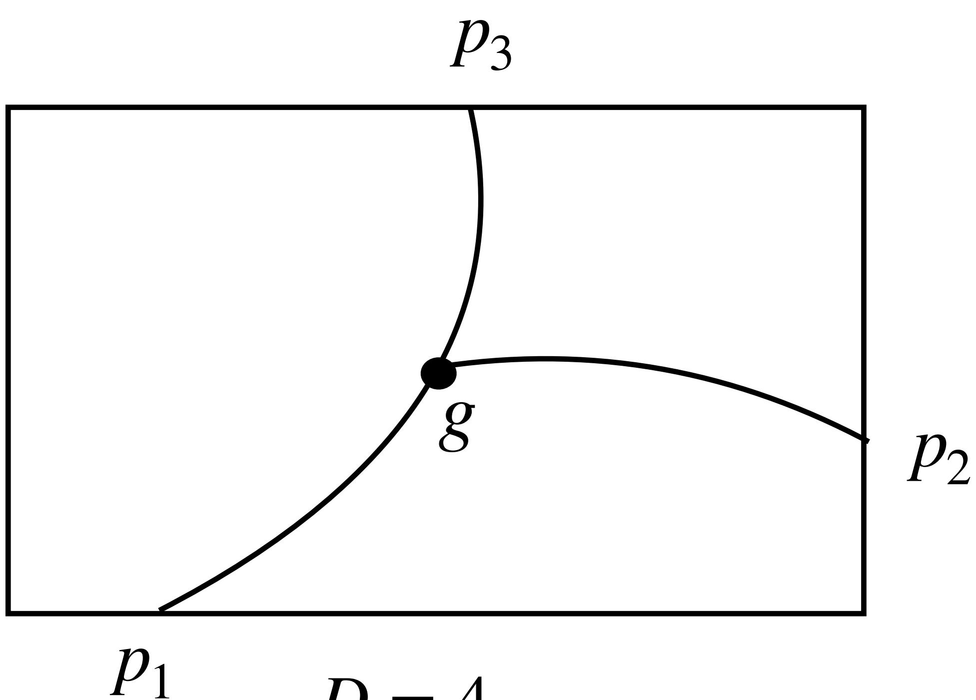
- amplitudes specified by asymptotic wave functions, which transform simply under space-time translations
- with manifest translation symmetry
- traditional amplitudes describe transitions between momentum eigenstates

D=4 Minkowski space probably not the right space
to see **all** symmetries
of scattering amplitudes

Scattering amplitudes in D=4
have interpretation
as Euklidian **D=2** conformal correlators

Basic Idea

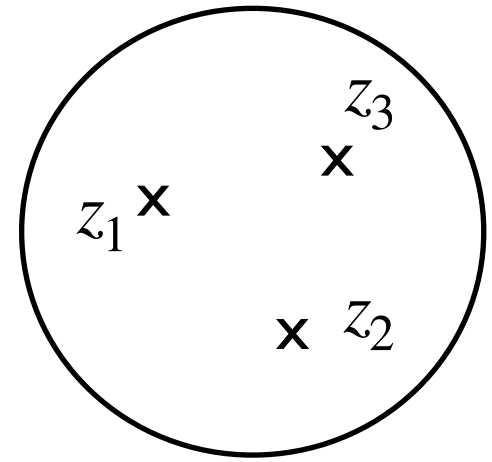
Amplitudes = conformal correlators of primary fields on celestial sphere



$D = 4$

$$z_k = \frac{p_k^1 + ip_k^2}{p_k^0 + p_k^3}$$

=



celestial sphere

$D = 2$

$$\sim \frac{g}{|z_1 - z_2|^{h_1+h_2-h_3}\ |z_2 - z_3|^{h_2+h_3-h_1}\ |z_1 - z_3|^{h_1+h_3-h_2}}$$

D=4 space-time QFT correlators

D=2 Euklidian CFT correlators

Lorentz symmetry

$$SO(1,3) \simeq SL(2,\mathbf{C})$$

$$z_i \to \frac{az_i + b}{cz_i + d}$$

global conformal symmetry on CS^2

Why ?

- Constrain S-matrix and understand amplitude relations

  - look for new physical principles and mathematical structures to constrain S-matrix

  - sensitivity to both UV and IR physics leads to powerful constraints on the analytic structure of amplitudes

