Okinawa Institute of Science and Technology, Quantum Gravity Unit
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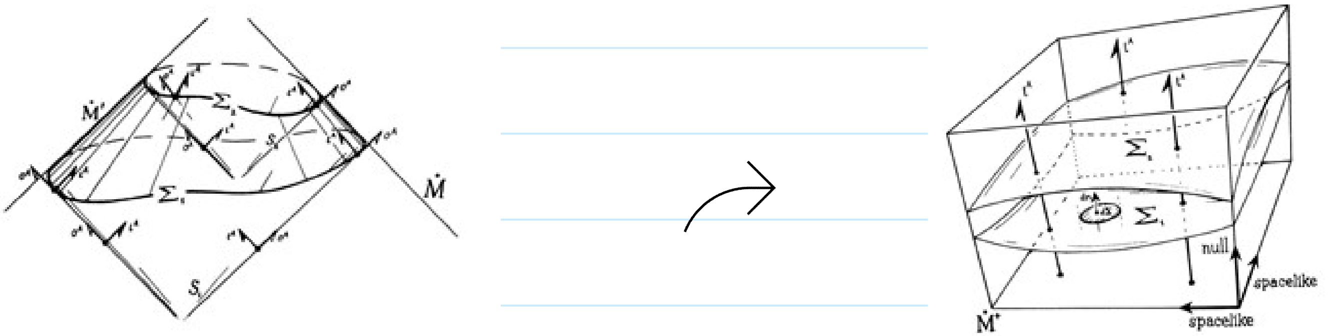
Coadjoint representation of BMS4 on celestial Riemann surfaces



Penrose 1967

Glenn Barnich

Physique théorique et mathématique

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Contents

- Coadjoint representation for semi-direct product groups
- Construction for BMS_4 on sphere & punctured plane
- Identification in asymptotically flat spacetimes at $\mathcal{J}^+$
- Perspectives

in collaboration with R. Ruzziconi, to appear

(C. Troessaert, B. Oblak, P. Mao)

Coadjoint representation & semi-direct product groops

adjoint $\quad \mathfrak{g} : [e_a, e_b] = f^c{}_{ab}\, e_c \qquad (\mathrm{ad}\, e_a)^b{}_c = f^b{}_{ac} \iff \mathrm{ad}\, e_a(e_b) = f^c_{ab}\, e_c$

coadjoint $\quad \mathfrak{g}^* : \langle e_*^b, e_a \rangle = \delta^b_a \qquad (\mathrm{ad}^* e_a) = -(\mathrm{ad}\, e_a)^T \iff \mathrm{ad}^* e_a(e_*^b) = -f^b{}_{ac}\, e_*^c$

group $\quad \mathrm{Ad}_g\, e_a = g\, e_a\, g^{-1}, \qquad \mathrm{Ad}^*_g = g\, e_*^b\, g^{-1}$

semi-direct product $\quad G \ltimes_\sigma A : \quad (f,\alpha)\cdot(g,\beta) = (f\cdot g,\ \alpha + \sigma_f(\beta)) \qquad \sigma$: representation

ISO(3), ISO(3,1), $\qquad A$: abelian ideal

BMS_3, BMS_4 ... $\qquad \mathfrak{g} \oplus_\Sigma A : \quad [(X,\alpha),(Y,\beta)] = ([X,Y],\ \Sigma_X \beta - \Sigma_Y \alpha)$

$$\mathrm{Ad}_{(f,\alpha)}(X,\beta) = (\mathrm{Ad}_f X,\ \sigma_f \beta - \Sigma_{\mathrm{Ad}_f X}\, \alpha)$$

$$\mathrm{ad}_{(X,\alpha)}(Y,\beta) = ([X,Y],\ \Sigma_X \beta - \Sigma_Y \alpha)$$

dual space $\quad \mathfrak{g}^* \oplus A^* \qquad \langle (j,p), (X,\alpha) \rangle = \langle j, X \rangle + \langle p, \alpha \rangle$

terminology $\quad j$: angular momentum $\qquad p$: linear momentum $\qquad$ BMS: add "super"

X : inf. rotation $\qquad \alpha$: inf. translation

ingredients $\quad \times : A \oplus A^* \to \mathfrak{g}^* \quad : \quad \langle \alpha \times p, X \rangle = \langle p, \Sigma_X \alpha \rangle$

change in angular momentum due to a translation

$$\sigma^* : G \times A^* \to A^* \quad : \quad \langle \sigma_f^* p, \alpha \rangle = \langle p, \sigma_{f^{-1}} \alpha \rangle$$

coadjoint representation $\quad \mathrm{Ad}^*_{(f,\alpha)}(j,p) = \left(\mathrm{Ad}^*_f j + \alpha \times \sigma^*_f p, \; \sigma^*_f p \right)$

$$\mathrm{ad}^*_{(X,\alpha)}(j,p) = \left(\mathrm{ad}^*_X j + \alpha \times p, \; \Sigma^*_X p \right)$$

Poincaré & BMS_4 algebra at $\mathcal{J}^+$

Poincaré generators $\mathbb{R}^{3,1}$ $\quad L^{ab} = -\left(x^a \frac{\partial}{\partial x_b} - x^b \frac{\partial}{\partial x_a}\right)$, $P^a = \frac{\partial}{\partial x^a}$ $\qquad \eta_{ab} = \mathrm{diag}(1,-1,-1,-1)$

structure constants
$$\begin{cases} [L^{ab}, L^{cd}] = -\left(\eta^{bc} L^{ad} - \eta^{ac} L^{bd} - \eta^{bd} L^{ac} + \eta^{ad} L^{bc}\right) \\ [P^a, L^{bc}] = -\left(\eta^{ab} P^c - \eta^{ac} P^b\right) \end{cases}$$

