Super-BMS algebras and the asymptotic structure of supergravity at spatial infinity

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Introduction

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2. The existence of a smooth null infinity, given reasonable initial data, is a non-trivial dynamical question. Obstructions to the smoothness of null infinity have indeed been identified (Friedrich, Valiente-Kroon). Does the BMS symmetry exhibited at null infinity exist independently of the existence of a smooth null infinity ?

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We describe here the extension to supergravity.

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- We insist throughout that the action be finite (on-shell and off-shell) for all allowed phase space configurations. In particular, there is a well-defined (finite) symplectic structure.
- Symmetries are phase space transformations that leave the action invariant and for which there is therefore a well-defined moment map.

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Indeed, $i\bar{\epsilon}_1\gamma^\mu\epsilon_2$ goes at infinity to an arbitrary "angle-dependent" translation, which involves four functions of the angles instead of one.

This means that the supersymmetry transformations should be restricted, and perhaps redefined, asymptotically.

Finite-dimensional supersymmetric extension

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This implies that the supersymmetry parameter ϵ should go to a constant at infinity

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An equivalent analysis can be performed at spatial infinity.

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and to make the supersymmetry transformations with non-vanishing supersymmetry parameters at infinity well-defined canonical transformations.

There is a well-defined moment map, and well-defined supersymmetry charges.

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These fermionic symmetries transform as spinors under Lorentz transformations.

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$$[\mathcal{Q}, \mathcal{Q}] \sim \gamma^0 \gamma^\mu P_\mu.$$

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From the algebraic point of view (independently of any dynamical realization), could the superalgebra be different ?

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The theory of representations of the Lorentz algebra sheds indeed an instructive light on this question.

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From the algebraic point of view (independently of any dynamical realization), could the superalgebra be different ?

The theory of representations of the Lorentz algebra sheds indeed an instructive light on this question.

We shall first review general features of irreducible, infinite-dimensional (but not necessary unitary) representations of the Lorentz group.

Lorentz generators

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The Lorentz algebra $so(3,1)$ reads

$$[A_1, A_2] = A_3, \quad [A_2, A_3] = A_1, \quad [A_3, A_1] = A_2 \quad \Leftrightarrow \quad [A_i, A_j] = \epsilon_{ijk} A_k,$$
$$[A_i, B_j] = \epsilon_{ijk} B_k,$$
$$[B_i, B_j] = -\epsilon_{ijk} A_k.$$

The A_k's are the generators of spatial rotations, while the B_k's are the generators of boosts. They transform as vectors under spatial rotations. It is useful to define

$$H_3 = iA_3, \qquad H_+ = iA_1 - A_2, \qquad H_- = iA_1 + A_2$$

and similarly

$$F_3 = iB_3, \qquad F_+ = iB_1 - B_2, \qquad F_- = iB_1 + B_2\,.$$

Irreducible representations

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General results on irreducible (infinite-dimensional, not necessarily unitary) representations of $so(3,1)$ by Gel'fand, Naimark, Harish-Chandra.

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Since $so(3)$ is a (compact) subalgebra of the Lorentz algebra $so(3,1)$, any representation R of $so(3,1)$ decomposes as a direct sum of representations R_l of $so(3)$,

$$R = \oplus R_l$$

where l (the weight/$so(3)$-spin of R_l) is an non-negative integer or half-integer. The sum can be infinite.

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If the representation R of $so(3,1)$ is irreducible, each representation R_l of $so(3)$ that appears in R occurs at most once, i.e., is non degenerate.

Irreducible representations

Irreducible representations

Let $l_0 \geq 0$ be the lowest $so(3)$-spin that appears in the decomposition of R. Then the $so(3)$-spins are $l = l_0, l_0 + 1, \cdots$, with a maximum value $l_0 + n$ (n integer) if the representation R is finite-dimensional, and no upper bound if R is infinite-dimensional.

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In a standard basis ξ_{lm} of R_l, one has

$$H_3 \xi_{lm} = m \xi_{lm},$$
$$H_- \xi_{lm} = \sqrt{(l+m)(l-m+1)}\, \xi_{l,m-1},$$
$$H_+ \xi_{lm} = \sqrt{(l+m+1)(l-m)}\, \xi_{l,m+1},$$

($m = -l, -l+1, \cdots, l-1, l$). These relations are unaffected if we rescale all ξ_{lm}'s by the same m-independent factor $h(l)$, $\xi_{lm} \to h(l) \xi_{lm}$.

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The B_i's are vector operators, so that by the Wigner-Eckart theorem, they can only change the $so(3)$-spin l by ± 1, i.e. $B_i \xi_{lm}$ is a linear combination of vectors $\xi_{l'm'}$ with $l' = l-1, l, l+1$.