- New way of looking at quantum field theory and quantum gravity

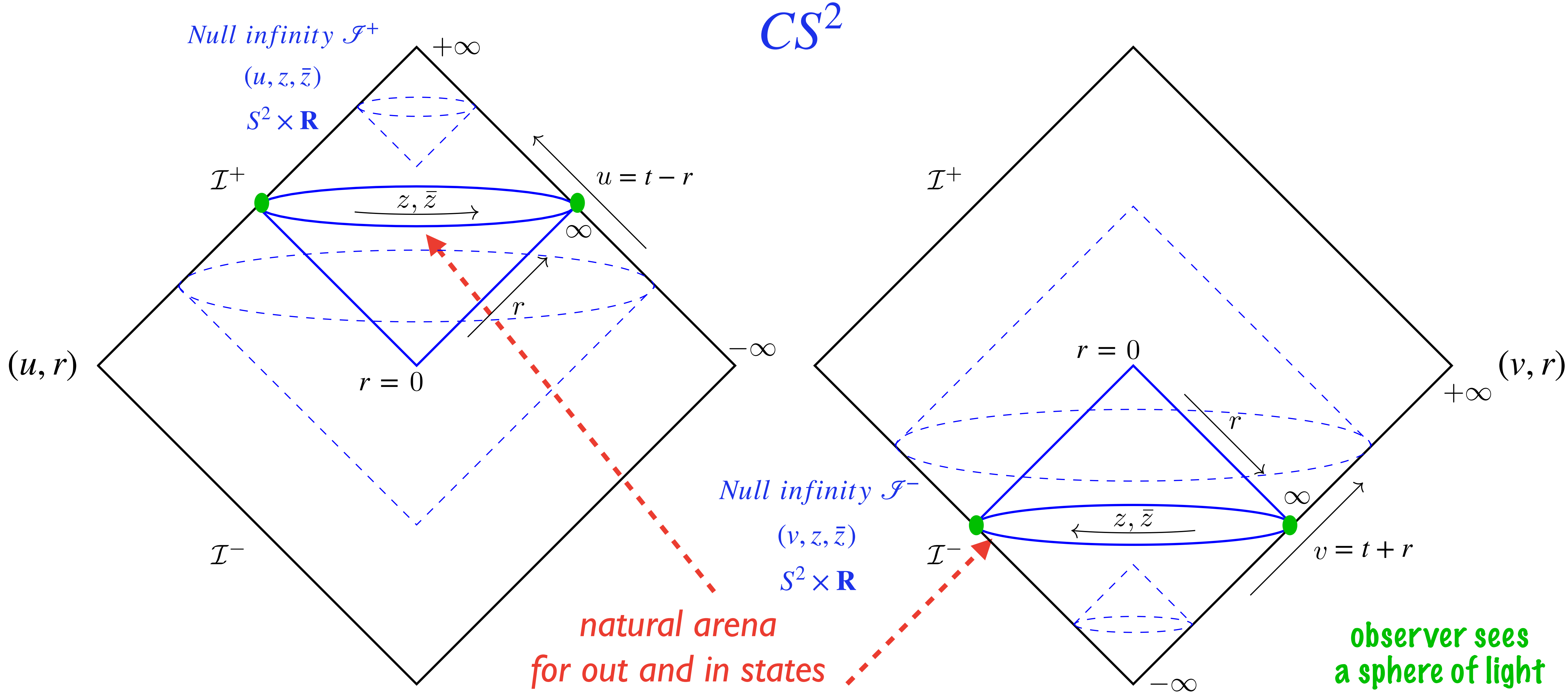
  - flat space-time holography

$$ds^2 = -dt^2 + d\vec{x}^2$$

Flat Minkowski metric in <u>retarded (or Bondi) coordinates</u> $(u, r, z, \bar{z})$

$$\begin{cases} x^0 = u + r \\ x^1 = \dfrac{r\,(z+\bar{z})}{1+|z|^2} \\ x^2 = -i\dfrac{r\,(z-\bar{z})}{1+|z|^2} \\ x^3 = \dfrac{r(1-|z|^2)}{1+|z|^2} \end{cases} \qquad r^2 = \vec{x}^2$$

$$ds^2 = -du^2 - 2\,dudr + \underbrace{\frac{4r^2}{(1+|z|^2)^2}}_{CS^2}\,dzd\bar{z}$$



Massless particle on celestial sphere

described by
- the point $z \in CS^2$ at which it enters or exits the celestial sphere
- SL(2,C) Lorentz quantum numbers $(h, \bar{h})$

$$z \in CS^2 \quad \Longrightarrow \quad p^\mu = \frac{\omega}{1 + |z|^2} q^\mu(z, \bar{z})$$

Null vector q

$$q_\mu q^\mu = 0$$

with:

$$q^\mu = (1 + |z|^2, z + \bar{z}, -i(z - \bar{z}), 1 - |z|^2)$$

$$\omega = p^0 = E$$

invert:

$$z = \frac{p^1 + ip^2}{p^0 + p^3}$$

$$(\vec{p})^2 = (p^0)^2$$

$$p^\mu \longrightarrow (\omega, z, \bar{z})$$

plane waves in Minkowski: $\exp\{\pm i p_\mu x^\mu\}$

boost eigenstates: $\exp\{\pm i E u\}$

Particles <-> Operators

in momentum basis: plane waves with momentum $p^\mu = \omega q^\mu(z)$

in conformal basis: conformal primary wave functions Φ

"state operator correspondence"

$$\Phi_{h,\bar{h}}\left(\frac{az+b}{cz+d}, \frac{\bar{a}\bar{z}+\bar{b}}{\bar{c}\bar{z}+\bar{d}}; \Lambda^\mu{}_\nu X^\nu\right) = (cz+d)^{2h}\,(\bar{c}\bar{z}+\bar{d})^{2\bar{h}}\,\Phi_{h,\bar{h}}(z,\bar{z};X^\mu)$$

with:

$h+\bar{h}=\Delta$ dimension

$h-\bar{h}=J$ spin

$$(h,\bar{h}) = \frac{1}{2}(\Delta+J, \Delta-J)$$

In the massless case, with or without spin,

transition from momentum space to conformal primary wavefunctions

with conformal dimension Δ

is implemented by Mellin transform:

$$|\,\Delta, z\rangle = \int_0^\infty d\omega\ \omega^{\Delta-1}\ |\,\omega, z\rangle$$

E.g.: plane wave $\exp\{\pm i p_\mu x^\mu\}$

$$\phi^{\pm}_{\Delta}(x,z,\bar{z}) = \int_0^\infty d\omega\ \omega^{\Delta-1}\ \exp\left\{\pm i\omega q_\mu x^\mu - \epsilon\omega\right\}$$

$$= \left\{x^\mu q_\mu(z,\bar{z}) \mp i\epsilon\right\}^{-\Delta}$$

solves D=4 Klein-Gordon equation

scalar: J=0 $h = \bar{h} = \frac{\Delta}{2}$ $\Delta = 1 + i\lambda,\ \lambda \in \mathbf{R}$

Pasterski, Shao (2017)

n-point amplitude on celestial sphere

$$\mathscr{A}(\{p_i, \epsilon_j\}) = i(2\pi)^4\ \delta^{(4)}\left(p_1 + p_2 - \sum_{k=3}^{n} p_k\right)\ A(\{p_i, \epsilon_j\})$$

Celestial amplitudes $\tilde{\mathscr{A}}$ of massless particles are obtained
from momentum-space amplitudes $\mathscr{A}$
by Mellin transforms w.r.t. particle energies $\Delta_j = 1 + i\lambda_j$

$$\tilde{\mathscr{A}}_{\{\Delta_l\}}(\{z_l, \bar{z}_l\}) = \left(\prod_{l=1}^{n}\int_0^\infty \omega_l^{\Delta_l - 1}\ d\omega_l\right)\ \delta^{(4)}(\omega_1 q_1 + \omega_2 q_2 - \sum_{k=3}^{N}\omega_k q_k)$$
$$\times A(\{\omega_i, z_i, \bar{z}_i\})$$

