Centerless BMS$_4$ charge algebra, the quantized angular momentum and (A)dS uplift

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“Flat Asymptotia” Virtual Workshop
Okinawa, March 15-18th, 2021

References

- G.C., Adrien Fiorucci and Romain Ruzziconi,
  The Λ-BMS_4 Charge Algebra
  2004.10769

- G.C., Roberto Oliveri, Ali Seraj,
  The Poincaré and BMS flux-balance laws with application to binary systems
  1912.03164

- G.C., Adrien Fiorucci and Romain Ruzziconi,
  The Λ-BMS_4 group of dS_4 and new boundary conditions for AdS_4.
  1905.00971

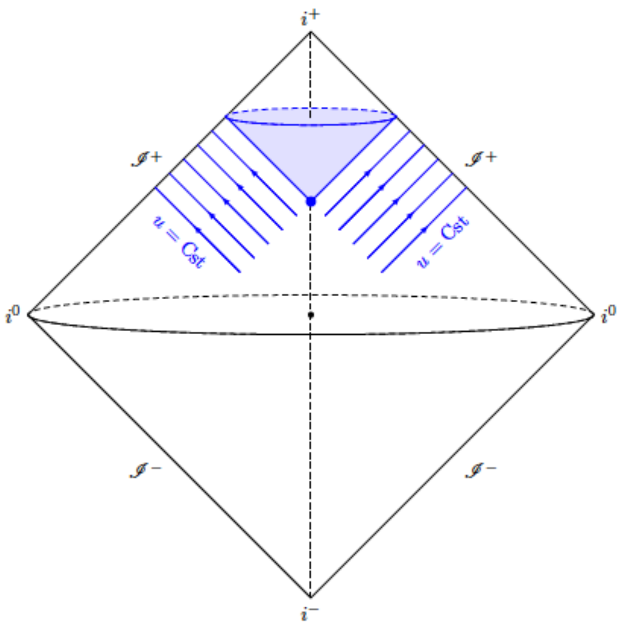
- G.C., Adrien Fiorucci and Romain Ruzziconi,
  Superboost transitions, refraction memory and super-Lorentz charge algebra
  1810.00377

- G.C., David Nichols, to be published.

Three main messages

- The extended BMS charge algebra (supertranslations and Diff(S^2) super-Lorentz transformations) is realized without center at the past and future of null infinity (at spatial and timelike infinity). The asymptotic symmetry group at spatial infinity therefore includes the extended BMS group.
- The extended BMS asymptotic symmetry algebra leads to a unique definition of the quantized angular momentum, which differs from many existing classical prescriptions.
- The extended BMS charge algebra admits a natural extension to (A)dS: the Λ-BMS algebroid. It is the asymptotic symmetry group of Al(A)dS spacetimes with "leaky boundary conditions": without intrinsic boundary conditions (except a boundary gauge fixing condition that does not constraint the Cauchy problem) but with external boundary conditions.

0. Lightning review



Flat case $\Lambda = 0$.

Gauge fixed approach :

Bondi / Newman-Unti coordinates

Formalism: Lagrangian (metric)

Bondi / Newman-Unti equivalent to each other [Barnich, Lambert, 2011]

Hamiltonian/Lagrangian equivalent to each other

Supertranslations

$$T(\theta,\phi)\partial_u + \frac{1}{2}\nabla^2 T\partial_r - \frac{1}{r}(\partial_\theta T\partial_\theta + \frac{1}{\sin^2\theta}\partial_\phi T\partial_\phi) + \ldots$$

- The associated Noether charge is the Bondi mass aspect $m(u,x^A)$ integrated over the celestial sphere
- The 4 lowest harmonics are the translations associated with Momenta.
- Supertranslations transitions are associated with displacement memory and are caused by any null radiation exiting null infinity

[Bondi, Metzner, van der Burg, Sachs, 1962] [Strominger, Zhiboedov, 2014]

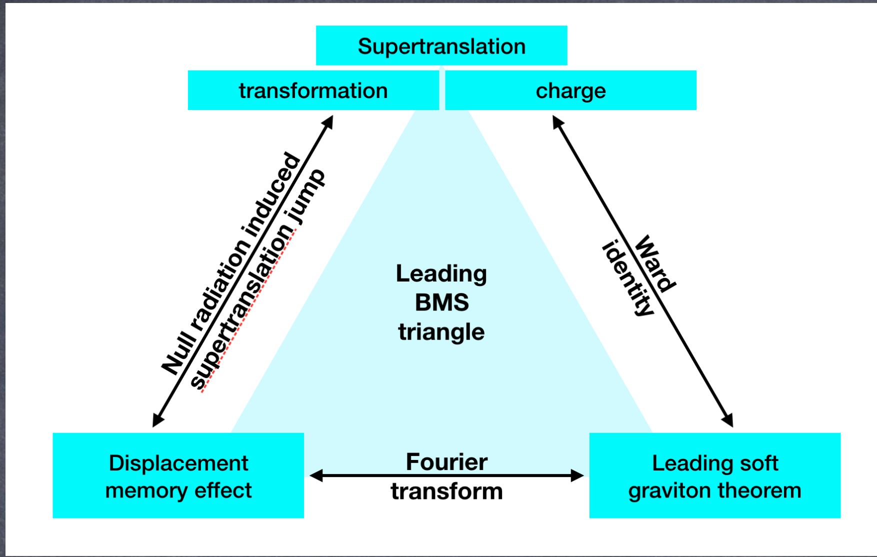
Super-Lorentz transformations

$$\frac{1}{2}uD_AR^A\partial_u + (-\frac{1}{2}(r+u)D_AR^A + \mathcal{O}(\frac{1}{r})\partial_r + (R^A - \frac{u}{2r}D^AD_BR^B + \mathcal{O}(\frac{1}{r^2}))\partial_A$$

- Associated Noether charge: Bondi angular momentum aspect $N_A(u,x^A)$ integrated over the celestial sphere after renormalization of radial divergence
- The 6 lowest harmonics are associated with the Lorentz charges: angular momentum and center-of-mass charge (orbital angular momentum).
- Lorentz transformations are asymptotic symmetries. Super-Lorentz transformations are asymptotic symmetries only after renormalization.
- Superrotations $R^A = \epsilon^{AB}\partial_B\Phi$ and superboosts $R^A = \gamma^{AB}\partial_B\Psi$
- Super-Lorentz transitions are cosmic events. But super-Lorentz charges are non-trivial already for standard asymptotically flat spacetimes.

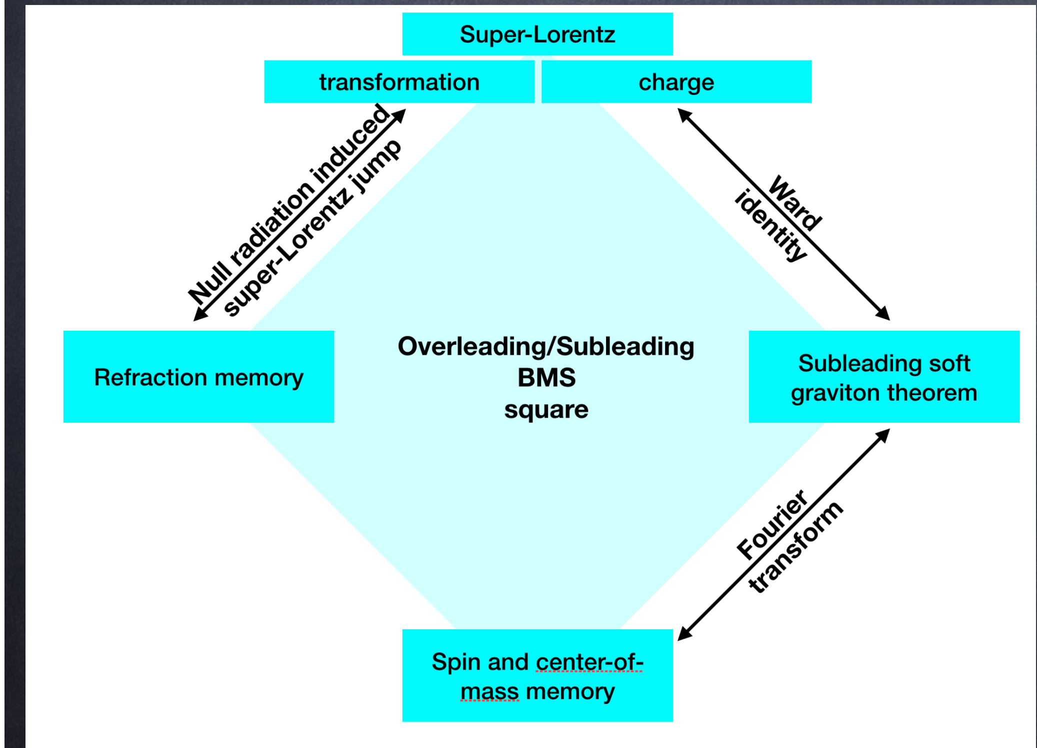
[Barnich, Troessaert] [Campiglia, Laddha] [G.C., Fiorucci, Ruzziconi] [Strominger, Zhiboedov] [Hawking,Perry,Strominger][Flanagan, Nichols]

