MHV Gluon Amplitudes from Celestial Current Algebras

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Sudip Ghosh

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Okinawa Institute of Science and Technology, Japan

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Introduction and Motivation

- Soft theorems in gauge and gravitational theories are equivalent to Ward identities for infinite dimensional asymptotic symmetries of asymptotically flat spacetimes [H.Bondi, M.G. van der Burg, A.W.Metzner; R.K.Sachs; G.Barnich, C.Troessaert; A.Strominger; T.He, V. Lysov, P. Mitra, A.Strominger; T.He, P. Mitra, A.P.Porfyriadis, A. Strominger; D.Kapec, V.Lysov, A.Strominger; M.Campiglia, A.Laddha; D. Kapec, P.Mitra, A.Raclariu, A.Strominger; $\cdots$]
- Such infinite dimensional symmetries impose plethora of constraints on S-matrix.
- Understanding the implications of these symmetries is important for flat space holography.
- For this it is useful to study scattering amplitudes using a conformal basis of asymptotic states, where the S-matrix takes the form of a correlator in a CFT defined on the celestial sphere at null infinity. [S.Pasterski, S.H.Shao, A.Strominger, 2016; S.Pasterski, S.H.Shao, 2017]

Conformal Basis of Asymptotic States

- Asymptotic states for massless particles in conformal basis are Mellin transform of Fock space creation (annihilation) operators. [S.Pasterski, S.H.Shao, 2017]

$$\mathcal{O}^{\epsilon}_{h,\bar{h}}(z,\bar{z}) = \int_0^{\infty} d\omega\ \omega^{\Delta-1} A(\epsilon\omega, z, \bar{z}, \sigma)$$

where $\epsilon = \pm 1$ for an outgoing (incoming) annihilation (creation) operator $A(\epsilon\omega, z, \bar{z}, \sigma)$, σ is the helicity and on-shell null momenta have been parametrised as

$$p^{\mu} = \epsilon\omega(1 + z\bar{z}, z + \bar{z}, -i(z - \bar{z}), 1 - z\bar{z})$$

$\rightarrow (z, \bar{z})$: coordinates on celestial sphere.

- Under Lorentz group which acts on celestial sphere as $SL(2,\mathbb{C})$, $\mathcal{O}^{\epsilon}_{h,\bar{h}}(z,\bar{z})$ transforms as a primary operator with weights $(h, \bar{h}) = \left(\frac{\Delta+\sigma}{2}, \frac{\Delta-\sigma}{2}\right)$.

- Δ can take complex values. [L.Donnay, S.Pasterski, A.Puhm, 2020]

Celestial Amplitude

- **Celestial amplitude** is the Mellin transform of the S-matrix

$$\left\langle \prod_{i=1}^{n} \mathcal{O}^{\epsilon_i}_{h_i,\bar{h}_i}(z_i,\bar{z}_i) \right\rangle = \prod_{i=1}^{n} \int_0^\infty d\omega_i \ \omega_i^{\Delta_i - 1} \underbrace{\mathcal{A}_n(\{\epsilon_i\omega_i, z_i, \bar{z}_i, \sigma_i\})}_{\text{S-matrix}}$$

- Under $SL(2,\mathbb{C})$ this transforms as n-point correlator of primary operators in 2-d Celestial CFT (CCFT).

$$\left\langle \prod_{i=1}^{n} \mathcal{O}'^{\epsilon_i}_{h_i,\bar{h}_i}(z_i,\bar{z}_i) \right\rangle$$
$$= \prod_{i=1}^{n} (cz_i + d)^{-2h_i} (\bar{c}\bar{z}_i + \bar{d})^{-2\bar{h}_i} \left\langle \prod_{i=1}^{n} \mathcal{O}^{\epsilon_i}_{h_i,\bar{h}_i}\left(\frac{az_i + b}{cz_i + d}, \frac{\bar{a}\bar{z}_i + \bar{b}}{\bar{c}\bar{z}_i + \bar{d}}\right) \right\rangle$$

where $ad - bc = \bar{a}\bar{d} - \bar{b}\bar{c} = 1$

- In the rest of this talk we will focus on tree-level scattering amplitudes of gluons in Yang-Mills theory.

- Gluon conformal primaries on the celestial sphere will be denoted as

$$\mathcal{O}^{a}_{h,\bar{h}}(z,\bar{z})$$

  where a: Lie-group index.

Main Result:

- A system of $(n-2)$ coupled first order linear PDEs for Mellin transformed n-point tree-level MHV gluon amplitudes.
- These PDEs follow from consideration of the celestial OPE between gluon primaries and infinite-dimensional symmetry algebras arising from the leading and subleading soft gluon theorems.
- These PDEs bear close resemblance to Knizhnik-Zamolodchikov (KZ) equations.

Outline

- Symmetries.
- MHV sector of Yang-Mills theory.
- Celestial gluon OPE in MHV sector.
- Null states and differential equations for MHV gluon amplitudes.
- Summary & Future Directions.