D=2 CFT correlators involve conformal wave packets

Gauge Amplitudes

four-gluon amplitude:

$$\tilde{\mathcal{A}}_4(-,-,+,+) = 8\pi\,\delta(r-\bar{r})\,\theta(r-1)\left(\prod_{i<j}^{4} z_{ij}^{\frac{h}{3}-h_i-h_j}\,\bar{z}_{ij}^{\frac{\bar{h}}{3}-\bar{h}_i-\bar{h}_j}\right)$$

$$\times\, r^{\frac{5}{3}}\,(r-1)^{\frac{2}{3}}\,\delta\left(-4+\sum_{i=1}^{4}\Delta_i\right)$$

all four points z_i must lie on a circle

$$r = \frac{z_{12}\,z_{34}}{z_{23}\,z_{41}}$$

conformal invariant cross-ratio on CS^2

$$r^{-1} = \sin^2\left(\frac{\theta}{2}\right)$$

Pasterski, Shao, Strominger (2017)

$$h_1 = \tfrac{i}{2}\lambda_1,\quad h_2 = \tfrac{i}{2}\lambda_2,\quad h_3 = 1+\tfrac{i}{2}\lambda_3\,,\, h_4 = 1+\tfrac{i}{2}\lambda_4$$

$$\bar{h}_1 = 1+\tfrac{i}{2}\lambda_1,\quad \bar{h}_2 = 1+\tfrac{i}{2}\lambda_2,\quad \bar{h}_3 = \tfrac{i}{2}\lambda_3\,,\, \bar{h}_4 = \tfrac{i}{2}\lambda_4$$

higher-point: involve Gaussian hypergeometric functions like string amplitudes

Schreiber, Volovich, Zlotnikov (2017)

Graviton Amplitudes

four-graviton amplitude:

$$\tilde{\mathcal{A}}_4(--,--,++,++) = 2\pi\,\delta(r-\bar{r})\,\theta(r-1)\,\left(\prod_{i<j}^{4} z_{ij}^{\frac{h}{3}-h_i-h_j}\,\bar{z}_{ij}^{\frac{\bar{h}}{3}-\bar{h}_i-\bar{h}_j}\right)$$

$$\times\, r^{\frac{11}{3}-\frac{\beta}{3}}\,(r-1)^{-\frac{1}{3}-\frac{\beta}{3}}\,\delta\left(-2+\sum_{i=1}^{4}\Delta_i\right)$$

St.St., Taylor (2018)

$$h_1 = -\frac{1}{2}+\frac{i}{2}\lambda_1,\;\; h_2 = -\frac{1}{2}+\frac{i}{2}\lambda_2,\;\; h_3 = \frac{3}{2}+\frac{i}{2}\lambda_3\;, h_4 = \frac{3}{2}+\frac{i}{2}\lambda_4$$

$$\bar{h}_1 = \frac{3}{2}+\frac{i}{2}\lambda_1,\;\; \bar{h}_2 = \frac{3}{2}+\frac{i}{2}\lambda_2,\;\; \bar{h}_3 = -\frac{1}{2}+\frac{i}{2}\lambda_3\;, \bar{h}_4 = -\frac{1}{2}+\frac{i}{2}\lambda_4$$

$$\beta := 2-\frac{1}{2}\sum_{i=1}^{4}\Delta_i$$

- *first calculation of graviton amplitude in the conformal basis*
- *important for the soft graviton theorems* $\Delta \to 1,0,\ldots$ *in celestial basis*

no holomorphic factorization (due to supertranslation operator P)

Celestial Conformal Field Theory (CCFT)

understand the nature of 2D CFT on celestial sphere, i.e. spectrum of fields and their interactions

- states, spectrum
- operator products (OPEs)
- energy momentum tensor, Virasoro algebra
- conformal block expansion
- crossing symmetry and conformal bootstrap
- ⋮

recent progress

Operator product expansion

Celestial conformal field theory (CCFT)

$$\mathcal{O}^a_{\Delta_1,-1}(z,\bar z)\mathcal{O}^b_{\Delta_2,+1}(w,\bar w) = \frac{C_{(-,+)-}(\Delta_1,\Delta_2)}{z-w}\sum_c f^{abc}\ \mathcal{O}^c_{(\Delta_1+\Delta_2-1),-1}(w,\bar w)$$
$$+\frac{C_{(-+)+}(\Delta_1,\Delta_2)}{\bar z-\bar w}\sum_c f^{abc}\ \mathcal{O}^c_{(\Delta_1+\Delta_2-1),+1}(w,\bar w)$$
$$+\ C_{(-+)--}(\Delta_1,\Delta_2)\ \frac{\bar z-\bar w}{z-w}\ \delta^{ab}\ \mathcal{O}_{(\Delta_1+\Delta_2),-2}(w,\bar w)$$
$$+\ C_{(-+)++}(\Delta_1,\Delta_2)\ \frac{z-w}{\bar z-\bar w}\ \delta^{ab}\ \mathcal{O}_{(\Delta_1+\Delta_2),+2}(w,\bar w)\ +reg\,.$$

Derive from collinear limits of D=4 EYM amplitudes

Fan, Fotopoulos, St.St., Taylor, Zhu (2019)

↕ D=4 S-matrix constrains OPE or vice versa

Derive from first principles and consistency conditions

Pate, Raclariu, Strominger, Yuan (2019)

} extended BMS symmetry

Symmetries

At null infinity $\mathscr{I}^{\pm}$ more (hidden) symmetries present
to constrain S-matrix

→ non-trivial consistency on amplitudes
(sensitiv to both UV and IR physics)

$$z_i \to \frac{az_i + b}{cz_i + d}$$

$$SL(2,\mathbf{C})_{z_i}: \ \tilde{\mathscr{A}}_n(\{\Delta_i, J_i\}) \longrightarrow (cz_i + d)^{\Delta_i + J_i} \, (\bar{c}\bar{z}_i + \bar{d})^{\Delta_i - J_i} \, \tilde{\mathscr{A}}_n(\{\Delta_i, J_i\})$$

$$P_{-1/2,-1/2} = e^{(\partial_h + \partial_{\bar{h}})/2} = P^0 + P^3$$

St.St., Taylor (2018)

$$P^{(j)}_{-1/2,-1/2}: \ \tilde{\mathscr{A}}_n(\{\Delta_i, J_i\}) \longrightarrow \tilde{\mathscr{A}}_n(\{\Delta_j + 1, J_i\})$$

comprises into translation operator P^{μ} *shifts conformal dimension* Δ_j

celestial gravitational amplitudes appear
as gauge amplitudes translated in space-time