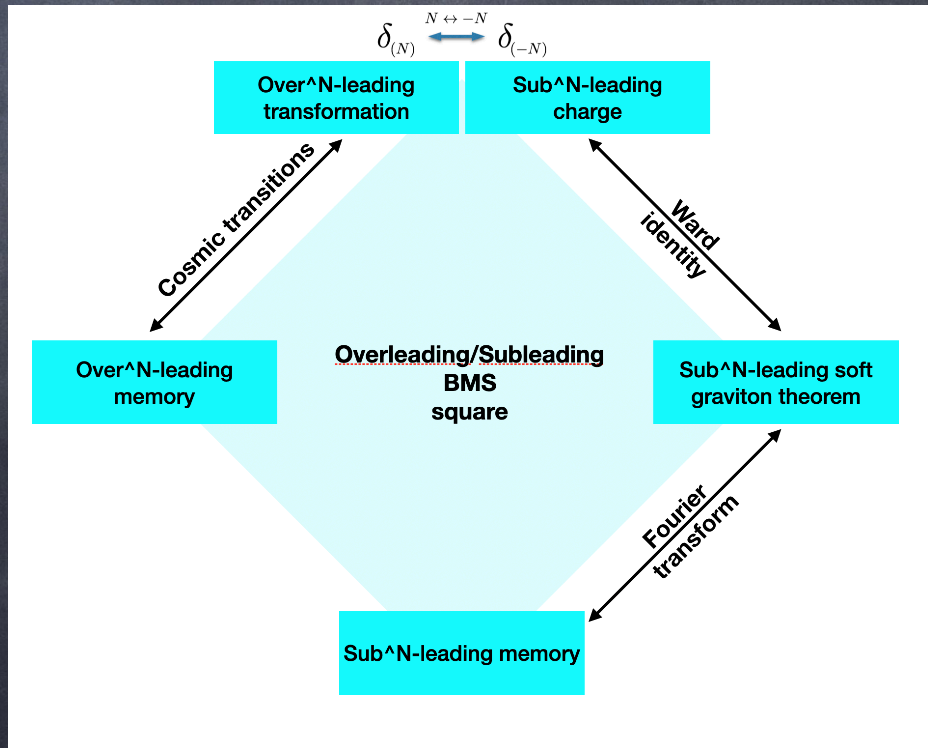
Leading, subleading, sub^N-leading



Lectures on the Infrared Structure of Gravity and Gauge Theory
Andrew Strominger

An Infinite Set of Ward Identities for Adiabatic Modes in Cosmology
Kurt Hinterbichler[a], Lam Hui[b] and Justin Khoury[c]





Superboost transitions, refraction memory and super-Lorentz charge algebra
Geoffrey Compère, Adrien Fiorucci, Romain Ruzziconi

Infinite towers of supertranslation and superrotation memories
Geoffrey Compère

1. Centerless BMS charge at timelike and spatial infinity - Centerless BMS current algebra at null infinity

Gauge theories:
Definitions in the Lagrangian formalism

- **Fields** : $\Phi^i = \{\phi, A_\mu, g_{\mu\nu}, \dots\}$

- **Lagrangian:** $L[\Phi^i]$

- **Gauge transformations:** $\delta_\lambda \Phi^i = R^i_\lambda[\Phi], \qquad \lambda = \lambda^\alpha(\Phi^i(x^\mu), x^\mu)$

- **Gauge algebra:**

$$\delta_{\lambda_1}\delta_{\lambda_2}\Phi^i - (1 \leftrightarrow 2) = R^i_{[\lambda_1,\lambda_2]}[\Phi], \qquad [\lambda_1,\lambda_2]^\alpha = C^\alpha_{\beta\gamma}(\lambda_1^\beta,\lambda_2^\gamma) + \delta_{\lambda_1}\lambda_2^\alpha - \delta_{\lambda_2}\lambda_1^\alpha$$

Algorithmic derivation of the infinitesimal surface charge:
Contribution from the Lagrangian

- Action principle

$$\delta L = \underbrace{\frac{\delta L}{\delta \Phi^i}}_{\text{EOM}} \delta \Phi^i + d\underbrace{\Theta}_{\text{Presymplectic potential}}[\delta \Phi^i; \Phi^i]$$

- Symplectic structure

$$\omega[\delta_1 \Phi^i, \delta_2 \Phi^i; \Phi^i] = \delta_1 \Theta[\delta_2 \Phi^i; \Phi^i] - \delta_2 \Theta[\delta_1 \Phi^i; \Phi^i]$$

- Infinitesimal surface charge

$$\omega[\delta_\lambda \Phi^i, \delta \Phi^i; \Phi^i] = dk_\lambda[\delta \Phi; \Phi] + \mathrm{EOM}$$

[Lee, Wald, 1990][Iyer, Wald, 1994]

$$\delta_\lambda \Phi^i = 0 \Rightarrow dk_\lambda[\delta \Phi; \Phi] = 0$$

Algorithmic derivation of the infinitesimal surface charge: Contributions from the Lagrangian and boundary terms

- Action principle

$$\delta S = \int d^n x L[\Phi] + \int d^n x L_B[\Psi]$$

EOM Presymplectic potential

$$\delta L = \underbrace{\frac{\delta L}{\delta \Phi^i}}_{\text{EOM}} \delta \Phi^i + d\underbrace{\Theta[\delta \Phi^i; \Phi^i]}_{\text{Presymplectic potential}}$$

$$\Theta = \Theta_L - d\Theta_B$$

- Symplectic structure

Only gauge invariant prescription

$$\omega[\delta_1 \Phi^i, \delta_2 \Phi^i; \Phi^i] = \delta_1 \Theta[\delta_2 \Phi^i; \Phi^i] - \delta_2 \Theta[\delta_1 \Phi^i; \Phi^i]$$

$$\omega = \omega_L - d\omega_B$$

- Infinitesimal surface charge

$$\omega[\delta_\lambda \Phi^i, \delta \Phi^i; \Phi^i] = dk_\lambda[\delta \Phi; \Phi] + \text{EOM}$$

$$k_\lambda = k^L_\lambda[\delta \Phi; \Phi] - \omega_B[\delta_\lambda \Psi; \delta \Psi; \Psi]$$

$$\delta_\lambda \Phi^i = 0 \Rightarrow dk_\lambda[\delta \Phi; \Phi] = 0$$

$$\delta_\lambda \Phi^i = 0,\ \delta_\lambda \Psi^i = 0 \Rightarrow dk_\lambda = 0$$

[Skenderis,Papadimitriou,2005][G.C., Marolf, 2008][Harlow, Wu, 2020][Freidel, Geiller, Pranzetti, 2020]

Flat BMS surface charge algebra

Surface charges

$$\mathrm{d}\boldsymbol{k}^{\text{flat}}_\xi[\delta\phi;\phi] = \boldsymbol{\omega}_{\text{ren,tot}}[\delta_\xi\phi,\delta\phi;\phi]\Big|_{\Lambda=0}.$$

(radial divergence subtracted after renormalization)

$$\delta\!\!\!/ H^{\text{flat}}_\xi[\phi] = \int_{S^2_\infty} \boldsymbol{k}^{\text{flat}}_\xi[\delta\phi;\phi] = \int_{S^2_\infty} 2(\mathrm{d}^2x)_{ru}\,(k^{\text{flat}}_\xi)^{ru}[\delta\phi;\phi].$$

$$\boxed{\delta\!\!\!/ H^{\text{flat}}_\xi[\phi] = \delta H^{\text{flat}}_\xi[\phi] + \Xi^{\text{flat}}_\xi[\delta\phi;\phi],}$$

where

$$H^{\text{flat}}_\xi[\phi] = \frac{1}{16\pi G}\int_{S^2_\infty} \mathrm{d}^2\Omega\left[4fM + 2Y^A N_A + \frac{1}{16}Y^A\partial_A(C_{BC}C^{BC})\right],$$

$$\Xi^{\text{flat}}_\xi[\delta\phi;\phi] = \frac{1}{16\pi G}\int_{S^2_\infty} \mathrm{d}^2\Omega\left[\frac{1}{2}f\left(N^{AB}+\frac{1}{2}q^{AB}R[q]\right)\delta C_{AB} - 2\partial_{(A}f\mathcal{U}_{B)}\delta q^{AB}\right.$$
$$\left. - fD_{(A}\mathcal{U}_{B)}\delta q^{AB} - \frac{1}{4}D_C D^C f C_{AB}\delta q^{AB}\right].$$

$$\mathcal{U}_A \equiv -\frac{1}{2}D^B C_{AB}.$$

Barnich-Troessaert bracket $\{H_\xi[\phi],H_\chi[\phi]\}_\star \equiv \delta_\chi H_\xi[\phi] + \Xi_\chi[\delta_\xi\phi,\phi].$

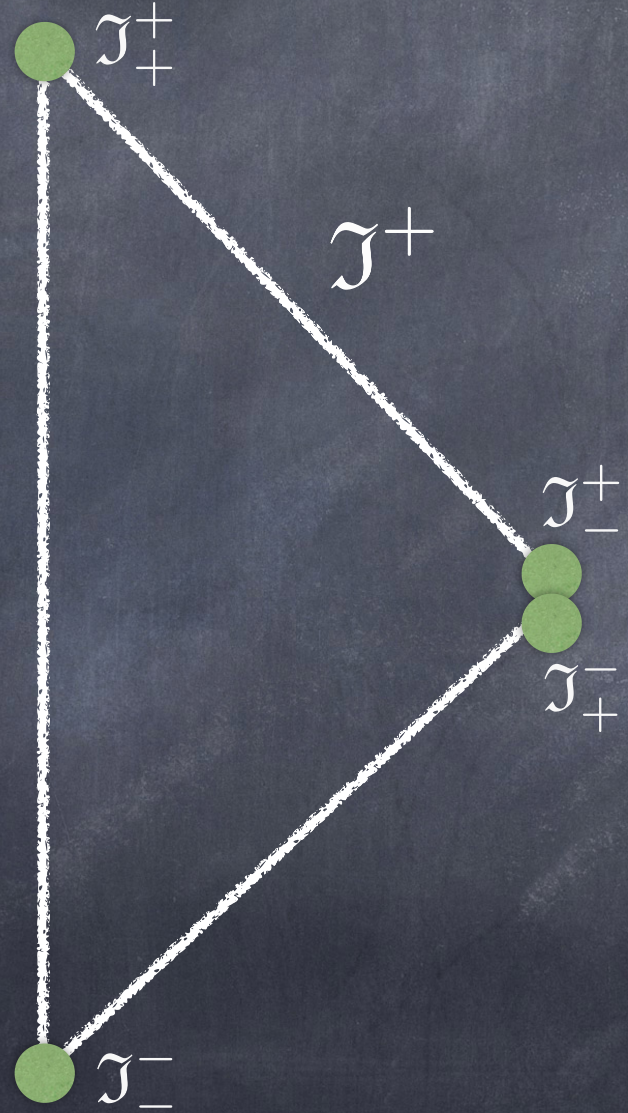
$$\boxed{\{H^{\text{flat}}_\xi[\phi],H^{\text{flat}}_\chi[\phi]\}_\star = H^{\text{flat}}_{[\xi,\chi]_\star}[\phi] + K^{\text{flat}}_{\xi;\chi}[\phi]}$$

Non-trivial 2-cocyle

$$K^{\text{flat}}_{\xi_1;\xi_2}[\phi] = \frac{1}{16\pi G}\int_{S^2_\infty} \mathrm{d}^2\Omega\left[\frac{1}{2}f_1 D_A f_2 D^A R[q] + \frac{1}{2}C^{BC} f_1 D_B D_C D_D Y_2^D - (1\leftrightarrow 2)\right].$$

[Barnich, Troessaert, 2011] [G.C., Fiorucci, Ruzziconi, 2018]

Flat BMS surface charge algebra at the corners ?