Symmetries

Poincare Algebra

Global symmetries of CCFT include the Poincare group.

- Lorentz/ global conformal generators :

$$L_n, \bar{L}_n, \quad n = 0, \pm 1.$$

  These satisfy the algebra

$$[L_m, L_n] = (m-n)L_{m+n}, \quad [\bar{L}_m, \bar{L}_n] = (m-n)\bar{L}_{m+n}$$

- Global spacetime translation generators: $P^\mu = (P^0, P^1, P^2, P^3)$. It's convenient for us to define

$$P_{-1,-1} = P_0 + P_3, \quad P_{-1,0} = P_1 - iP_2, \quad P_{0,-1} = P_1 + iP_2, \quad P_{0,0} = P_0 - P_3$$

  Their algebra is

$$[P_{m,n}, P_{m',n'}] = 0$$

- Commutators between Lorentz and spacetime-translation generators

$$[L_n, P_{m',n'}] = \left(\frac{n-1}{2} - m'\right)P_{m'+n,n'}, \; [\bar{L}_n, P_{m',n'}] = \left(\frac{n-1}{2} - n'\right)P_{m',n'+n}$$

Poincare Primary

- Lorentz generators act on a gluon conformal primary as

$$\Big[L_n, \mathcal{O}^a_{h,\bar{h}}(z,\bar{z})\Big] = z^n(z\partial + (n+1)h)\mathcal{O}^a_{h,\bar{h}}(z,\bar{z})$$
$$\Big[\bar{L}_n, \mathcal{O}^a_{h,\bar{h}}(z,\bar{z})\Big] = \bar{z}^n(\bar{z}\partial + (n+1)\bar{h})\mathcal{O}^a_{h,\bar{h}}(z,\bar{z})$$

  where $n = 0, \pm 1$.

- Action of translation generators on gluon conformal primary

$$\Big[P_{m,n}, \mathcal{O}^a_{h,\bar{h}}(z,\bar{z})\Big] = \epsilon z^{m+1}\bar{z}^{n+1}\mathcal{O}^a_{h+\frac{1}{2},\bar{h}+\frac{1}{2}}(z,\bar{z}), \quad m,n = 0,\pm 1$$

  where $\epsilon = \pm 1$ for an outgoing (incoming) gluon.

- These yield the primary state conditions under the Poincare group

$$L_1\mathcal{O}^a_{h,\bar{h}}(0) = \bar{L}_1\mathcal{O}^a_{h,\bar{h}}(0) = 0, \quad L_0\mathcal{O}^a_{h,\bar{h}}(0) = h\mathcal{O}^a_{h,\bar{h}}(0), \ \bar{L}_0\mathcal{O}^a_{h,\bar{h}}(0) = \bar{h}\mathcal{O}^a_{h,\bar{h}}(0)$$
$$P_{0,-1}\mathcal{O}^a_{h,\bar{h}}(0) = P_{-1,0}\mathcal{O}^a_{h,\bar{h}}(0) = P_{0,0}\mathcal{O}^a_{h,\bar{h}}(0) = 0$$

Leading Conformal Soft Gluon Theorem

- Consider the leading conformal soft gluon operator with helicity $\sigma = 1$ [L.Donnay, A.Puhm, A.Strominger, 2019; M.Pate, A.M.Raclariu, A.Strominger, 2019; W.Fan, A.Fotopoulos, T.R.Taylor, 2019, T.Adamo, L.Mason, A.Sharma, 2019]

$$j^a(z) = \lim_{\Delta \to 1} (\Delta - 1) \mathcal{O}^a_{\Delta,+}(z, \bar{z})$$

$j^a(z)$ is a $(1, 0)$ primary.

- Correlation function of $j^a(z)$ with other gluon primaries is given by the leading soft gluon theorem which in conformal basis is the Ward identity [T.He, P.Mitra, A. Strominger 2015; W.Fan, A.Fotopoulos, T.R.Taylor, 2019]

$$\left\langle j^a(z) \prod_{i=1}^{n} \mathcal{O}^{a_i}_{h_i, \bar{h}_i}(z_i, \bar{z}_i) \right\rangle = - \sum_{k=1}^{n} \frac{T^a_k}{z - z_k} \left\langle \prod_{i=1}^{n} \mathcal{O}^{a_i}_{h_i, \bar{h}_i}(z_i, \bar{z}_i) \right\rangle$$

where $T^a \to$ Lie-algebra generators.