Soft theorems

In usual QFT soft theorems $E_s \to 0$ play an important role
in consistency and structure of amplitudes
(in fact, soft theorems completely constrain almost all amplitudes)

in D=4: $p_s \to 0$

$$M_{n+1} \longrightarrow \left(\underbrace{\frac{1}{\epsilon^3} S_{\rm G}^{(0)}}_{\text{Weinberg (1965)}} + \underbrace{\frac{1}{\epsilon^2} S_{\rm G}^{(1)} + \frac{1}{\epsilon} S_{\rm G}^{(2)}}_{\text{Cachazo, Strominger (2014)}} + \ldots \right) M_n$$

$$A_{n+1} \longrightarrow \left(\frac{1}{\epsilon^2} S_{\rm YM}^{(0)} + \frac{1}{\epsilon} S_{\rm YM}^{(1)} + \ldots \right) A_n$$

soft theorems imply
Ward identities for asymptotic symmetries

Strominger (2013)

on CS^2: $\omega_s \to 0$ typical IR poles in Mellin transform

Cf.: $$\int_0^\infty d\omega_s\, \omega_s^{\Delta_s-1}\, \frac{e^{-J\omega_s}}{\omega_s} = \frac{1}{\Delta_s-1} - \frac{J}{\Delta_s} + \frac{1}{2}\frac{J^2}{\Delta_s+1} + \ldots$$

$$\mathcal{M}_{n+1} \longrightarrow \Big(\underbrace{\omega_s^{-1}\, S_G^{(0)}}_{\Delta_s \to 1} + \underbrace{\omega_s^{0}\, S_G^{(1)}}_{\Delta_s \to 0} + \underbrace{\omega_s\, S_G^{(2)}}_{\Delta_s \to -1} + \ldots \Big)\, \mathcal{M}_n$$

$$\mathcal{A}_{n+1} \longrightarrow \Big(\underbrace{\omega_s^{-1}\, S_{YM}^{(0)}}_{\Delta_s \to 1} + \underbrace{\omega_s^{0}\, S_{YM}^{(1)}}_{\Delta_s \to 0} + \ldots \Big)\, \mathcal{A}_n$$

E.g.: Yang-Mills $s = n+1$

$$\mathcal{A}_{n+1} = \omega_s^{-1}\, \frac{z_{n1}}{z_{ns} z_{s1}} + \omega_s^{0} \left\{ \frac{1}{\omega_1}\frac{1}{z_{s1}} \left(\bar{z}_{s1}\partial_{\bar{z}_1} - 2\bar{h}_1 \right) + \frac{1}{\omega_n}\frac{1}{z_{ns}} \left(\bar{z}_{sn}\partial_{\bar{z}_n} - 2\bar{h}_n \right) \right\}$$

$$\times\, \mathcal{A}_n(\{z_1, \bar{z}_1, \Delta_1, J_1\}, \ldots, \{z_n, \bar{z}_n, \Delta_n, J_n\}) + \ldots$$

Soft expansion

$$\mathscr{M}_{n+1} \longrightarrow \Big(\underbrace{\omega_s^{-1}\ S_{\mathrm{G}}^{(0)}}_{\Delta_s \to 1} + \underbrace{\omega_s^{0}\ S_{\mathrm{G}}^{(1)}}_{\Delta_s \to 0} + \underbrace{\omega_s\ S_{\mathrm{G}}^{(2)}}_{\Delta_s \to -1} + \dots \Big)\ \mathscr{M}_n$$

$$\mathscr{A}_{n+1} \longrightarrow \Big(\underbrace{\omega_s^{-1}\ S_{\mathrm{YM}}^{(0)}}_{\Delta_s \to 1} + \underbrace{\omega_s^{0}\ S_{\mathrm{YM}}^{(1)}}_{\Delta_s \to 0} + \dots \Big)\ \mathscr{A}_n$$

$$\Delta \to 1, 0 \dots$$

in Mellin space "soft-limits" reproduce Weinberg's soft theorem and more symmetries become manifest