Algebra of BMS fluxes?

$$\bar{F}_\xi[\phi] = \bar{H}^{\text{flat}}_\xi[\phi]\Big|_{\mathscr{I}^+_+} - \bar{H}^{\text{flat}}_\xi[\phi]\Big|_{\mathscr{I}^+_-} = \int_{-\infty}^{+\infty} \mathrm{d}u\,\partial_u \bar{H}^{\text{flat}}_\xi[\phi].$$

Non-radiative asymptotic boundary conditions at the corners

Minkowski in an arbitrary super-Lorentz and supertranslation frame:

$$ds^2 = -\frac{\mathring{R}}{2}\mathrm{d}u^2 - 2\mathrm{d}\rho\mathrm{d}u + (\rho^2 q_{AB} + \rho C^{\text{vac}}_{AB} + \frac{1}{8}C^{\text{vac}}_{CD}C^{CD}_{\text{vac}}q_{AB})\mathrm{d}x^A\mathrm{d}x^B + D^B C^{\text{vac}}_{AB}\mathrm{d}x^A\mathrm{d}u$$

where

$$C^{\text{vac}}_{AB}[\Phi, C] = (u + C)N^{\text{vac}}_{AB} + C^{(0)}_{AB}, \quad \begin{cases} N^{\text{vac}}_{AB} &= \left[\frac{1}{2}D_A\Phi D_B\Phi - D_A D_B\Phi\right]^{\text{TF}} \\ C^{(0)}_{AB} &= -2D_A D_B C + q_{AB}D^2 C. \end{cases}$$

$$q_{AB} = q^{\text{vac}}_{AB} \equiv e^{-\Phi}\partial_A G^a \partial_B G^b \gamma_{ab}. \qquad \mathring{R} = D^2\Phi,$$

This is the 4d analogue of 3d metric: $ds^2 = \Theta(\phi)du^2 - 2dudr + 2\left[\Xi(\phi) + \frac{u}{2}\partial_\phi\Theta(\phi)\right]dud\phi + r^2 d\phi^2$

We therefore impose as boundary conditions at past/future of scri:

$$C_{AB} = (u+C_\pm)N^{\text{vac}}_{AB} - 2D_A D_B C_\pm + q_{AB}D^E D_E C_\pm + o(u^0)$$

$$N^{AB} = N^{AB}_{\text{vac}} + o(u^{-1}) \text{ as } u \to \pm\infty$$

The supertranslation frame can then be fixed if needed by a boundary condition at spatial infinity.

[Barnich, Troessaert, 2011, 2016]
[G.C., Fiorucci, Ruzziconi, 2018]

Ambiguity shift in the charges

$$\delta\!\!\!/ H^{\text{flat}}_{\xi}[\phi] = \delta H^{\text{flat}}_{\xi}[\phi] + \Xi^{\text{flat}}_{\xi}[\delta\phi;\phi],$$

$$\bar{H}^{\text{flat}}_{\xi(T,Y)}[\phi] \equiv H^{\text{flat}}_{\xi(T,Y)}[\phi] + \Delta H^{\text{flat}}_{\xi(T,Y)}[\phi]$$

This leads in particular to an ambiguity in the definition of angular momentum (discussed later). This ambiguity can be fixed by requesting the BMS algebra to be realized without anomaly (field-dependent center) at the corners.

Under the standard Dirac bracket,

$$\{\bar{H}^{\text{flat}}_{\xi}[\phi], \bar{H}^{\text{flat}}_{\chi}[\phi]\} = \bar{H}^{\text{flat}}_{[\xi,\chi]_\star}[\phi] + R_{\xi,\chi}[\phi],$$

$$R_{\xi,\chi}[\phi] \equiv K^{\text{flat}}_{\xi;\chi}[\phi] - \Delta H^{\text{flat}}_{[\xi,\chi]_\star}[\phi] + \delta_\chi \Delta H^{\text{flat}}_{\xi}[\phi] - \Xi_\chi[\delta_\xi\phi;\phi]$$

With hindsight we define the shift

$$\Delta H^{\text{flat}}_{\xi(T,Y)}[\phi] \equiv \int_{S^2_\infty} \frac{\mathrm{d}^2\Omega}{16\pi G}\left[\frac{1}{2} f C_{AB} N^{AB}_{\text{vac}} + \frac{1}{2} Y^A C_{AB} D_C C^{BC} + \frac{1}{8} Y^A \partial_A (C^B_C C^C_B)\right].$$

[G.C., Fiorucci, Ruzziconi, 2020]

Final BMS charges

$$\bar{H}^{\text{flat}}_{\xi(T,Y)}[\phi] = \int_{S^2_\infty} \frac{\mathrm{d}^2\Omega}{16\pi G}\Big[4T\bar{M} + 2Y^A\bar{N}_A\Big],$$

$$\bar{M} = M + \frac{1}{8}C_{AB}N^{AB}_{\text{vac}},$$
$$\bar{N}_A = N_A^{[B.T.]} - u\partial_A\bar{M} + \frac{1}{4}C_{AB}D_C C^{BC} + \frac{3}{32}\partial_A(C_{BC}C^{BC}).$$

- The BMS algebra is realized without central extension at the corners (spatial infinity, timelike infinity)

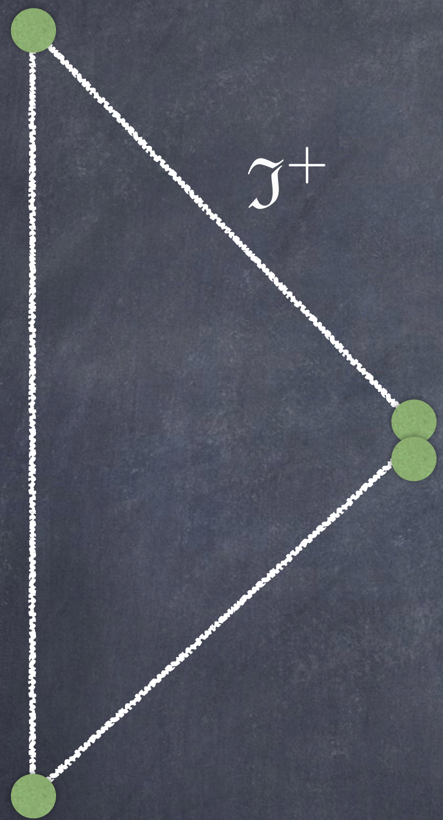
$$\boxed{\{\bar{H}^{\text{flat}}_\xi[\phi], \bar{H}^{\text{flat}}_\chi[\phi]\}\Big|_{\mathscr{I}^+_\pm} = \bar{H}^{\text{flat}}_{[\xi,\chi]_\star}[\phi]\Big|_{\mathscr{I}^+_\pm}.}$$

$$R_{\xi,\chi}[\phi] = 0$$

- The BMS charges can therefore be directly quantized
- All BMS charges are identically zero for Minkowski in an arbitrary BMS frame

The charge prescription differs from [Barnich, Troessaert, 2011] [Wald,Zoupas, 1999]
It agrees with [Pasterski, Strominger, Zhiboedov, 2014-2015][Hawking, Perry, Strominger, 2015] when $N^{AB}_{\text{vac}} = 0$. The shift with the vacuum news $N^{AB}_{\text{vac}} = 0$ is necessary to allow super-Lorentz to be represented without anomaly.