- Since gluons transform in adjoint representation

$$T^a_k \mathcal{O}^{a_i}_{h_i, \bar{h}_i}(z_i, \bar{z}_i) = i f^{a a_i b} \, \mathcal{O}^b_{h_i, \bar{h}_i}(z_i, \bar{z}_i) \delta_{ik}$$

Leading Soft Gluon Current Algebra

- Consider the modes of $j^a(z)$ on the celestial sphere

$$j^a_m = \oint \frac{dz}{2\pi i}\, z^m\, j^a(z), \quad m \in \mathbb{Z}$$

  These satisfy a level zero Kac-Moody current algebra

$$\left[j^a_m, j^b_n\right] = -i f^{abc} j^c_{m+n}, \quad (m,n) \in \mathbb{Z}$$

- Action of j^a_m on gluon conformal primaries

$$\left[j^a_m, \mathcal{O}^b_{h,\bar{h}}(z,\bar{z})\right] = -i f^{abc} z^n \mathcal{O}^c_{h,\bar{h}}(z,\bar{z})$$

- Commutator with Poincare generators

$$[L_m, j^a_n] = -n\, j^a_{m+n}, \quad [\bar{L}_m, j^a_n] = 0, \quad m = 0, \pm 1$$
$$[P_{m,n}, j^a_n] = 0$$

OPE with leading soft gluon current

- Useful to consider the OPE between j^a and a hard gluon $\mathcal{O}^a_{h,\bar{h}}$. This can be derived from the Kac-Moody Ward identity and is given by

$$j^a(z)\mathcal{O}^b_{h,\bar{h}}(w,\bar{w}) = -\frac{T_1^a}{z-w}\mathcal{O}^b_{h,\bar{h}}(w,\bar{w}) + \sum_{p=1}^{\infty}(z-w)^{p-1}\left(j^a_{-p}\mathcal{O}^b_{h,\bar{h}}\right)(w,\bar{w})$$

where $j^a_{-n}\mathcal{O}^b_{h,\bar{h}}$ for $n \geq 1$ are Kac-Moody descendants.

- Using this OPE we can define Kac-Moody current algebra primary as

$$j^a_n\mathcal{O}^b_{h,\bar{h}}(0) = 0, \quad \forall n \geq 1; \quad j^a_0\mathcal{O}^b_{h,\bar{h}}(0) = -if^{abc}\mathcal{O}^c_{h,\bar{h}}(0)$$

- Correlators involving Kac-Moody descendants are determined by leading soft gluon theorem. For e.g.,

$$\left\langle j^a_{-p}\mathcal{O}^{a_1}_{h_1,\bar{h}_1}(z_1,\bar{z}_1)\prod_{i=2}^{n}\mathcal{O}^{a_i}_{h_i,\bar{h}_i}(z_i,\bar{z}_i)\right\rangle = \mathscr{J}^a_{-p}(1)\left\langle\prod_{i=1}^{n}\mathcal{O}^{a_i}_{h_i,\bar{h}_i}(z_i,\bar{z}_i)\right\rangle$$

$$= \sum_{k=2}^{n}\frac{T_k^a}{z_k - z_1}\left\langle\prod_{i=1}^{n}\mathcal{O}^{a_i}_{h_i,\bar{h}_i}(z_i,\bar{z}_i)\right\rangle$$

Subleading Conformal Soft Gluon Theorem

- Consider the subleading conformal soft gluon operator with helicity $\sigma = 1$ [L.Donnay, A.Puhm, A.Strominger, 2019; M.Pate, A.M.Raclariu, A.Strominger, 2019; W.Fan, A.Fotopoulos, T.R.Taylor, 2019, T.Adamo, L.Mason, A.Sharma, 2019]

$$S_1^{+a}(z,\bar{z}) = \lim_{\Delta \to 0} \Delta \, \mathcal{O}^a_{\Delta,+}(z,\bar{z})$$

  This is a $(1/2, -1/2)$ primary.

- Correlator of $S_1^{+a}(z,\bar{z})$ with other gluon primaries is given by the subleading soft gluon theorem which in conformal basis takes the form [E. Himwich, A.Strominger, 2019; M.Pate, A.M.Raclariu, A.Strominger, E.Y.Yuan, 2019]

$$\left\langle S_1^{+a}(z,\bar{z}) \prod_{i=1}^{n} \mathcal{O}^{a_i}_{h_i,\bar{h}_i}(z_i,\bar{z}_i) \right\rangle$$
$$= \sum_{k=1}^{n} \frac{\epsilon_k}{(z-z_k)} \left[2\bar{h}_k - 1 - (\bar{z}-\bar{z}_k)\partial_{\bar{z}_k}\right] T_k^a P_k^{-1} \left\langle \prod_{i=1}^{n} \mathcal{O}^{a_i}_{h_i,\bar{h}_i}(z_i,\bar{z}_i) \right\rangle$$

  where P_k^{-1} is defined as

$$P_k^{-1} \mathcal{O}^{a_i}_{h_i,\bar{h}_i}(z_i,\bar{z}_i) = \mathcal{O}^{a_i}_{h_i-\frac{1}{2},\bar{h}_i-\frac{1}{2}}(z_i,\bar{z}_i)\delta_{ik}$$

Subleading Soft Gluon Currents

- Treating $z, \bar{z}$ as independent, S_1^{+a} can be expressed as [S.Banerjee, **S.G.**, 2020]

$$S_1^{+a}(z,\bar{z}) = J^a(z) + \bar{z}\, K^a(z)$$

where $J^a(z)$ and $K^a(z)$ are currents with weights $(1/2,-1/2)$ and $(1/2,1/2)$ respectively.