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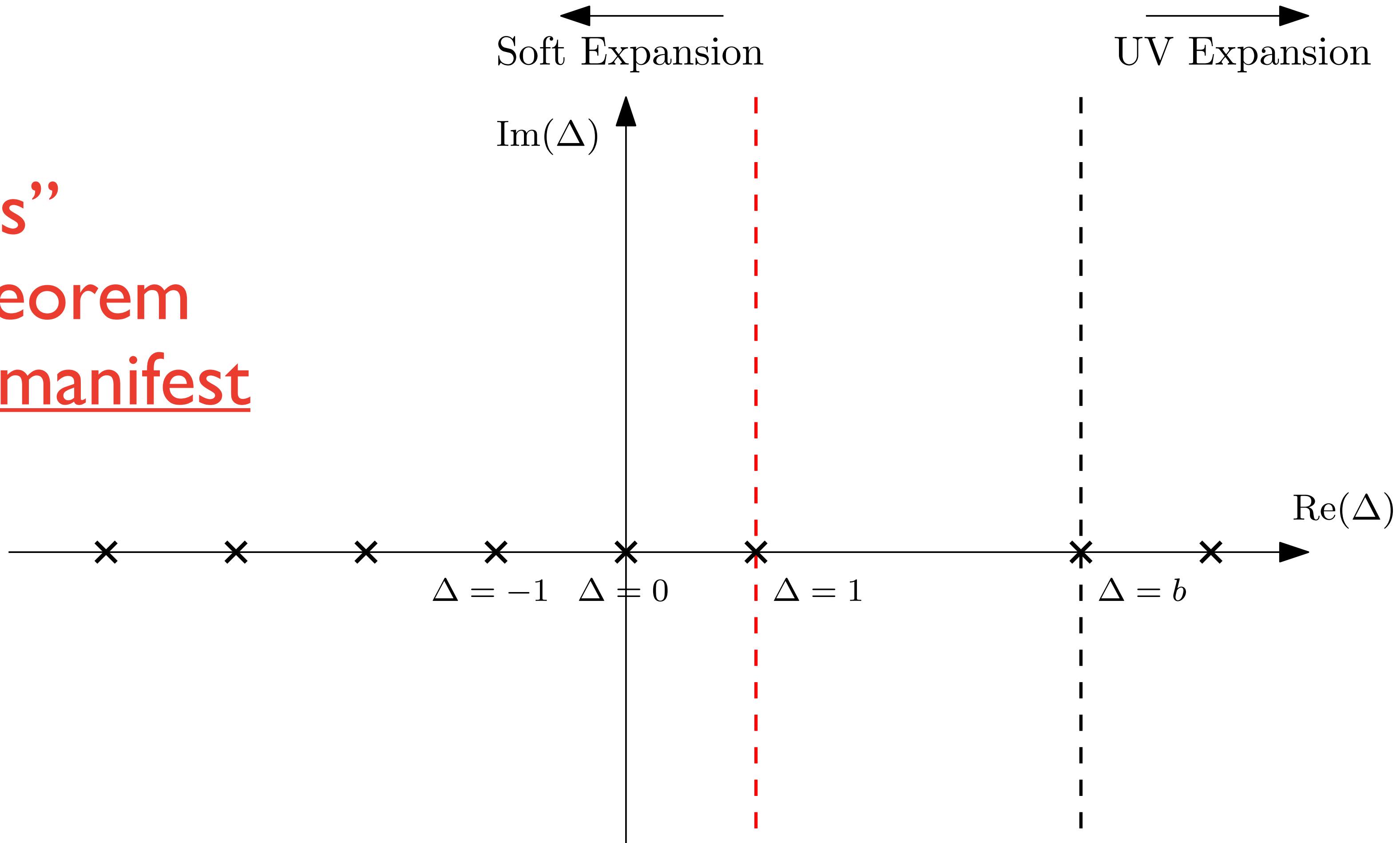


Kapec, Mitra, Raclariu, Strominger (2016)

Donnay, Puhm, Strominger (2018)

Recap: conformally soft modes ($\Delta \to 1$) of gauge bosons generate a Kac-Moody symmetry and correspond to large gauge transformations in D=4

Here: relate soft graviton modes to Ward identities and BMS symmetries

explicit field realization

(i) energy-momentum tensor T(z):

soft-graviton $\Delta \to 0$

$$T(z) := \tilde{\mathcal{O}}_{\Delta=2,J=+2}(z,\bar{z}) = \frac{3}{\pi}\int d^2w\ (z-w)^{-4}\ \mathcal{O}_{\Delta=0,J=-2}(w,\bar{w})$$

$(h,\bar{h}) = (2{,}0)$

Fotopoulos, Taylor (2019)

shadow transformation:

$$\tilde{\mathcal{O}}^a_{\tilde{\Delta},\tilde{J}}(z,\bar{z}) = \tilde{\mathcal{O}}^a_{2-\Delta,-J}(z,\bar{z}) = \frac{(\Delta+J-1)}{\pi}\int_{\mathbf{C}} \frac{d^2w}{(z-w)^{2-\Delta-J}(\bar{z}-\bar{w})^{2-\Delta+J}}\ \mathcal{O}^a_{\Delta,J}(w,\bar{w})$$

Ferrara, Grillo, Parisi, Gatto (1972)
Dolan, Osborn (2012)

then:

$$\langle T(z) \prod_{i=1}^{n} O_{\Delta_i}(z_i, \bar{z}_i) \rangle = \sum_{i=1}^{n} \left(\frac{h_{O_i}}{(z-z_i)^2} + \frac{\partial_{z_i}}{z-z_i} \right) \langle \prod_{i=1}^{n} O_{\Delta_i}(z_i, \bar{z}_i) \rangle$$

OPE:

$$T(w)T(z) = \frac{2T(z)}{(w-z)^2} + \frac{\partial_z T(z)}{w-z} + \dots$$

$$T(w)\bar{T}(z) = reg\,.$$

$\rightarrow \quad L_n, \bar{L}_m \qquad c = 0$

Fotopoulos, St.St., Taylor, Zhu (2019)

$\frac{1}{2k+C_2} \simeq \frac{1}{2\dim(g)},$

$k = 0$

Note: Sugawara construction of the energy momentum tensor

$$T(w) = \frac{1}{2\dim(g)} \lim_{\Delta_1,\Delta_2 \to 0} [\Delta_2(\Delta_1+\Delta_2)] \lim_{u \to w} \sum_a \mathcal{O}^a_{\Delta_2,+1}(u,\bar{u}) \tilde{\mathcal{O}}^a_{2-\Delta_1,+1}(w,\bar{w})$$

Fan, Fotopoulos, St.St., Taylor (2020)

(ii) supertranslation operator P:

soft-graviton $\Delta \to 1$

$$P(z,\bar{z}) := \partial_{\bar{z}} \mathcal{O}_{\Delta=1,J=+2}(z,\bar{z})$$

$$(h,\bar{h}) = (\tfrac{3}{2},\tfrac{1}{2})$$

then:

$$\left\langle P(z_0) \prod_{j=1}^{n} \mathcal{O}_{\Delta_j,l_j}(z_j,\bar{z}_j) \right\rangle = \frac{1}{4} \sum_{i=1}^{n} \frac{c_i(\Delta_i)}{c_i(\Delta_i+1)} \frac{1}{z_0 - z_i} \left\langle \prod_{n=1}^{n} \mathcal{O}_{\Delta_j,l_j}(z_j,\bar{z}_j) \right\rangle \Bigg|_{\Delta_i \to \Delta_i + 1}$$

OPE:

$$T(w)P(z) = \frac{3}{2(w-z)^2} P(z) + \frac{1}{w-z} \partial_z P(z) + reg\,.$$

In addition to Virasoro symmetry, we construct all supertranslation generators acting on primary fields