BMS4 flux asymptotic symmetry algebra



$$\bar{F}_\xi[\phi] = \bar{H}^{\text{flat}}_\xi[\phi]\Big|_{\mathscr{I}^+_+} - \bar{H}^{\text{flat}}_\xi[\phi]\Big|_{\mathscr{I}^+_-} = \int_{-\infty}^{+\infty} \mathrm{d}u\, \partial_u \bar{H}^{\text{flat}}_\xi[\phi].$$

Supermomenta $\bar{P}_T \equiv \bar{F}_{\xi(T,0)}[\phi]$

Super-Lorentz $\bar{J}_Y \equiv \bar{F}_{\xi(0,Y)}[\phi],$

Standard bracket:

$$\{\bar{P}_{T_1}, \bar{P}_{T_2}\} = 0, \quad \{\bar{J}_{Y_1}, \bar{P}_{T_2}\} = \bar{P}_{Y_1(T_2)}, \quad \{\bar{J}_{Y_1}, \bar{J}_{Y_2}\} = \bar{J}_{[Y_1,Y_2]}$$

where $Y_1(T_2) \equiv (Y_1^A \partial_A - \frac{1}{2} D_A Y_1^A) T_2.$

[Campiglia, Peraza, 2020]

[G.C., Fiorucci, Ruzziconi, 2020]

Since charges are conserved at spatial infinity,

the extended BMS4 algebra therefore belongs to the Asymptotic Symmetry Group at spatial infinity (after renormalization)

The extended BMS4 algebra is therefore quantized

(consistently with its role in the subleading soft graviton theorem)

2. The quantized angular momentum

The α-ambiguous angular momentum

There exists a one parameter family of Lorentz and supertranslation covariant definitions that are consistent with the standard BMS algebra:

$$\mathcal{J}_i^{(\alpha)} \equiv -\frac{1}{2}\oint_S \epsilon^{AD}\partial_D n_i \left(\bar{N}_A - \frac{\alpha c^3}{4G} C_{AB} D_C C^{BC}\right)$$

$$\bar{N}_A = N_A^{[B.T.]} - \cancel{u\partial_A M} + \frac{1}{4} C_{AB} D_C C^{BC} + \frac{3}{32}\partial_A(\cancel{C_{BC}C^{BC}}).$$

[G.C., Oliveri, Seraj, 2019]

Supertranslations act as

$$\delta_{T,Y} C_{AB} = \left(T + \frac{u}{2} D_A Y^A\right)\dot{C}_{AB} + \mathcal{L}_Y C_{AB} - \frac{1}{2} D_C Y^C C_{AB} - 2D_A D_B T + \gamma_{AB}\Delta T,$$

$$\bar{N}_A \mapsto \bar{N}_A + 3\frac{m}{c} D_A T + \frac{T}{c}\partial_A m,$$

The standard BMS commutator is obeyed for any α at $\mathscr{I}^+_\pm$ $\quad \delta_T \mathcal{J}_i^{(\alpha)} = \mathcal{P}_{T'}, \qquad T' = -\epsilon^{AD}\partial_D n_i \partial_A T$

A formula invariant under supertranslations can be written down by adding a supertranslation frame term (C^- dependent). See Ashtekar's talk.

$$\mathcal{J}_i^{(\alpha)} \equiv -\frac{1}{2}\oint_S \epsilon^{AD}\partial_D n_i \left(\bar{N}_A - \frac{2m}{c}\partial_A C^- - \frac{\alpha c^3}{4G} C_{AB} D_C C^{BC}\right) \qquad \delta_T C^- = T$$

[Javadinezhad,Kol,Porrati,2018][Keller, Wang, Yau,2018]
[Ashtekar,De Lorenzo, Khera,2019][G.C., Oliveri, Seraj, 2019]

Here, I simply fix the supertranslation frame at past retarded time such that

$$C^- = 0$$

The α-ambiguous angular momentum

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$$\mathcal{J}_i^{(\alpha)} \equiv -\frac{1}{2} \oint_S \epsilon^{AD} \partial_D n_i \left(\bar{N}_A - \frac{\alpha c^3}{4G} C_{AB} D_C C^{BC} \right)$$

[G.C., Oliveri, Seraj, 2019]

Definitions used in the literature:

$\alpha = 1$

[Thorne, 1980][Dray-Streubel,84]
[Ashtekar, Streubel,81]
[Wald,Zoupas,1999][Penrose]
[Barnich, Troessaert][Flanagan, Nichols][Keller,Wang,Yau]

$$\dot{\mathcal{J}}_i = -\frac{G}{c^5}\left(\frac{2}{5}\epsilon_{ijk} I^{(2)}_{jl} I^{(3)}_{kl}\right) - \frac{G}{c^7}\left(\frac{1}{63}\epsilon_{ijk} I^{(3)}_{jlm} I^{(4)}_{klm} + \frac{32}{45}\epsilon_{ijk} J^{(2)}_{jl} J^{(3)}_{kl}\right) + O\left(c^{-9}\right),$$

$\alpha = 3$

[Landau-Lifshitz]
[Damour]
[Bonga, Poisson]

$\alpha = 0$

[Strominger, Zhiboedov, 2014]
[Pasterski, Strominger, Zhiboedov, 2015]
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[Landau-Lifshitz]
[Damour]
[Bonga, Poisson]

The change of definition leads to a numerically 0.01%-0.1% effect for binary coalescences [Elhashash, Nichols, 2021]

$\alpha = 0$

[Strominger, Zhiboedov, 2014]
[Pasterski, Strominger, Zhiboedov, 2015]
[G.C., Fiorucci, Ruzziconi, 2020]

The quantized angular momentum (the unique J compatible with quantized enhanced symmetry)

There exists a one parameter family of Lorentz and supertranslation covariant definitions that are consistent with the standard BMS algebra:

$$\mathcal{J}_i^{(\alpha)} \equiv -\frac{1}{2}\oint_S \epsilon^{AD}\partial_D n_i \left(\bar{N}_A - \frac{\alpha c^3}{4G} C_{AB} D_C C^{BC}\right)$$

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[Barnich, Troessaert][Flanagan, Nichols][Keller,Wang,Yau]

$$\dot{\mathcal{J}}_i = -\frac{G}{c^5}\left(\frac{2}{5}\epsilon_{ijk} I^{(2)}_{jl} I^{(3)}_{kl}\right) - \frac{G}{c^7}\left(\frac{1}{63}\epsilon_{ijk} I^{(3)}_{jlm} I^{(4)}_{klm} + \frac{32}{45}\epsilon_{ijk} J^{(2)}_{jl} J^{(3)}_{kl}\right) + O\left(c^{-9}\right),$$

$\alpha = 3$

[Landau-Lifshitz]
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[G.C., Fiorucci, Ruzziconi, 2020]

Extended BMS4 algebra -> The unique quantized J

3. Extension of the BMS group to (A)dS

Question

- Three-dimensional case :

Asymptotically AdS_3 :
$\mathrm{Diff}(S^1) \times \mathrm{Diff}(S^1)$
[Brown-Henneaux '86]

$$\xrightarrow[\Lambda \to 0]{}$$

Asymptotically flat :
$BMS_3 = \mathrm{Diff}(S^1) \ltimes \mathrm{Vect}(S^1)$
[Ashtekar-Bicak-Schmidt '96]
[Barnich-Compère '07]

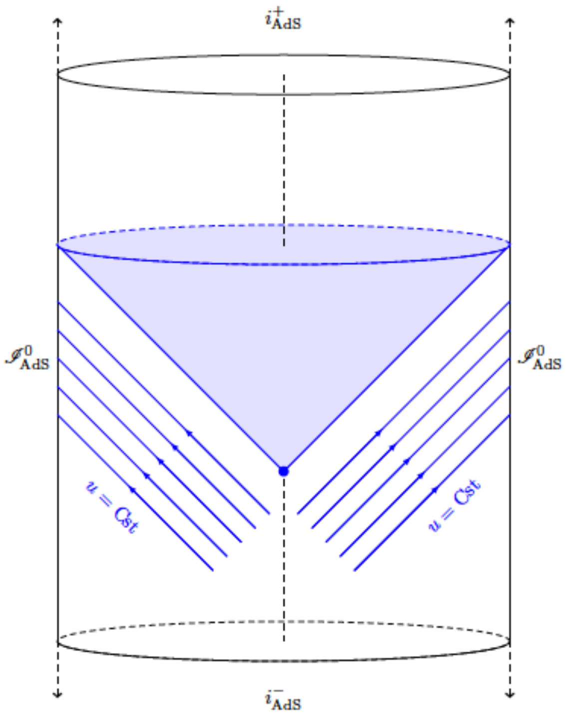
- Four-dimensional case :

Asymptotically AdS_4 :
? ? ?

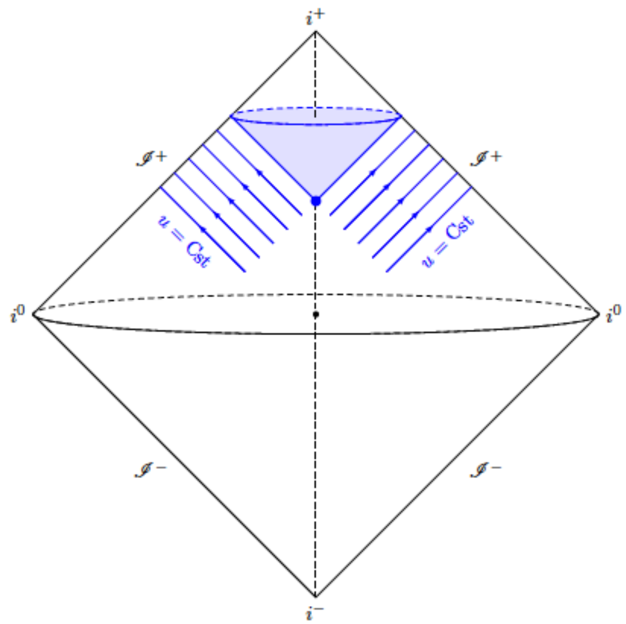
$$\xrightarrow[\Lambda \to 0]{}$$

Asymptotically flat :
$BMS_4 = \mathrm{Diff}(S^2) \ltimes \mathcal{T}$
[Bondi-van der Burg-Metzner '62]
[Sachs '62]
[Barnich-Troessaert, '11][Campiglia-Laddha, '15]

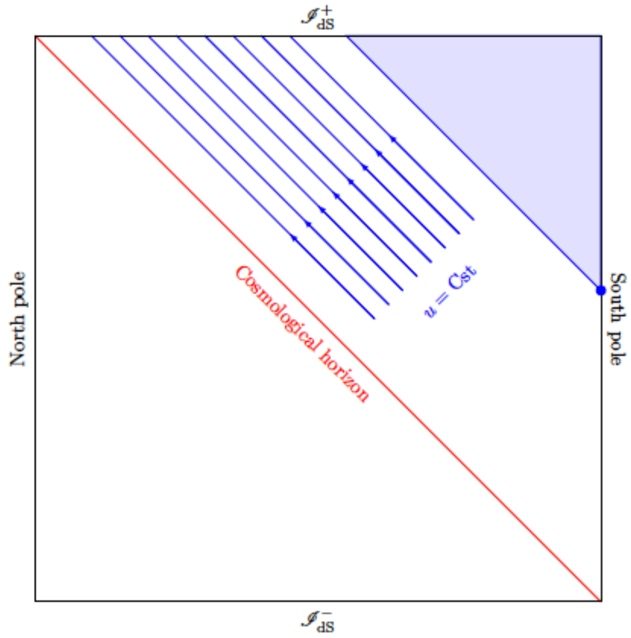
Universal BMS structure (keeping all dynamics)



AdS case $\Lambda < 0$.