- Ward identity for $J^a(z)$

$$\left\langle J^a(z) \prod_{i=1}^{n} \mathcal{O}^{a_i}_{h_i,\bar{h}_i}(z_i,\bar{z}_i) \right\rangle = \sum_{k=1}^{n} \frac{\epsilon_k}{(z-z_k)} \left[2\bar{h}_k - 1 + \bar{z}_k \partial_{\bar{z}_k}\right] T_k^a P_k^{-1} \left\langle \prod_{i=1}^{n} \mathcal{O}^{a_i}_{h_i,\bar{h}_i}(z_i,\bar{z}_i) \right\rangle$$

- Ward identity for $K^a(z)$

$$\left\langle K^a(z) \prod_{i=1}^{n} \mathcal{O}^{a_i}_{h_i,\bar{h}_i}(z_i,\bar{z}_i) \right\rangle = -\sum_{k=1}^{n} \frac{\epsilon_k}{(z-z_k)} \partial_{\bar{z}_k} T_k^a P_k^{-1} \left\langle \prod_{i=1}^{n} \mathcal{O}^{a_i}_{h_i,\bar{h}_i}(z_i,\bar{z}_i) \right\rangle$$

Subleading Soft Gluon Current Algebra

- Useful to consider modes of $J^a(z)$ and $K^a(z)$

$$J^a_m = \oint \frac{dz}{2\pi i}\ z^m\, J^a(z), \quad K^a_m = \oint \frac{dz}{2\pi i}\ z^m\, K^a(z) \quad m \in \mathbb{Z}$$

- Their action on conformal primaries is

$$\left[J^a_n, \mathcal{O}^b_{h,\bar{h}}(z,\bar{z})\right] = i\epsilon f^{abc} z^n \left(2\bar{h} - 1 + \bar{z}\partial_{\bar{z}}\right) \mathcal{O}^c_{h-\frac{1}{2},\bar{h}-\frac{1}{2}}(z,\bar{z})$$
$$\left[K^a_n, \mathcal{O}^b_{h,\bar{h}}(z,\bar{z})\right] = -i\epsilon f^{abc} z^n \partial_{\bar{z}}\, \mathcal{O}^c_{h-\frac{1}{2},\bar{h}-\frac{1}{2}}(z,\bar{z})$$

- Commutator with Kac-Moody currents

$$[j^a_m, J^a_n] = -i f^{abc} J^a_{m+n}, \quad [j^a_m, K^a_n] = -i f^{abc} K^c_{m+n}$$

- Commutators of J^a_n, K^a_n with Poincare generators can also be easily derived.
- Commutators among (J^a_n, K^a_n) do not close. However when acting on gluon primaries and descendants these commutators have simple expressions. For details see [S.Banerjee, **S.G.**, 2020].

  A closed algebra can be obtained if we consider an infinite tower of conformally soft theorems [A.Guevara, E. Himwich, M.Pate, A. Strominger, 2021].

OPE with subleading soft gluon

- OPE between $S_1^{+a}(z,\bar{z})$ and a hard gluon primary $\mathcal{O}^b_{h,\bar{h}}(w,\bar{w})$

$$S_1^{+a}(z,\bar{z})\mathcal{O}^b_{h,\bar{h}}(w,\bar{w})$$
$$= \frac{\epsilon_1}{z-w}\left(2\bar{h}-1\right) T_1^a P_1^{-1}\mathcal{O}^b_{h,\bar{h}}(w,\bar{w}) + \sum_{p=1}^{\infty}(z-w)^{p-1}\left(J^a_{-p}\mathcal{O}^b_{h,\bar{h}}\right)(w,\bar{w})$$
$$-\left(\bar{z}-\bar{w}\right)\left(\epsilon_1\frac{\partial_{\bar{w}}}{z-w}T_1^a P_1^{-1}\mathcal{O}^b_{h,\bar{h}}(w,\bar{w}) + \sum_{p=1}^{\infty}(z-w)^{p-1}\left(K^a_{-p}\mathcal{O}^b_{h,\bar{h}}\right)(w,\bar{w})\right)$$

where $J^a_{-p}\mathcal{O}^b_{h,\bar{h}}, K^a_{-p}\mathcal{O}^b_{h,\bar{h}}$ for $p \geq 1$ are descendants.