Fotopoulos, St.St., Taylor, Zhu (2019)

$P_{-1/2,-1/2} = e^{(\partial_h + \partial_{\bar{h}})/2} = P^0 + P^3$

use $P_{-1/2,-1/2}$ as starting point for constructing all supertranslation generators

$$P_{n-\frac{1}{2},-\frac{1}{2}} = \frac{1}{i\pi(n+1)} \oint dw\, w^{n+1} \left[T(w), P_{-\frac{1}{2},-\frac{1}{2}}\right]$$

$$P_{n-\frac{1}{2},m-\frac{1}{2}} = \frac{1}{i\pi(m+1)} \oint d\bar{w}\, \bar{w}^{m+1} \left[\bar{T}(\bar{w}), P_{n-\frac{1}{2},-\frac{1}{2}}\right]$$

$\longrightarrow \quad P_{k,l}, \bar{P}_{k,l} \quad \longrightarrow \quad \mathscr{P}(z,\bar{z}) := \sum_{n,m\in\mathbb{Z}} P_{n-\frac{1}{2},m-\frac{1}{2}}\, z^{-n-1} \bar{z}^{-m-1}$

later: explicit field realisation in terms of a supercurrent

$\Longrightarrow$ local (or extended) BMS algebra:

$m, n \in \mathbf{Z},\ i, j, k, l \in \mathbf{Z} + \frac{1}{2}$

$$[P_{ij}, P_{k,l}] = 0\ ,$$

$$[L_n, P_{k,l}] = \left(\frac{1}{2}n - k\right) P_{n+k,l} + n(n^2-1)\ C_{n,k}\ ,$$

$$[\bar{L}_n, P_{k,l}] = \left(\frac{1}{2}n - l\right) P_{k,n+l} + n(n^2-1)\ \bar{C}_{n,l}\ .$$

Barnich (2017)

Conformal soft-theorems $\longleftrightarrow$ Ward identities $\longleftrightarrow$ BMS algebra

BMS group

BMS$^{\pm}$ group

= symmetry of asymptotically flat D=4 space-time at null infinity $\mathscr{I}^{\pm}$

- transforms one asymptotically flat solution to a new physically inequivalent one

- large diffeomorphisms, which take one asymptotically flat solution into another (not isometries of flat space)

BMS group = Lorentz group + supertranslations

originally proposed in order to investigate the flow of energy at infinity due to propagating gravitational waves

Bondi, Burg, Metzner (1962)
Sachs (1962)

Extended BMS group on celestial sphere

global BMS symmetry on celestial sphere	local BMS symmetry on celestial sphere
<u>Lorentz group:</u> global conformal transformations on celestial sphere SL(2,C) $z \to \frac{az+b}{cz+d}$ $L_{-1} = \partial$, $L_0 = z\partial + h$, $L_1 = z^2\partial + 2hz$	local conformal transformations = superrotations T(z) $[L_m, L_n] = (m-n)\, L_{m+n}$ $[\bar{L}_m, \bar{L}_n] = (m-n)\, \bar{L}_{m+n}$
<u>global space-time translation:</u> Abelian subgroup of supertranslations $P_{-1/2,-1/2} = e^{(\partial_h + \partial_{\bar{h}})/2}$ $P_{1/2,1/2} = z\ e^{(\partial_h + \partial_{\bar{h}})/2}$ $P_{-1/2,1/2} = \bar{z}\ e^{(\partial_h + \partial_{\bar{h}})/2}$ $P_{-1/2,1/2} = \|z\|^2\ e^{(\partial_h + \partial_{\bar{h}})/2}$	local space-time translations =supertranslations P(z) $P_{n-\frac{1}{2},m-\frac{1}{2}}$ $n, m \in \mathbf{Z}$

→ Symmetries of the celestial OPEs and correlators
S-matrix (non-trivial consistency)

Supermultiplets

construct D=2 superfields:
bosonic & fermionic conformal primary wave functions

scalar:
$J = 0$

$$\varphi^{\pm}_{\Delta}(X^\mu, z, \bar{z}) = \int_0^\infty d\omega\, \omega^{\Delta-1} e^{\pm i\omega q\cdot X - \epsilon\omega} = \frac{(\mp i)^\Delta\, \Gamma(\Delta)}{(-q\cdot X \mp i\epsilon)^\Delta}$$

$$(h, \bar{h}) = (\frac{\Delta}{2}, \frac{\Delta}{2})$$

fermion:
$J = -\frac{1}{2}$

$$\psi^{\pm}_{\Delta,\alpha}(X, z, \bar{z}) = |q\rangle_\alpha \int_0^\infty d\omega\ \omega^{\Delta+\frac{1}{2}-1} e^{\pm i\omega q\cdot X - \epsilon\omega}$$
$$= |q\rangle_\alpha\ \varphi^{\pm}_{\Delta+\frac{1}{2}}(X, z, \bar{z})$$

$$(h, \bar{h}) = (\frac{\Delta}{2} - \frac{1}{4}, \frac{\Delta}{2} + \frac{1}{4})$$

solves Weyl equation
$$\bar{\sigma}^\mu \partial_\mu \psi_\Delta = 0$$

<u>proceed as before</u>: extract OPEs from collinear singularities of scattering amplitudes

$$\mathcal{O}_{\Delta_1,-\frac{3}{2}}(z_1, \bar{z}_1)\mathcal{O}_{\Delta_2,+\frac{3}{2}}(z_2, \bar{z}_2) =$$

$$= \frac{z_{12}}{\bar{z}_{12}} B\left(\Delta_1 - \frac{1}{2}, \Delta_2 + \frac{5}{2}\right) \mathcal{O}_{\Delta_1+\Delta_2,+2}(z_2\bar{z}_2) + \frac{\bar{z}_{12}}{z_{12}} B\left(\Delta_1 + \frac{5}{2}, \Delta_2 - \frac{1}{2}\right) \mathcal{O}_{\Delta_1+\Delta_2,-2}(z_2\bar{z}_2) + \text{reg}\,.$$