Flat case $\Lambda = 0$.



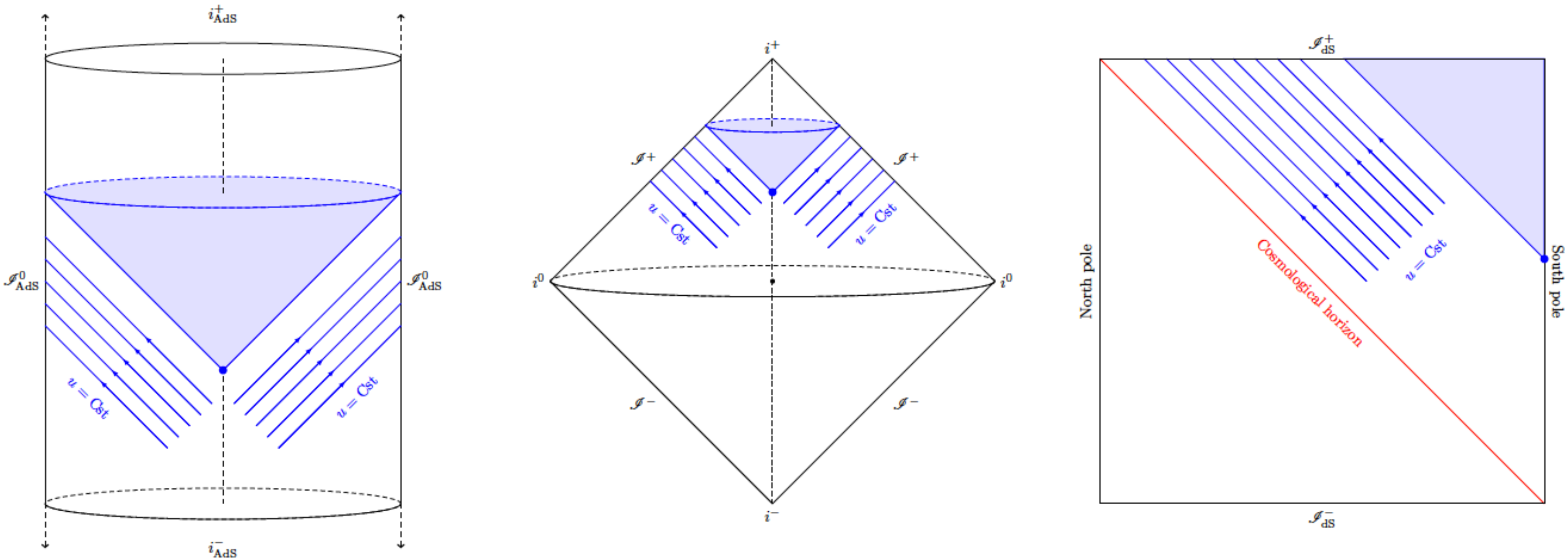
dS case $\Lambda > 0$.

Boundary structure: codimension 1 codimension 2

"Boundary gauge fixing": Fixing a foliation and a measure

$$ds^2 = sign(\Lambda) du^2 + q_{AB} dx^A dx^B \qquad \sqrt{q}$$

A dictionary exists between distinct bulk gauges



AdS case $\Lambda < 0$.

Flat case $\Lambda = 0$.

dS case $\Lambda > 0$.

Fefferman-Graham

$$g^{(0)}_{ab}, T^{ab}$$

Bondi

$$C_{AB}, M, N_A,$$
$$D_{AB}, E_{AB}, \ldots$$

Starobinsky

$$g^{(0)}_{ab}, T^{ab}$$

Definitions

Starobinsky/
Fefferman-Graham
(SFG) gauge

$$(\rho, x^a)$$

$$g_{\rho a} = 0,$$

$$g_{\rho\rho} = -\frac{3}{\Lambda}\frac{1}{\rho^2}$$

Bondi gauge

$$(u, r, x^A)$$

$$g_{rr} = 0, \quad g_{rA} = 0$$

$$\partial_r \left(\frac{\det(g_{AB})}{r^4} \right) = 0$$

The dictionary between Bondi and Starobinsky/Fefferman-Graham gauge has been worked out

- One can solve the large radius expansion of Einstein's equations in both gauges
- A diffeomorphism exists between the two gauges when $\Lambda \neq 0$
- The (2-covariant) map between the free fields in each gauge can be formulated

[Poole, Skenderis, Taylor, 2018]
[G.C., Fiorucci, Ruzziconi, 2019]

Solution space (Al(A)dS$_4$)

SFG gauge

$$ds^2 = -\frac{3}{\Lambda}\frac{d\rho^2}{\rho^2} + \gamma_{ab}(\rho, x^c)dx^a dx^b.$$

$$\gamma_{ab} = \frac{1}{\rho^2} g^{(0)}_{ab} + \frac{1}{\rho} g^{(1)}_{ab} + g^{(2)}_{ab} + \rho\, g^{(3)}_{ab} + \mathcal{O}(\rho^2)$$

[FG theorem]

$$T_{ab} = \frac{\sqrt{3|\Lambda|}}{16\pi G} g^{(3)}_{ab}$$

Bondi gauge

$$ds^2 = e^{2\beta}\frac{V}{r}du^2 - 2e^{2\beta}dudr + g_{AB}(dx^A - U^A du)(dx^B - U^B du)$$

$$g_{AB} = r^2 q_{AB} + r\, C_{AB} + D_{AB} + \frac{1}{r}E_{AB} + \frac{1}{r^2}F_{AB} + \mathcal{O}(r^{-3})$$

$$l = l^A_A = \frac{1}{2}q^{AB}\partial_u q_{AB} = \partial_u \ln\sqrt{q}.$$

[Blanchet-Damour]

$$U^A = U^A_0(u, x^B) + \overset{(1)}{U^A}(u, x^B)\frac{1}{r} + \overset{(2)}{U^A}(u, x^B)\frac{1}{r^2} + \overset{(3)}{U^A}(u, x^B)\frac{1}{r^3} + \overset{(L3)}{U^A}(u, x^B)\frac{\ln r}{r^3} + o(r^{-3})$$

$$\beta(u, r, x^A) = \beta_0(u, x^A) + \frac{1}{r^2}\Big[-\frac{1}{32}C^{AB}C_{AB}\Big] + \frac{1}{r^3}\Big[-\frac{1}{12}C^{AB}\mathcal{D}_{AB}\Big] + \frac{1}{r^4}\Big[-\frac{3}{32}C^{AB}\mathcal{E}_{AB} - \frac{1}{16}\mathcal{D}^{AB}\mathcal{D}_{AB} + \frac{1}{128}(C^{AB}C_{AB})^2\Big] + \mathcal{O}(r^{-5}).$$

$$\frac{V}{r} = \frac{\Lambda}{3}e^{2\beta_0}r^2 - r(l + D_A U^A_0) - e^{2\beta_0}\Big[\frac{1}{2}\Big(R[q] + \frac{\Lambda}{8}C_{AB}C^{AB}\Big) + 2D_A\partial^A\beta_0 + 4\partial_A\beta_0\partial^A\beta_0\Big] - \frac{2M}{r} + o(r^{-1})$$

$$\frac{\Lambda}{3}C_{AB} = e^{-2\beta_0}\Big[(\partial_u - l)q_{AB} + 2D_{(A}U^0_{B)} - D^C U^0_C q_{AB}\Big].$$

Holographic fields $\Lambda \neq 0$

SFG gauge | Bondi gauge (2+1 boundary split)

$$g^{(0)}_{ab}\, dx^a dx^b = \frac{\Lambda}{3} e^{4\beta_0} du^2 + q_{AB}(dx^A - U_0^A du)(dx^B - U_0^B du)$$

T^{ab}

$$g^{(0)}_{ab} T^{ab} = 0.$$

$$T_{ab} = \frac{\sqrt{3|\Lambda|}}{16\pi G} \begin{bmatrix} -\frac{4}{3} M^{(\Lambda)} & -\frac{2}{3} N^{(\Lambda)}_B \\ -\frac{2}{3} N^{(\Lambda)}_A & J_{AB} + \frac{2}{\Lambda} M^{(\Lambda)} q_{AB} \end{bmatrix}$$

$$M^{(\Lambda)} = M + \frac{1}{16}(\partial_u + l)(C_{CD}C^{CD}),$$

$$N^{(\Lambda)}_A = N_A - \frac{3}{2\Lambda} D^B (N_{AB} - \frac{1}{2} l C_{AB}) - \frac{3}{4}\partial_A (\frac{1}{\Lambda} R[q] - \frac{3}{8} C_{CD}C^{CD}),$$