- This yields a nontrivial constraint on the OPE of hard gluons.
- Using this OPE we also get

$$J_0^a\mathcal{O}^b_{h,\bar{h}}(0) = i\epsilon f^{abc}(2\bar{h}-1)\mathcal{O}^c_{h-\frac{1}{2},\bar{h}-\frac{1}{2}}(0), \quad J_n^a\mathcal{O}^a_{h,\bar{h}}(0) = 0, \quad \forall n > 0$$
$$K_0^a\mathcal{O}^b_{h,\bar{h}}(0) = -i\epsilon f^{abc}\bar{\partial}\mathcal{O}^c_{h-\frac{1}{2},\bar{h}-\frac{1}{2}}(0), \quad K_n^a\mathcal{O}^a_{h,\bar{h}}(0) = 0, \quad \forall n > 0$$

Correlators with subleading current descendants

- Correlators involving subleading current algebra descendants

$$\left\langle J^{a}_{-p}\mathcal{O}^{a_1}_{h_1,\bar{h}_1}(z_1,\bar{z}_1)\prod_{i=1}^{n}\mathcal{O}^{a_i}_{h_i,\bar{h}_i}(z_i,\bar{z}_i)\right\rangle = \mathcal{J}^{a}_{-p}(z_1,\bar{z}_1)\left\langle\prod_{i=1}^{n}\mathcal{O}^{a_i}_{h_i,\bar{h}_i}(z_i,\bar{z}_i)\right\rangle$$

$$= -\left(\sum_{j=2}^{n}\epsilon_j\frac{(2\bar{h}_j-1+(\bar{z}_j-\bar{z}_1)\bar{\partial}_j)}{(z_j-z_1)^p}T^{a}_jP^{-1}_j\right)\left\langle\prod_{i=1}^{n}\mathcal{O}^{a_i}_{h_i,\bar{h}_i}(z_i,\bar{z}_i)\right\rangle, \quad p\geq 1$$

and

$$\left\langle K^{a}_{-p}\mathcal{O}^{a_1}_{h_1,\bar{h}_1}(z_1,\bar{z}_1)\prod_{i=1}^{n}\mathcal{O}^{a_i}_{h_i,\bar{h}_i}(z_i,\bar{z}_i)\right\rangle = \mathcal{K}^{a}_{-p}(z_1,\bar{z}_1)\left\langle\prod_{i=1}^{n}\mathcal{O}^{a_i}_{h_i,\bar{h}_i}(z_i,\bar{z}_i)\right\rangle$$

$$= \left(\sum_{j=2}^{n}\epsilon_j\frac{\bar{\partial}_j}{(z_j-z_1)^p}T^{a}_jP^{-1}_j\right)\left\langle\prod_{i=1}^{n}\mathcal{O}^{a_i}_{h_i,\bar{h}_i}(z_i,\bar{z}_i)\right\rangle, \quad p\geq 1$$

MHV sector of Yang-Mills theory

MHV Amplitudes

- MHV amplitudes are of the form

$$\mathcal{A}_n(-,-,+,+,\cdots,+)$$

- At tree level following amplitudes in Yang-Mills theory vanish

$$\mathcal{A}_n(+,+,+,\cdots,+) = \mathcal{A}_n(-,+,+,\cdots,+) = 0$$

- Thus MHV amplitudes are the first set of nontrivial amplitudes at tree level.

- We shall refer to the set of tree-level MHV gluon amplitudes as the MHV sector. For our purposes the most important property of this sector is that it is closed under soft and collinear limits.

MHV Amplitudes: Soft Limits

Consider taking a negative helicity gluon soft within a n-point tree level MHV amplitude. The $(n-1)$-point amplitude obtained from this via soft factorisation vanishes.

$$\mathcal{A}_n(1^-, 2^+, 3^+, \cdots, (n-1)^+, n^-) \xrightarrow{p_n \to 0} \mathcal{S}^- \mathcal{A}_{n-1}(1^-, 2^+, 3^+, \cdots, (n-1)^+) \\ = 0$$

where $\mathcal{S}^-$: soft factor.

$\implies$ No negative helicity soft gluon in MHV sector.

Soft theorems associated to positive helicity soft gluon dictate the symmetry structure of MHV sector.

MHV Amplitudes: Collinear Limits

Collinear Limits:

$$\mathcal{A}_n(1^-,\cdots,i^+,j^+,\cdots,n^-) \xrightarrow{p_i||p_j} \mathsf{Split}^-_{+,+}(p_i,p_j)\mathcal{A}_{n-1}(1^-,\cdots,P^+,\cdots,n^-)$$

$$\mathcal{A}_n(1^-,\cdots,i^+,j^-,\cdots,n^+) \xrightarrow{p_i||p_j} \mathsf{Split}^+_{+,-}(p_i,p_j)\mathcal{A}_{n-1}(1^-,\cdots,P^-,\cdots,n^+)$$

$$\mathcal{A}_n(1^+,\cdots,i^-,j^-,\cdots,n^+) \xrightarrow{p_i||p_j} \mathsf{Split}^+_{-,-}(p_i,p_j)\mathcal{A}_{n-1}(1^+\cdots,P^-,\cdots,n^+)$$
$$= 0$$

where $\mathsf{Split}^{\sigma}_{\sigma_i,\sigma_j}(p_i,p_j)$: Splitting functions; $\quad P^\mu = p_1^\mu + p_2^\mu$

$\implies$ MHV sector is closed under collinear limits.