Soft fermionic theorems

Dumitrescu, He Mitra, Strominger (2015)
Lysov (2015); Avery, Schwab (2015)

on CS^2: $\omega_s \to 0$ residua of Mellin amplitude

$$\mathcal{M}_{n+1} \longrightarrow \Big(\underbrace{\omega_s^{-1/2}\, S_F^{(0)}}_{\Delta \to 1/2} + \underbrace{\omega_s^{1/2}\, S_F^{(1)}}_{\Delta \to -1/2} + \dots \Big)\, \mathcal{M}_n$$

E.g.: Gravitino amplitude

$$\mathcal{M}_{n+1}(p_s \ell_s + \tfrac{3}{2}, p_1 \ell_1, \cdots, p_n \ell_n) =$$
$$= \omega_s^{-1/2} \sum_{i=1}^{n} (-1)^{\sigma_i} \frac{\bar{z}_{si}}{z_{si}} \frac{z_{ri}}{z_{rs}}\, \omega_i^{1/2}\, \mathcal{M}_n(p_1 \ell_1, \cdots, p_i \ell_i - \tfrac{1}{2}, \cdots, p_n, \ell_n) + \mathcal{O}(\omega_s^{1/2})$$

Soft theorems $\Delta \to \frac{1}{2}$ ⟶ Ward identities

intricate pattern of supersymmetric Ward identities that relates fermionic and bosonic soft theorems, at both leading and sub-leading levels $\Delta = 0, \frac{1}{2}, 1$

CCFT Supersymmetry Currents

Recall: shadow transformation $T(z) := \tilde{\mathcal{O}}_{\Delta=2,J=+2}(z,\bar{z}) = \frac{3}{\pi}\int d^2w\ (z-w)^{-4}\ \mathcal{O}_{\Delta=0,J=-2}(w,\bar{w})$

$$S(z) = \lim_{\Delta\to\frac{1}{2}} \frac{\Delta-\frac{1}{2}}{\pi}\int d^2z'\ \frac{\mathcal{O}_{\Delta,-\frac{3}{2}}(z',\bar{z}')}{(z-z')^3} \qquad (h,\bar{h}) = (\frac{3}{2},0)$$

$$\bar{S}(\bar{z}) = \lim_{\Delta\to\frac{1}{2}} \frac{\Delta-\frac{1}{2}}{\pi}\int d^2z'\ \frac{\mathcal{O}_{\Delta,+\frac{3}{2}}(z',\bar{z}')}{(\bar{z}-\bar{z}')^3} \qquad (h,\bar{h}) = (0,\frac{3}{2})$$

apply on leading soft gravitino theorem (cf. before)

$$S(z)\mathcal{O}_{\Delta,J-\frac{1}{2}}(w,\bar{w}) = \frac{1}{z-w}\mathcal{O}_{\Delta+\frac{1}{2},J}(w,\bar{w}) \ + \dots$$

$$\bar{S}(\bar{z})\mathcal{O}_{\Delta,J}(w,\bar{w}) = \frac{1}{\bar{z}-\bar{w}}\mathcal{O}_{\Delta+\frac{1}{2},J-\frac{1}{2}}(w,\bar{w}) \ + \dots$$

OPE:

$$T(z)S(w) = \frac{3}{2}\frac{S(w)}{(z-w)^2} + \frac{\partial S(w)}{z-w} \ + \dots$$

$$\bar{T}(\bar{z})\bar{S}(\bar{w}) = \frac{3}{2}\frac{\bar{S}(\bar{z})}{(\bar{z}-\bar{w})^2} + \frac{\bar{\partial}\bar{S}(\bar{w})}{\bar{z}-\bar{w}} \ + \dots$$

Extended Super BMS Algebra

Laurent expansion of fields:

$$S(z) = \sum_{n\in\mathbb{Z}+\frac{1}{2}} \frac{G_n}{z^{n+\frac{3}{2}}}, \qquad G_n = \oint dz\, z^{n+1/2}\, S(z)$$

$$\bar{S}(\bar{z}) = \sum_{n\in\mathbb{Z}+\frac{1}{2}} \frac{\bar{G}_n}{\bar{z}^{n+\frac{3}{2}}}, \qquad \bar{G}_n = \oint d\bar{z}\, \bar{z}^{n+1/2}\, \bar{S}(\bar{z})$$

$G_{\pm\frac{1}{2}}, \bar{G}_{\pm\frac{1}{2}}$ related to SUSY generators in D=4

Recall: Supertranslation operator: $\mathscr{P}(z,\bar{z}) \equiv \sum_{n,m\in\mathbf{Z}} P_{n-\frac{1}{2},m-\frac{1}{2}}\, z^{-n-1}\bar{z}^{-m-1} = \underbrace{S(z)\bar{S}(\bar{z}) + \bar{S}(\bar{z})S(z)}_{\Delta\to\frac{1}{2}}$

proceed as for the bosonic case extract the Super BMS algebra

$$\{G_m, \bar{G}_n\} = P_{m,n}$$
$$\{G_m, G_n\} = \{\bar{G}_m, \bar{G}_n\} = 0$$
$$[P_{k,l}, G_n] = [P_{k,l}, \bar{G}_m] = 0$$
$$[L_m, G_k] = \left(\frac{1}{2}m - k\right) G_{m+k} \qquad m,n\in\mathbf{Z},\ i,j,k,l\in\mathbf{Z}+\tfrac{1}{2}$$
$$[\bar{L}_m, \bar{G}_l] = \left(\frac{1}{2}m - l\right) \bar{G}_{m+l}$$
$$[L_m, \bar{G}_n] = [\bar{L}_m, G_n] = 0$$

superrotations L_n ($\Delta\to 0$)

supertranslations $P_{k,l}$ ($\Delta\to 1$)

supersymmetry G_k ($\Delta\to\frac{1}{2}$)

D=4 SUSY/SUGRA chiral and gauge multiplets → supersymmetric generalization of BMS symmetry