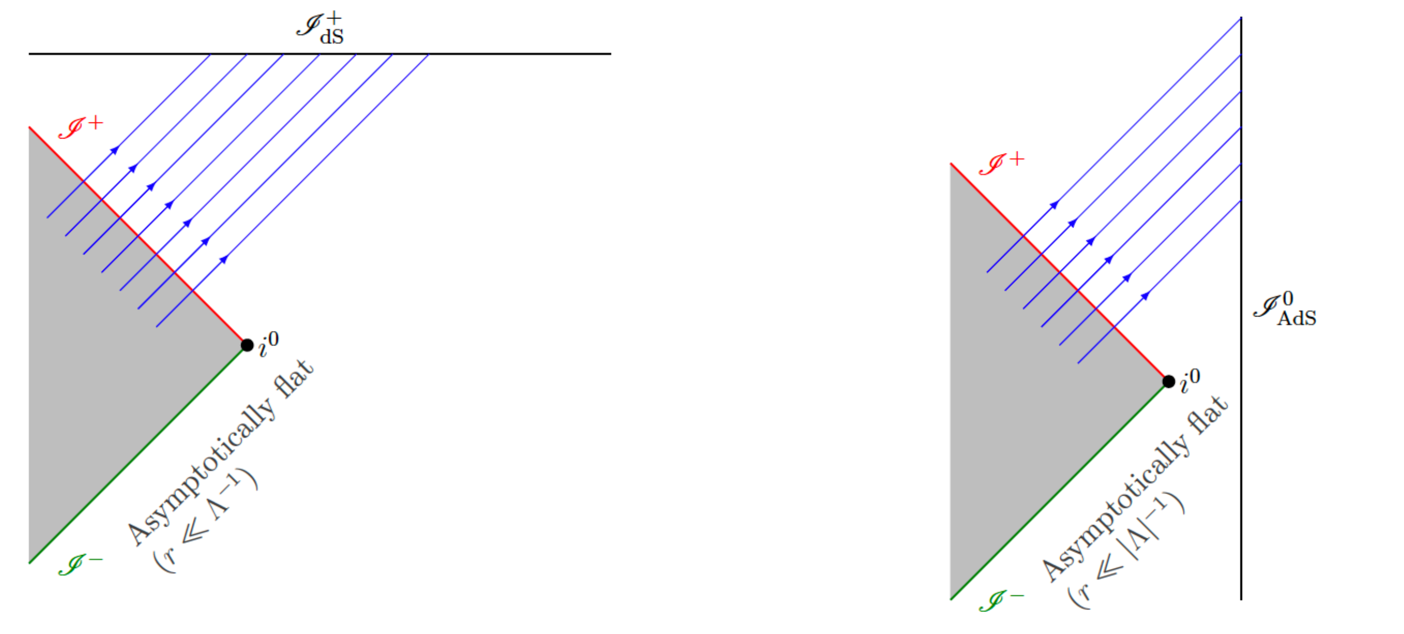
$$\begin{aligned} J_{AB} = & - \mathcal{E}_{AB} - \frac{3}{\Lambda^2}\Big[\partial_u (N_{AB} - \frac{1}{2} l C_{AB}) - \frac{\Lambda}{2} q_{AB} C^{CD}(N_{CD} - \frac{1}{2} l C_{CD})\Big] \\ & + \frac{3}{\Lambda^2}(D_A D_B l - \frac{1}{2} q_{AB} D_C D^C l) \\ & - \frac{1}{\Lambda}(D_{(A} D^C C_{B)C} - \frac{1}{2} q_{AB} D^C D^D C_{CD}) \\ & + C_{AB}\Big[\frac{5}{16} C_{CD} C^{CD} + \frac{1}{2\Lambda} R[q]\Big]. \end{aligned}$$

$$\boxed{\text{3 boundary ODEs} = \text{3 flux-balance laws}}$$

$$D^{(0)}_a T^{ab} = 0 \quad = \quad \begin{aligned} & (\partial_u + \frac{3}{2} l) M^{(\Lambda)} + \frac{\Lambda}{6} D^A N^{(\Lambda)}_A + \frac{\Lambda^2}{24} C_{AB} J^{AB} = 0. \\ & (\partial_u + l) N^{(\Lambda)}_A - \partial_A M^{(\Lambda)} - \frac{\Lambda}{2} D^B J_{AB} = 0. \end{aligned}$$

Why the Bondi mass differs from (A)dS and Minkowski?

$$
\begin{aligned}
M^{(\Lambda)} &= M + \frac{1}{16}(\partial_u + l)(C_{CD}C^{CD}), \\
N_A^{(\Lambda)} &= N_A - \frac{3}{2\Lambda}D^B(N_{AB} - \frac{1}{2}lC_{AB}) - \frac{3}{4}\partial_A(\frac{1}{\Lambda}R[q] - \frac{3}{8}C_{CD}C^{CD}),
\end{aligned}
$$



(a) Asymptotically locally dS_4 case. (b) Asymptotically locally AdS_4 case.

More holographic fields for $\Lambda = 0$

SFG gauge

infinite number of boundary ODEs / flux-balance laws

Bondi gauge

$$\frac{\Lambda}{3}e^{4\beta_0}du^2 + q_{AB}(dx^A - U_0^A du)(dx^B - U_0^B du)$$

$$M, N_A, C_{AB}, D_{AB}, E_{AB}, F_{AB}, \cdots$$

Constraints

$$\partial_u q_{AB} = l q_{AB} + 2D_{(A}U^0_{B)} - D^C U^0_C q_{AB}$$

$$\begin{aligned}
\partial_u M &= -\frac{1}{8}N_{AB}N^{AB} + \frac{1}{4}D_A D_B N^{AB} + \frac{1}{8}D_A D^A \mathring{R},\\
\partial_u N_A &= D_A M + \frac{1}{16}D_A(N_{BC}C^{BC}) - \frac{1}{4}N^{BC}D_A C_{BC}\\
&\quad - \frac{1}{4}D_B(C^{BC}N_{AC} - N^{BC}C_{AC}) - \frac{1}{4}D_B D^B D^C C_{AC}\\
&\quad + \frac{1}{4}D_B D_A D_C C^{BC} + \frac{1}{4}C_{AB}D^B \mathring{R}.
\end{aligned}$$

$$\partial_u D_{AB} = 0,$$
$$\partial_u E_{AB} = \cdots$$
$$\partial_u F_{AB} = \cdots$$

see [Barnich,Troessart, 2012]

Boundary gauge condition: Definition of Λ-BMS

$$g^{(0)}_{tt} = \frac{\Lambda}{3}, \qquad g^{(0)}_{tA} = 0, \qquad \det(g_{(0)}) = \frac{\Lambda}{3}\bar{q}.$$

$$ds^2_{(0)} = \frac{\Lambda}{3} dt^2 + q_{AB} dx^A dx^B$$

- Can always be reached
- Does not constraint the Cauchy problem

The residual diffeomorphisms in a given bulk gauge and in this boundary gauge form the Λ-BMS group.

Symmetry generators

Preserving SFG gauge: Diff(S³)xWeyl

$$\xi^u = f,$$
$$\xi^A = Y^A + I^A, \quad I^A = -\partial_B f \int_r^\infty dr' (e^{2\beta} g^{AB}),$$
$$\xi^r = -\frac{r}{2}(\mathscr{D}_A Y^A - 2\omega + \mathscr{D}_A I^A - \partial_B f U^B + \frac{1}{2} f g^{-1} \partial_u g),$$

$$g = \det(g_{AB})$$

$$\partial_r f = 0 = \partial_r Y^A$$

Preserving further boundary gauge:

$$\delta_\xi \sqrt{q} = 0 \longrightarrow \omega = 0.$$

$$\delta_\xi \beta_0 = 0 \longrightarrow \left(\partial_u - \frac{1}{2} l\right) f = \frac{1}{2} D_A Y^A,$$

$$l = l^A_A = \frac{1}{2} q^{AB} \partial_u q_{AB} = \partial_u \ln \sqrt{q}.$$

$$\delta_\xi U_0^A = 0 \longrightarrow \partial_u Y^A = -\frac{\Lambda}{3} \partial^A f.$$

Flat spacetime limit: $Y^A = V^A(x^B), \ f = T(x^A) + \frac{u}{2} D_A V^A$

(Soft) Algebra / Algebroid

$\bar{\xi} = f\partial_u + Y^A\partial_A$

$[\bar{\xi}_1, \bar{\xi}_2] = \hat{\bar{\xi}},$

$\hat{\bar{\xi}} = \hat{f}\partial_u + \hat{Y}^A\partial_A$

$$\hat{f} = Y_1^A\partial_A f_2 + \frac{1}{2} f_1 D_A Y_2^A - (1 \leftrightarrow 2),$$
$$\hat{Y}^A = Y_1^B\partial_B Y_2^A - \frac{\Lambda}{3} f_1 q^{AB}\partial_B f_2 - (1 \leftrightarrow 2).$$

The structure constants of the Λ-BMS algebra are field-dependent.

In the flat limit, the structure constants are field-independent and reproduce the generalized BMS algebra.