Since leading collinear limit corresponds to leading OPE in Celestial CFT, MHV sector is also closed under OPEs.

Null states & differential equations for MHV gluon amplitudes

Positive helicity gluon OPE in MHV sector

- OPE between positive helicity outgoing gluon primaries in MHV sector [S.Ebert, A.Sharma, D.Wang, 2020; S.Banerjee, **S.G.**, 2020]

$$
\begin{aligned}
&\mathcal{O}^{a}_{\Delta,+}(z,\bar{z})\mathcal{O}^{a_1}_{\Delta_1,+}(z_1,\bar{z}_1) \\
&= -iB(\Delta-1,\Delta_1-1)\left[\frac{f^{aa_1x}}{z-z_1} + \frac{\Delta-1}{\Delta+\Delta_1-2}\, f^{aa_1x} L_{-1}\right. \\
&\left. + i\left(\frac{\Delta-1}{\Delta+\Delta_1-2}\,\delta^{ax}\delta^{a_1y} + \frac{\Delta_1-1}{\Delta+\Delta_1-2}\,\delta^{ay}\delta^{a_1x}\right) j^{y}_{-1}\right]\mathcal{O}^{x}_{\Delta+\Delta_1-1,+}(z_1,\bar{z}_1) + \cdots
\end{aligned}
$$

where $B(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$ is the Euler Beta function. The dots denote contribution from descendants at higher orders.

Subleading conformal soft limit

- Taking the subleading conformal soft limit $\Delta \to 0$

$$\lim_{\Delta\to 0} \Delta\, \mathcal{O}^{a}_{\Delta,+}(z,\bar{z})\mathcal{O}^{a_1}_{\Delta_1,+}(z_1,\bar{z}_1) = \Bigg[\frac{(\Delta_1-2)}{z-z_1}\, if^{aa_1x} - if^{aa_1x}L_{-1} + (\delta^{a_1y} - (\Delta_1-1)\delta^{ay}\delta^{a_1x})\, j^{y}_{-1}\Bigg] O^{x}_{\Delta_1-1,+}(z_1,\bar{z}_1) + \cdots$$

- But the subleading soft gluon theorem implies

$$\lim_{\Delta\to 0} \Delta\, \mathcal{O}^{a}_{\Delta,+}(z,\bar{z})\mathcal{O}^{a_1}_{\Delta_1,+}(z_1,\bar{z}_1) = \Bigg[\frac{(\Delta_1-2)}{z-z_1}\, if^{aa_1x}P_1^{-1} + \delta^{a_1x}J^{a}_{-1}\Bigg]\mathcal{O}^{x}_{\Delta_1,+}(z_1,\bar{z}_1) + \cdots$$

where $P_1^{-1}\mathcal{O}^{x}_{\Delta_1,+} = \mathcal{O}^{x}_{\Delta_1-1,+}$.

- Consistency of OPE with subleading soft gluon theorem yields

$$\left[\delta^{a_1 x} J^{a}_{-1} + i f^{a a_1 x} L_{-1} P_1^{-1} - \left(\delta^{ax}\delta^{a_1 y} - (\Delta_1 - 1)\delta^{ay}\delta^{a_1 x}\right) j^{y}_{-1} P_1^{-1}\right] \mathcal{O}^{x}_{\Delta_1,+}(z_1, \bar{z}_1)$$
$$= 0$$

- Multiplying the above by $i f^{a a_1 b}$ and using

$$f^{a a_1 b} f^{a a_1 c} = C_A \delta^{bc}$$

where C_A is the quadratic Casimir of the adjoint representation, we can express the above relation between descendants as

$$\left[C_A L_{-1} - \Delta_1\, j^{b}_{-1} j^{b}_{0} - J^{b}_{-1} j^{b}_{0} P_{-1,-1}\right] \mathcal{O}^{a_1}_{\Delta_1 - 1,+}(z_1, \bar{z}_1) = 0$$

where

$$j^{a}_{0} \mathcal{O}^{b}_{h,\bar{h}}(z, \bar{z}) = -i f^{abc} \mathcal{O}^{c}_{h,\bar{h}}(z, \bar{z})$$

and $P_{-1,-1}\mathcal{O}^{a}_{h,\bar{h}} = \mathcal{O}^{a}_{h+1/2,\bar{h}+1/2}$ for an outgoing gluon.

Null State

- The above relation holds for arbitrary values of Δ and Δ_1. Then shifting $\Delta_1 \to \Delta_1 + 1$ yields the null state relation

$$\boxed{\Psi^{a}(z,\bar{z}) = \Big[C_A L_{-1} - (\Delta_1+1) j^{b}_{-1} j^{b}_{0} - J^{b}_{-1} j^{b}_{0} P_{-1,-1}\Big] \mathcal{O}^{a_1}_{\Delta_1,+}(z,\bar{z}) = 0}$$

- Ψ^{a} is also a primary under the Poincare group and the leading soft gluon current algebra since

$$L_1 \Psi^{a} = \bar{L}_1 \Psi^{a} = P_{0,-1} \Psi^{a} = P_{-1,0} \Psi^{a} = 0$$
$$j^{a}_{0} \Psi^{b} = -i f^{abc} \Psi^{c}, \quad j^{a}_{m} \Psi^{b} = 0, \quad \forall m \geq 1$$

These primary state conditions also uniquely determine the null state $\Psi^{a}(z,\bar{z})$.