N=1 supersymmetric extension of BMS algebra on CS^2

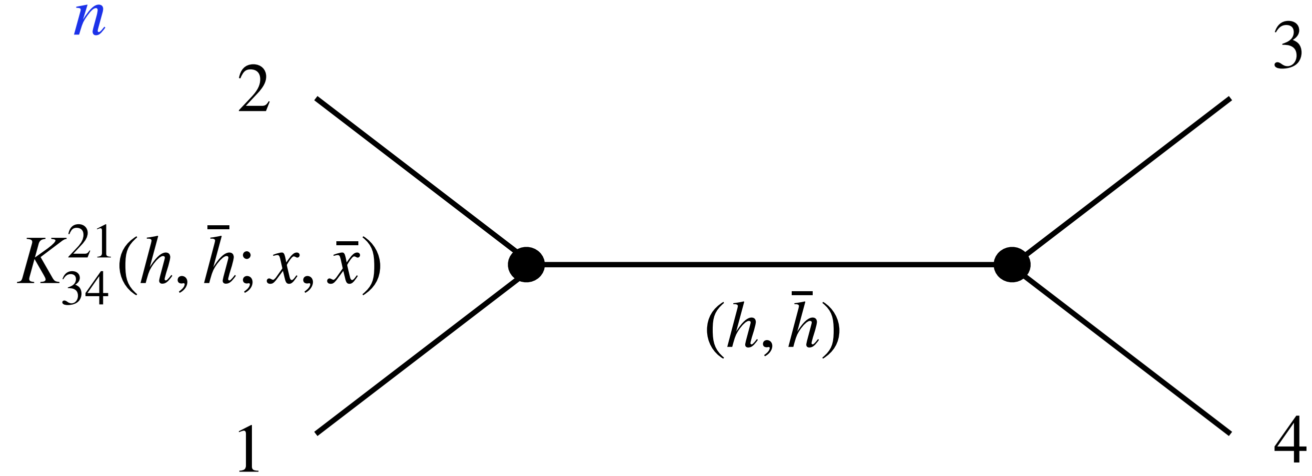
Conformal Blocks

standard CFT: conformal block decomposition of correlation functions
→ comprises full spectrum

$$G^{21}_{34}(x,\bar{x}) = \lim_{z_{1'},\bar{z}_{1'}\to\infty} z_{1'}^{2h_{1'}}\bar{z}_{1'}^{2\bar{h}_{1'}}\left\langle \tilde{\phi}^{a_1}_{\tilde{\Delta}_1,+}(z_{1'},\bar{z}_{1'})\phi^{a_2}_{\Delta_2,-}(1,1)\phi^{a_3}_{\Delta_3,+}(z'=x,\bar{z}'=\bar{x})\phi^{a_4}_{\Delta_4,+}(0,0)\right\rangle$$

$$\stackrel{!}{=} \sum_n C^n_{34}C^n_{12}\; A^{21}_{34}(n;x,\bar{x}) = \sum_n C^n_{34}C^n_{12}\; \mathcal{F}^{21}_{34}(n;x)\times\bar{\mathcal{F}}^{21}_{34}(n;\bar{x})$$

$$\equiv \sum_{h,\bar{h}} a_{h,\bar{h}}\; K^{21}_{34}(h,\bar{h};x,\bar{x})$$



Di Francesco, Mathieu, Senechal (1997)
Osborn (2012)

conformal block:

$$K^{21}_{34}(h,\bar{h};x,\bar{x}) = x^{h-h_3-h_4}\,{}_2F_1\left[\begin{matrix} h-h_{12},\, h+h_{34} \\ 2h \end{matrix};x\right] \times \bar{x}^{\bar{h}-\bar{h}_3-\bar{h}_4}\,{}_2F_1\left[\begin{matrix} \bar{h}-\bar{h}_{12},\, \bar{h}+\bar{h}_{34} \\ 2\bar{h} \end{matrix};\bar{x}\right]$$

we have: $G^{21}_{34}(x,\bar{x})_s \;\sim\; f^{a_1a_2b}f^{a_3a_4b}\; I_s + f^{a_1a_3b}f^{a_2a_4b}\;\tilde{I}_s$

$$I_s(x,\bar{x}) = B\left(1+i\lambda_2+i\lambda_4, i\lambda_2+i\lambda_3\right)\; F_1\left[\begin{matrix}1+i\lambda_2+i\lambda_4; 2-i\lambda_1, -i\lambda_1\\ 1-i\lambda_1+i\lambda_2\end{matrix}; x,\bar{x}\right]$$

$$\tilde{I}_s(x,\bar{x}) = B\left(i\lambda_2+i\lambda_4, i\lambda_2+i\lambda_3\right)\; F_1\left[\begin{matrix}i\lambda_2+i\lambda_4; 2-i\lambda_1, -i\lambda_1\\ -i\lambda_1+i\lambda_2\end{matrix}; x,\bar{x}\right]$$

$$G^{21}_{34}(x,\bar{x})_s = \sum_{m,n=0}^{\infty}\left(a_{mn}\,f^{a_1a_2b}f^{a_3a_4b} + \tilde{a}_{mn}\,f^{a_1a_3b}f^{a_2a_4b}\right)$$

$$\times K^{21}_{34}\Big(\underbrace{m+1+\frac{i\lambda_2}{2}-\frac{i\lambda_1}{2}}_{h},\underbrace{n+1+\frac{i\lambda_2}{2}-\frac{i\lambda_1}{2}}_{\bar{h}}\Big)$$

$\Delta = 2+M+i(\lambda_2-\lambda_1)$ $\qquad M \geq 0$

$J = -M, -M+2,\ldots, M-2, M$

Fan, Fotopoulos, St.St., Taylor, Zhu (2021)

infinite tower of primary fields $\Leftrightarrow$ infinite number of symmetries

cf. also Guevara, Himwich, Pate, Strominger (2021)

Further Directions

- understand infinite number of higher spin states and possible further extensions of symmetry algebra

  - group representation ?

  - yield further symmetry constraints on amplitudes !

- understand symmetries at quantum level

  - perhaps protected by non-renormalization theorems ?

  - Virasoro central charge (-one-loop regulator ?)

- high-energy (large λ) limit: string world-sheet = celestial sphere

  celestial $CFT_2 \simeq$ string (free) world-sheet CFT_2

- understanding the nature of 2D CFT on celestial sphere would enable a holographic description of flat spacetime

- uplift AdS_3/CFT_2 holography to $\mathscr{M}_4$

  towards flat space-time holography