Λ-BMS: Surface charges

$$\mathrm{d}\boldsymbol{k}_{\xi,\mathrm{ren}}[\delta\phi;\phi] = \boldsymbol{\omega}_{\mathrm{ren}}[\delta_\xi\phi,\delta\phi;\phi],$$

$$\boxed{\delta\!\!\!/ H_\xi[\phi] = \int_{S^2_\infty} 2(\mathrm{d}^2x)_{\rho t}\left[\delta\left(\sqrt{|g^{(0)}|}T^t_{(\mathrm{tot})b}\right)\xi^b_{(0)} - \frac{1}{2}\sqrt{|g^{(0)}|}\,\xi^t_{(0)}\,T^{bc}_{(\mathrm{tot})}\delta g^{(0)}_{bc}\right].}$$

- Charges are finite thanks to renormalization
- Charges are neither conserved or integrable $\delta\!\!\!/ H_\xi[\phi] = \delta H_\xi[\phi] + \Xi_\xi[\delta\phi;\phi],$
- Charges associated with Weyl are zero
- They obey the surface charge algebra

$$\boxed{\{H_\xi[\phi], H_\chi[\phi]\}_\star = H_{[\xi,\chi]_\star}[\phi].}$$

where the Barnich-Troessaert brackets are

$$[\xi,\chi]_\star = [\xi,\chi] - \delta_\xi\chi + \delta_\chi\xi.$$

$$\{H_\xi[\phi], H_\chi[\phi]\}_\star \equiv \delta_\chi H_\xi[\phi] + \Xi_\chi[\delta_\xi\phi,\phi].$$

Completeness of Λ-BMS

Number of flux-balance laws: d-1

$$(\partial_u + \frac{3}{2}l)M^{(\Lambda)} + \frac{\Lambda}{6}D^A N_A^{(\Lambda)} + \frac{\Lambda^2}{24}C_{AB}J^{AB} = 0.$$

$$(\partial_u + l)N_A^{(\Lambda)} - \partial_A M^{(\Lambda)} - \frac{\Lambda}{2}D^B J_{AB} = 0.$$

Number of generators: d-1 : f , Y^A

Number of charges: d-1 : $M^{(\Lambda)},\ N_A^{(\Lambda)}$

In comparison with Diff(S^3) studied in [Anninos,Ng,Strominger,2011] the Λ-BMS groupoid is the subset of Diff(S^3) associated with non-trivial flux-balance laws / Ward identities

Symplectic structure and (A)dS equivalent of Bondi shear/news

Action with holographic counterterms

$$S = \frac{1}{16\pi G}\int_{\mathscr{M}} d^4x\sqrt{-g}\,(R[g]-2\Lambda) + \frac{1}{16\pi G}\int_{\mathscr{I}} d^3x\sqrt{|\gamma|}\,(2K + \frac{4}{\ell} - \ell R[\gamma]).$$

[Balasubramanian, Kraus]

Symplectic structure gets a contribution from the counterterm

$$\delta L = \frac{\delta L}{\delta g_{\mu\nu}}\delta g_{\mu\nu} + d\Theta, \qquad \omega = \delta\Theta$$

$$\omega = \omega_{EH}[\delta g, \delta g; g] - d\omega_{EH}[\delta\gamma, \delta\gamma; \gamma]$$

[Skenderis,Papadimitriou,2005]
[G.C.,Marolf,2008]

At the boundary, the orthogonal component of the symplectic structure is

$$\omega^\rho = \frac{1}{2\ell^2}\int_{\mathscr{I}} d^3x\,\delta\Big(\sqrt{|g_{(0)}|}T^{ab}\Big)\wedge\delta g^{(0)}_{ab}.$$

$$g^{(0)}_{tt} = \frac{\Lambda}{3}, \qquad g^{(0)}_{tA} = 0, \qquad \det(g_{(0)}) = \frac{\Lambda}{3}\bar{q}.$$

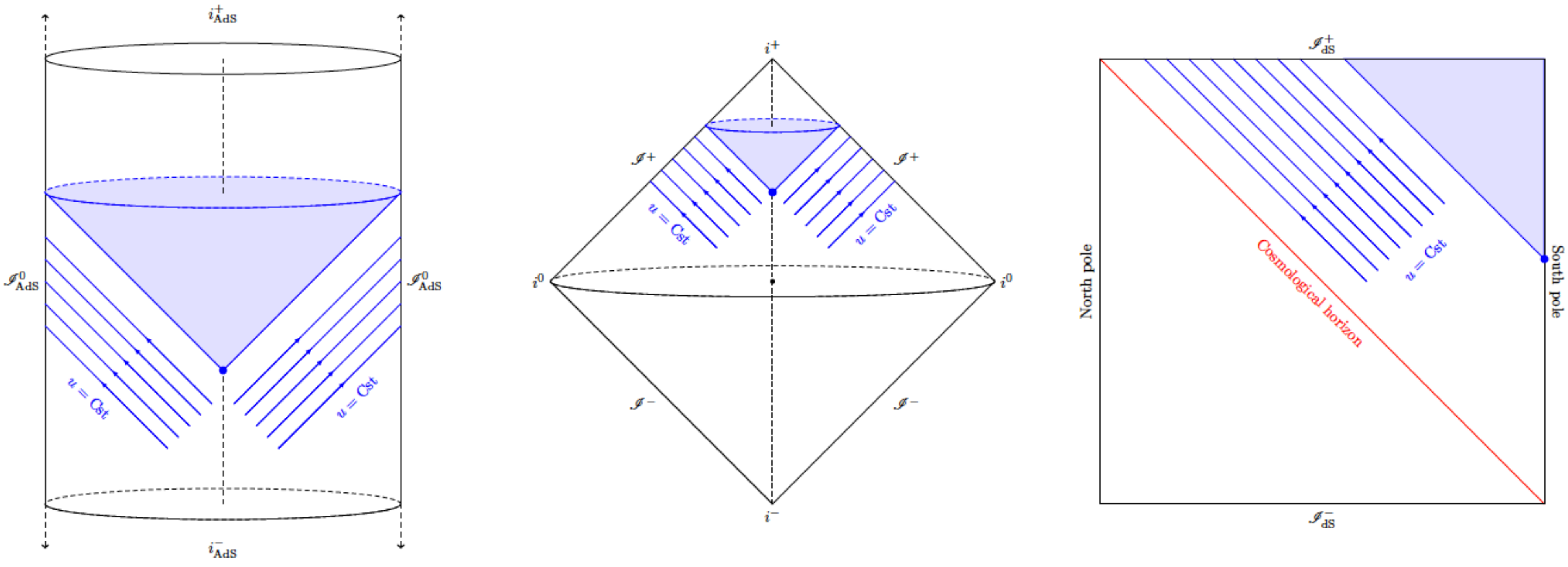
$$T_{ab} = \frac{\sqrt{3|\Lambda|}}{16\pi G}\begin{bmatrix} -\frac{4}{3}M^{(\Lambda)} & -\frac{2}{3}N^{(\Lambda)}_B \\ -\frac{2}{3}N^{(\Lambda)}_A & J_{AB} + \frac{2}{\Lambda}M^{(\Lambda)}q_{AB} \end{bmatrix}$$

$$\longrightarrow \quad \omega^\rho = \frac{3}{32\pi G\ell^4}\int_{\mathscr{I}} d^3x\,\sqrt{\bar{q}}\,\delta J^{AB}\wedge\delta q_{AB}.$$

Therefore, energy is transferred due to changes of both J^{AB} and q_{AB}. They are the analogue in dS/AdS of the Bondi shear/news.

[G.C., Fiorucci, Ruzziconi, 2019]

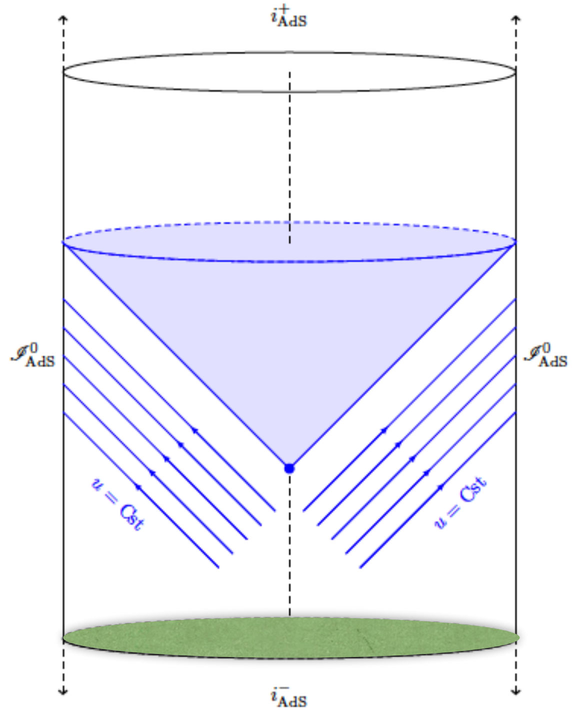
Symplectic flux and the Cauchy problem



AdS case $\Lambda < 0$. Flat case $\Lambda = 0$. dS case $\Lambda > 0$.

In AdS, the Cauchy problem requires an additional boundary condition (standard or "leaky")

Example of "leaky" boundary condition

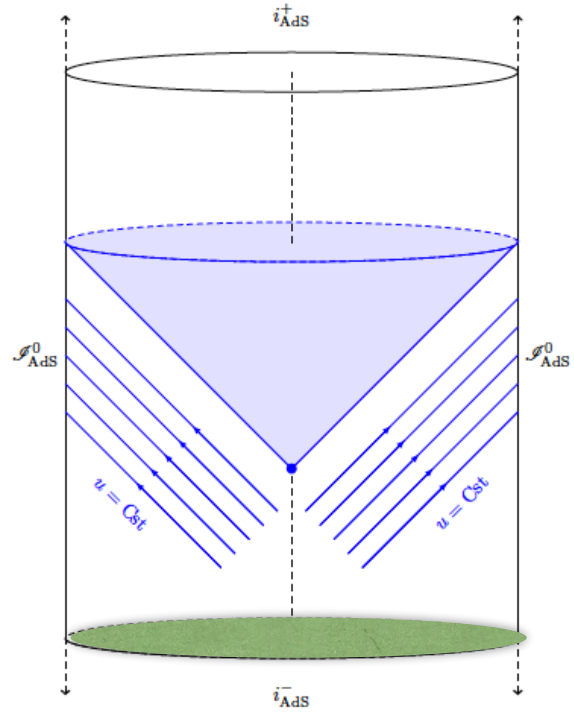


AdS case $\Lambda < 0$.

Glued at the boundary

(identification of codimension 2 boundary metrics)

$$q_{AB} = q'_{AB}$$



AdS case $\Lambda < 0$.

Fix the initial data of both AdS's.

Then the Cauchy problem of the first AdS is well-defined.