Mixed helicity gluon OPE

- The above null state relation exists only for positive helicity gluon primaries.
- This can be seen by considering the mixed helicity OPE in the MHV sector [S.Banerjee, **S.G.**, 2020]

$$\begin{aligned}&\mathcal{O}^{a_1}_{\Delta_1,+}(z_1,\bar{z}_1)\mathcal{O}^{a_2}_{\Delta_2,-}(z_2,\bar{z}_2)\\&\sim B(\Delta_1-1,\Delta_2+1)\Big[-\frac{if^{a_1a_2x}}{z_{12}}\\&+\Delta_1\,\delta^{bx}\,j^{a}_{-1}+\frac{(\Delta_1-1)}{(\Delta_1+\Delta_2)}\,\delta^{bx}\,J^{a}_{-1}P_{-1,-1}\Big]\mathcal{O}^{x}_{\Delta_1+\Delta_2-1,-}(z_2,\bar{z}_2)+\cdots\end{aligned}$$

- This OPE is manifestly consistent with the leading and subleading soft gluon theorems.

Differential Equation

Decoupling of $\Psi^{a}(z,\bar{z})$ leads to the following PDE for Mellin transformed MHV gluon amplitude [S.Banerjee, **S.G.**, 2020]

$$\Bigg[\frac{C_A}{2}\frac{\partial}{\partial z_i} - h_i \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{T_i^{a}T_j^{a}}{z_i - z_j}$$

$$+\frac{1}{2}\sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{\epsilon_j\left(2\bar{h}_j - 1 - (\bar{z}_i - \bar{z}_j)\frac{\partial}{\partial \bar{z}_j}\right)}{z_i - z_j} T_j^{a} P_j^{-1} T_i^{a} P_{-1,-1}(i)\Bigg]\left\langle \prod_{k=1}^{n} \mathcal{O}^{a_k}_{h_k,\bar{h}_k}(z_k,\bar{z}_k)\right\rangle_{\text{MHV}}$$

$$=0$$

where $2h_i = \Delta_i + 1, 2\bar{h}_i = \Delta_i - 1$ and

$$P_j^{-1}\mathcal{O}^{a_k}_{h_k,\bar{h}_k}(z_k,\bar{z}_k) = \mathcal{O}^{a_k}_{h_k-\frac{1}{2},\bar{h}_k-\frac{1}{2}}(z_k,\bar{z}_k)\delta_{jk}$$

$$P_{-1,-1}(i)\mathcal{O}^{a_i}_{h_i,\bar{h}_i}(z_i,\bar{z}_i) = \epsilon_i \mathcal{O}^{a_i}_{h_i+\frac{1}{2},\bar{h}_i+\frac{1}{2}}(z_i,\bar{z}_i)$$

- There are $(n-2)$ such PDEs for each of the $(n-2)$ positive helicity gluons in the n-point MHV amplitude.
- These can be used to determine the leading celestial OPE between two gluon primaries. [S.Banerjee, **S.G.**, 2020]

Comments on PDE for MHV gluon amplitudes

- The first two terms of this PDE resemble the Knizhnik-Zamolodchikov (KZ) equation obeyed by current algebra primaries in Wess-Zumino-Witten (WZW) model

$$\left[\left(k + \frac{C_A}{2}\right)\frac{\partial}{\partial z_i} - \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{T_i^a\, T_j^a}{z_i - z_j}\right]\left\langle \phi^{a_1}_{h_1,\bar{h}_1}(z_1,\bar{z}_1)\cdots\phi^{a_n}_{h_n,\bar{h}_n}(z_n,\bar{z}_n)\right\rangle = 0$$

  where $\phi^{a_i}_{h_i,\bar{h}_i}(z_i,\bar{z}_i)$ are primary operators, k is the level of the current algebra.
- In our case $k = 0$. Also there is the prefactor h_i in front of the second term in our PDE.
- In WZW model, scaling dimensions of primaries are fixed in terms of the level of the current algebra and the representation of the zero mode algebra. But our PDEs hold for any $(h, \bar{h})$ of the gluon primary. This is consistent with the fact that in CCFT, scaling dimensions can take arbitrary (complex) values.
- The third term is an additional correction due to the subleading soft gluon theorem. This has no analog in the standard KZ equation.