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One finds explicitly, upon suitable normalization of the ξ_{lm}'s,

$$F_3 \xi_{lm} = c_l \sqrt{l^2 - m^2}\, \xi_{l-1,m} - a_l m \xi_{l,m} - c_{l+1} \sqrt{(l+1)^2 - m^2}\, \xi_{l+1,m}$$

$$\begin{aligned} F_+ \xi_{lm} = {} & c_l \sqrt{(l-m)(l-m-1)}\, \xi_{l-1,m+1} - a_l \sqrt{(l-m)(l+m+1)}\, \xi_{l,m+1} \\ & + c_{l+1} \sqrt{(l+m+1)(l+m+2)}\, \xi_{l+1,m+1} \end{aligned}$$

$$\begin{aligned} F_- \xi_{lm} = {} & -c_l \sqrt{(l+m)(l+m-1)}\, \xi_{l-1,m-1} - a_l \sqrt{(l+m)(l-m+1)}\, \xi_{l,m-1} \\ & - c_{l+1} \sqrt{(l-m+1)(l-m+2)}\, \xi_{l+1,m-1} \end{aligned}$$

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where

$$a_l = \frac{i l_0 l_1}{l(l+1)}, \qquad c_l = \frac{i}{l}\sqrt{\frac{(l^2 - l_0^2)(l^2 - l_1^2)}{4l^2 - 1}}$$

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Thus, any representation of the Lorentz group is determined by a pair of numbers (l_0, l_1) where l_0 (the lowest $so(3)$-spin) is a non-negative integer or half-integer, and where l_1 is an arbitrary complex number.

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Thus, any representation of the Lorentz group is determined by a pair of numbers (l_0, l_1) where l_0 (the lowest $so(3)$-spin) is a non-negative integer or half-integer, and where l_1 is an arbitrary complex number.

It is important to realize that l_0 and l_1 enter symmetrically the formulas giving the coefficients a_l and c_l. However, only l_0 is required to be a non-negative integer or half-integer.

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The representation is finite-dimensional if this series stops. Assume that it stops at $l_{max} = l_0 + n$ for some integer n. This will occur if and only if $c_{l_{max}+1} = c_{l_0+n+1} = 0$, which will be the case if and only if $l_0 + n + 1 = |l_1|$. Thus, a representation is finite-dimensional if and only if $|l_1| - l_0 - 1$ is a non-negative integer. $|l_1| - 1$ is then the highest $so(3)$-spin occuring in the representation.

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Only the finite-dimensional trivial (scalar) representation is unitary ($l_0 = 0$, $l_1 = \pm 1$)

BMS$_4$ supertranslations

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Here, Y^A is a rotation Killing vector, $B^A \equiv -\partial^A b$ is the conformal Killing vector on the 2-sphere associated with the boost $b_i(x^i \frac{\partial}{\partial x^0} + x^0 \frac{\partial}{\partial x^i})$ and $b = b^i \frac{x^i}{r}$.

BMS$_4$ supertranslations

The first example of an infinite-dimensional representation of the Lorentz group arising in the asymptotic context is given by the BMS$_4$ supertranslations.

The supertranslations $\tau(x^A)$ are described by functions on the 2-sphere ($(x^A) \equiv (\theta, \varphi)$) that transform under the Lorentz group as

$$X_{Y,b}\tau = Y^A \partial_A \tau - \partial^A b \partial_A \tau - b\tau$$

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It is a multiplier representation of the type

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with $k = 1$.

BMS$_4$ supertranslations

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It is called the "tail" of the finite-dimensional representation with $l_0 = 1$ and $l_1 = 2$. (Incidentally, it is unitary and belongs to the principal series.)

The full representation is, however, not completely reducible (it is indecomposable) because the subspace of spherical harmonics with $\ell \geq 2$ is not invariant.

BMS_4 supertranslations and linearized theory

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(All symmetries are improper gauge symmetries (non trivial diffeomorphisms) in the full theory.)

Spinor representation

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The parity of χ is preserved by the Lorentz transformations. The representation that contains the constant spinors is described by even χ's and we take therefore χ to be even.

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Both are improper gauge symmetries in the full theory and combine to provide the infinite-dimensional spinor representation.

Closure of fermionic symmetries

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Symbolically,

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There is no “square root” of the pure supertranslations.

Could there be a deformation of this superalgebra such that pure BMS supertranslations are generated through the bracket of two fermionic symmetries ?

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Now, the decomposition of the tensor product $\text{Head}^F \otimes (\text{Head}^F \oplus_\sigma \text{Tail}^F)$ can be performed, and contains exactly once the reducible, indecomposable vector representation (in addition to another infinite-dimensional representation).

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So, it would be algebraically consistent to have

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The decomposition of $\mathrm{Tail}^F \otimes \mathrm{Tail}^F$ is harder to compute.

Conclusions and comments

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The second yields a much bigger graded extension $SBMS_4$ of the BMS_4 algebra, with an infinite number of fermionic generators.

Both contain the BMS_4 algebra and the super-Poincaré algebra, with $[\text{Head}^F, \text{Head}^F] = \text{Head}^B$, but in all cases, these are the only non-trivial brackets of two fermionic generators : the pure supertranslations have no square root.

Conclusions and comments

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Since there is no algebraic obstruction to taking the square root of pure BMS supertranslations, it would be interesting to explore whether the corresponding deformed superalgebra could be dynamically realized by deforming the asymptotic conditions and the asymptotic symmetries.

Same techniques on the sphere at spatial infinity as on the celestial sphere(s).

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THANK YOU !