Related example: [Almheiri, Hartman, Maldacena, Shaghoulian, Tajdini, 2019]

Flat space renormalization: corner (codimension 2) contributions

Flat limit:

$$\mathbf{\Theta}_{\text{ren}}[\delta\phi;\phi]|_{\mathscr{I}} = \frac{\mathrm{d}u\,\mathrm{d}^2\Omega}{16\pi G}\left[\frac{3}{2\Lambda}\partial_u(N^{TF}_{AB}\delta q^{AB}) + \frac{1}{2}\Big(N^{AB}_{TF} + \frac{1}{2}R[q]q^{AB}\Big)\delta C_{AB} - D_{(A}\mathcal{U}_{B)}\delta q^{AB}\right] + \frac{\delta \boldsymbol{L}_\circ}{\delta q^{AB}}\delta q^{AB} + \mathcal{O}(\Lambda),$$

Divergence is cured by adding a corner term to the action

$$\int_{\mathscr{I}} \mathrm{d}^2x\,\mathrm{d}u\,L_\circ = \int_{\partial\mathscr{I}_+} \mathrm{d}^2x\,L_C - \int_{\partial\mathscr{I}_-} \mathrm{d}^2x\,L_C.$$

$$\boldsymbol{L}_\circ = \mathrm{d}\boldsymbol{L}_C$$

$$\boldsymbol{L}_C = \frac{d^2\Omega}{\sqrt{\mathring{q}}}L_C$$

We define the corner presymplectic potential as

$$\delta L_\circ = \partial_u\left(\frac{\delta L_C}{\delta q_{AB}}\delta q_{AB}\right) + \partial_u \Theta^C_\circ,$$

(non-standard)

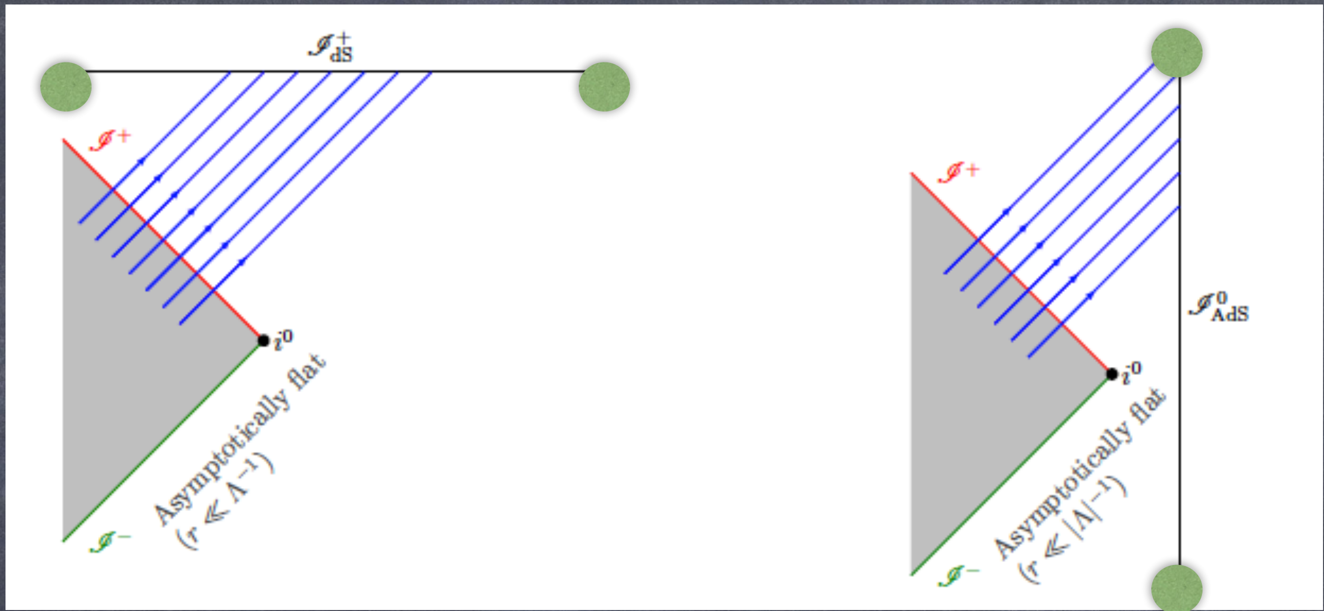
And the corner presymplectic form as

$$\mathbf{\Theta}^C_\circ = \mathrm{d}^2x\,\Theta^C_\circ, \qquad \boldsymbol{\omega}^C_\circ(\delta_1 q_{AB}, \delta_2 q_{AB}; q_{AB}) = \delta_1 \mathbf{\Theta}^C_\circ(\delta_2 q_{AB}; q_{AB}) - (1 \leftrightarrow 2).$$

Our new prescription:

$$\boxed{\mathbf{\Theta}_{\text{ren,tot}}|_{\mathscr{I}} = \mathbf{\Theta}_{\text{ren}}|_{\mathscr{I}} - \delta \mathrm{d}\boldsymbol{L}_C + \mathrm{d}\mathbf{\Theta}^C_\circ.}$$

With our prescription for taking into account the corner boundary term, the divergence is cancelled



The corner action is a kinetic term for the boundary metric!

$$L_C[q_{AB}, \boldsymbol{T}, \mathring{q}] \equiv \frac{\sqrt{\mathring{q}}}{64\pi G} C^{AB} C_{AB} = \frac{\sqrt{\mathring{q}}\ell^4}{64\pi G} \partial_u q_{AB} q^{AC} q^{BD} \partial_u q_{CD}.$$

It can be written in a covariant way in terms of the boundary foliation and measure

$$\boldsymbol{L}_\circ = \mathrm{d}\boldsymbol{L}_C$$

$$L_\circ = \frac{\sqrt{\mathring{q}}\ell}{64\pi G} \mathcal{L}_{\boldsymbol{T}} \left[\mathcal{L}_{\boldsymbol{T}} g^{(0)}_{ab} P^{ac} P^{bd} \mathcal{L}_{\boldsymbol{T}} g^{(0)}_{cd} \right].$$

$$P_{ab} = g^{(0)}_{ab} + \eta T_a T_b,$$

$$\eta \equiv -\mathrm{sgn}(\Lambda).$$

Flat Limit

- Solution space admits a smooth limit
- The asymptotic symmetry group Λ-BMS reduces to (generalized) BMS group
- After adding the corner term, the symplectic structure admits a smooth limit

$$\boxed{\boldsymbol{\omega}_{\text{ren,tot}}[\delta_1\phi,\delta_2\phi;\phi]\Big|_{\mathscr{I}} \overset{(\Lambda\to 0)}{=} \frac{\mathrm{d}u\,\mathrm{d}^2\Omega}{16\pi G}\left[\frac{1}{2}\delta_1\left(N^{AB}+\frac{1}{2}R[q]q^{AB}\right)\wedge\delta_2 C_{AB}-\delta_1\left(D_{(A}\mathcal{U}_{B)}\right)\wedge\delta_2 q^{AB}\right].}$$

This is exactly the renormalized symplectic structure discussed earlier

Conclusions

- The extended BMS algebra is realized without anomaly at the past of future null infinity. The asymptotic symmetry group at spatial infinity therefore includes the extended BMS group.
- The definition of extended BMS generators leads to a unique definition of the quantized angular momentum, which differs from many existing prescriptions.
- The flat BMS group admits a natural extension to (A)dS: the Λ-BMS group(oid). It is the asymptotic symmetry group of leaky Al(A)dS spacetimes without any boundary condition except a boundary gauge fixing condition (determined by a foliation and a measure) that does not constraint the Cauchy problem.
- The Λ-BMS_4 algebra is realized by the Dirac bracket of surface charges. There is a 1-1 map between the number of symmetries and number of flux-balance laws.
- The flat limit of the Λ-BMS_4 surface charge algebra reproduces the BMS4 surface charge algebra provided a covariant corner term (a kinetic term for the metric on the celestial sphere) is included in the variational principle and, accordingly, in the symplectic structure.

Corner term contribution to the Dirac bracket in (A)dS

$$\boldsymbol{\omega}_{\text{ren,tot}}|_{\mathscr{I}} = \boldsymbol{\omega}_{\text{ren}}|_{\mathscr{I}} + \mathrm{d}\boldsymbol{\omega}_{\circ}^{C}.$$

$$\mathrm{d}\boldsymbol{k}_{\xi,\text{ren}}^{\text{tot}}[\delta\phi;\phi]\Big|_{\mathscr{I}} = \boldsymbol{\omega}_{\text{ren,tot}}[\delta_\xi\phi,\delta\phi;\phi]\Big|_{\mathscr{I}},$$

$$\delta\!\!\!/ H_{\xi,\text{tot}}^{\Lambda\text{-BMS}}[\phi] = \delta H_{\xi}^{\Lambda\text{-BMS}}[\phi] + \Xi_{\xi}^{\Lambda\text{-BMS}}[\delta\phi;\phi] + \int_{S^2_\infty} \mathrm{d}^2\Omega\ \omega_{\circ}^{C}[\delta_\xi q_{AB},\delta q_{AB}].$$

It leads to a non-trivial 2-cocycle

$$\begin{aligned}
&\{H_{\xi}^{\Lambda\text{-BMS}}[\phi], H_{\chi}^{\Lambda\text{-BMS}}[\phi]\}_{\star,\text{tot}} \\
&\qquad = \delta_\chi H_{\xi}^{\Lambda\text{-BMS}}[\phi] + \Xi_{\chi}^{\Lambda\text{-BMS}}[\delta_\xi\phi;\phi] + \int_{S^2_\infty} \mathrm{d}^2\Omega\, \omega_{\circ}^{C}[\delta_\chi q_{AB},\delta_\xi q_{AB}] \\
&\qquad = H_{[\xi,\chi]_\star}^{\Lambda\text{-BMS}}[\phi] + \underbrace{\int_{S^2_\infty} \mathrm{d}^2\Omega\, \omega_{\circ}^{C}[\delta_\chi q_{AB},\delta_\xi q_{AB}]}_{K_{\xi;\chi}^{\Lambda\text{-BMS,tot}}[q_{AB}]},
\end{aligned}$$