Differential Equation in Fock Space

- The PDE for the celestial MHV gluon correlator can also be written for the corresponding Fock space S-matrix.

$$\left\langle \prod_{k=1}^{n} A^{a_k}(\epsilon_k \omega_k, z_k, \bar{z}_k, \sigma_k) \right\rangle_{\text{MHV}}$$

where $A^{a_k}(\epsilon_k \omega_k, z_k, \bar{z}_k, \sigma_k)$: Fock space gluon creation (annihilation) operators.

- With the substitution

$$\Delta_i \to -\omega_i \partial_{\omega_i}, \quad P_{-1,-1}(i) \to \epsilon_i \omega_i, \quad P_i^{-1} \to \omega_i^{-1}$$

the PDE for MHV gluon amplitudes in momentum space takes the form [S.Banerjee, **S.G.**, 2020]

$$\Bigg[C_A \frac{\partial}{\partial z_i} + \left(\omega_i \frac{\partial}{\partial \omega_i} - 1 \right) \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{T_i^a T_j^a}{z_i - z_j} + \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{\epsilon_i \omega_i}{\epsilon_j \omega_j} \frac{\left(\sigma_j + \omega_j \frac{\partial}{\partial \omega_j} + (\bar{z}_i - \bar{z}_j) \frac{\partial}{\partial \bar{z}_j} \right)}{z_i - z_j} T_i^a T_j^a \Bigg] \left\langle \prod_{k=1}^{n} A^{a_k}(\epsilon_k \omega_k, z_k, \bar{z}_k, \sigma_k) \right\rangle_{\text{MHV}} = 0$$

Summary and Future Directions

Summary

- Derived system of $(n-2)$ coupled first order linear PDEs for Mellin transform of tree-level n-point MHV gluon amplitudes in pure Yang-Mills theory. These are a consequence of decoupling of null states.

- These equations bear resemblance to KZ equations. However, there is an additional contribution from subleading soft gluon theorem.

- These PDEs hold for MHV gluon amplitudes in momentum space as well.

- Null state relations and associated PDEs also exist for tree-level MHV graviton amplitudes in Einstein gravity. The MHV sector in this case is governed by supertranslations and a $\overline{SL(2,\mathbb{C})}$ current algebra which follow from leading and subleading soft gravitons theorems for positive helicity soft gravitons respectively. [S.Banerjee, **S.G.**, P.Paul, 2020; also see Shamik's talk @ PCTS Celestial Holography 2020 Workshop]

Future Directions

- Mellin transform of color stripped gluon MHV amplitudes are given by Aomoto-Gelfand hypergeometric functions [A.Schreiber, A.Volovich, M.Zlotnikov, 2017]. These are know to satisfy certain differential equations. What is their relation to our PDEs?

- Extension of current analysis to NMHV gluon amplitudes. Symmetry algebras due to both positive and negative helicity soft gluons will be relevant here.

- Implications for Sugawara construction in CCFT. [W.Fan, A.Fotopoulos, S.Stieberger, T.R.Taylor, 2020]

- Will be interesting to study MHV sector in Einstein-Yang-Mills theory. The subleading soft gluon theorem here gets corrected.

- Infinite dimensional asymptotic symmetries impose powerful constraints on the S-matrix in gauge theories and gravity. Can we incorporate these into the S-matrix bootstrap programme?

THANK YOU

Extra Slides

Leading Celestial Gluon OPE

Together with global subleading soft gluon symmetry and global time translations, the PDE for MHV gluon amplitudes can be used to completely fix the leading celestial OPE between gluon primaries

- Outgoing (Incoming) OPEs

$$\mathcal{O}^{a_1,\epsilon}_{\Delta_1,+}(z_1,\bar{z}_1)\mathcal{O}^{a_2,\epsilon}_{\Delta_2,\sigma_2}(z_2,\bar{z}_2) \sim \frac{if^{a_1a_2x}}{z_{12}}\,\epsilon B(\Delta_1-1,\Delta_2-\sigma_2)\mathcal{O}^{x}_{\Delta_1+\Delta_2-1,\sigma_2}(z_2,\bar{z}_2)$$

- Outgoing-Incoming OPEs

$$\begin{aligned}&\mathcal{O}^{a_1,\epsilon}_{\Delta_1,+}(z_1,\bar{z}_1)\mathcal{O}^{a_2,-\epsilon}_{\Delta_2,\sigma_2}(z_2,\bar{z}_2)\\ &\quad\sim \frac{if^{a_1a_2x}}{z_{12}}\Big[-B(\Delta_2-\sigma_2,2+\sigma_2-\Delta_1-\Delta_2)\mathcal{O}^{x,\epsilon}_{\Delta_1+\Delta_2-1,\sigma_2}(z_2,\bar{z}_2)\\ &\quad+B(\Delta_1-1,2+\sigma_2-\Delta_1-\Delta_2)\mathcal{O}^{x,-\epsilon}_{\Delta_1+\Delta_2-1,\sigma_2}(z_2,\bar{z}_2)\Big]\end{aligned}$$

where $\epsilon=\pm1$ for outgoing (incoming) gluons and $\sigma_2=\pm1$.