Boost & rotation generators
$$\begin{cases} L_z = L^{12}, \quad L^{\pm} = \pm i\left(L^{23} + L^{13}\right) ; \quad K_z = L^{30}, \quad K^{\pm} = \mp i L^{20} - L^{10} \\ H = P^0, \quad P_z = -\frac{1}{2} P^3, \quad P^{\pm} = -\frac{1}{2}\left(i P^2 \pm P^1\right) \end{cases}$$

structure constants $\quad [L^+, L^-] = 2 i L_z$, $[L_z, L^{\pm}] = \pm i L^{\pm}$, ...

spherical & retarded time $r=\sqrt{\sum_i (x^i)^2}$, $u=x^0-r$, $r\cos\theta = x^3$, $r\sin\theta\, e^{i\phi} = x^1+ix^2$

$$\boxed{L_z = \partial_\phi,\quad L^\pm = e^{\pm i\phi}\left[\partial_\theta \pm i\cot\theta\,\partial_\phi\right]},\qquad K_z = -\left(1+\tfrac{u}{r}\right)\cancel{\cos\theta\, r\partial_r} + \cos\theta\,(u\partial_u) + \left(1+\cancel{\tfrac{u}{r}}\right)\sin\theta\,\partial_\theta$$

$$K^\pm = e^{\pm i\phi}\left[\cancel{\left(1+\tfrac{u}{r}\right)\sin\theta\, r\partial_r} - \sin\theta\,(u\partial_u) + \left(1+\cancel{\tfrac{u}{r}}\right)\cos\theta\,\partial_\theta \pm \left(1+\cancel{\tfrac{u}{r}}\right)\frac{i}{\sin\theta}\partial_\phi\right]$$

$$H=\partial_u,\quad -2P_z = \cos\theta\left(-\cancel{\partial_r}+\partial_u\right)+\cancel{\tfrac{1}{r}\sin\theta\,\partial_\theta},\quad \pm 2P^\pm = e^{\pm i\phi}\left[\sin\theta\left(\cancel{\partial_r}-\partial_u\right)+\cancel{\tfrac{1}{r}\cos\theta\,\partial_\theta} \pm \cancel{\tfrac{1}{r\sin\theta}\partial_\phi}\right]$$

Simplification 1: $\mathcal{I}^+$ $r = \text{cte} \to \infty$ Simplification 2: cut $u=0$ of $\mathcal{I}^+$

$$\boxed{\begin{array}{lll} K_z = \sin\theta\,\partial_\theta, & K^\pm = e^{\pm i\phi}\left[\cos\theta\,\partial_\theta \pm \dfrac{i}{\sin\theta}\partial_\phi\right] & \\ H=1 \sim {}_0Y_{00} & P_z = -\tfrac{1}{2}\cos\theta \sim {}_0Y_{10} & P^\pm = \mp\tfrac{1}{2}e^{\pm i\phi}\sin\theta \sim {}_0Y_{1\pm 1} \end{array}}$$

4 lowest harmonics

Poincaré algebra $f = H, P_z, P^\pm$

$[L_z, f] = L_z(f)$, $[L^\pm, f] = L^\pm(f)$

$[K_z, f] = K_z(f) - \cos\theta\, f$, $[K^\pm, f] = K^\pm(f) + e^{\pm i\phi}\sin\theta\, f$

BMS$_4$ algebra: $f \in C^\infty(S^2)$

Sachs Phys Rev 1962

Simplification 3: stereographic coordinates on the sphere

$$\boxed{\zeta = \cot\frac{\theta}{2}\, e^{-i\phi} \qquad ds^2 = -2(P_S \bar{P}_S)\, d\zeta\, d\bar{\zeta} \qquad P_S = \frac{1}{R\sqrt{2}}(1+\zeta\bar{\zeta})}$$

Lorentz algebra $\quad l_m = \zeta^{1-m}\partial\,,\quad \bar{l}_m = \bar{\zeta}^{1-m}\bar{\partial} \qquad sl(2,\mathbb{R}) \oplus sl(2,\mathbb{R})$

$$L_z = -i(l_0 - \bar{l}_0)\,,\quad K_z = -(l_0 + \bar{l}_0)\,,\quad L^+ = l_1 + \bar{l}_{-1}\,,\quad L^- = \bar{l}_1 + l_{-1}\,,\quad K^+ = -(\bar{l}_{-1} - l_1)\,,\quad K^- = -(l_{-1} - \bar{l}_1)$$

action of Lorentz on (super)-translations

$$H = 1\,,\quad P_z = \frac{1-\zeta\bar{\zeta}}{2(1+\zeta\bar{\zeta})}\,,\quad P^+ = -\frac{\bar{\zeta}}{1+\zeta\bar{\zeta}}\,,\quad P^- = \frac{\zeta}{1+\zeta\bar{\zeta}}$$

$$[K_z, f] = K_z(f) + \frac{1-\zeta\bar{\zeta}}{1+\zeta\bar{\zeta}} f \qquad [K^+, f] = K^+(f) + \frac{2\bar{\zeta}}{1+\zeta\bar{\zeta}} f\,,\quad [K^-, f] = K^-(f) + \frac{2\zeta}{1+\zeta\bar{\zeta}} f$$

Coadjoint representation of BMS_4: general structure

2d conformally flat S aim: unified description for sphere & punctured plane

$$ds^2 = -2(P\bar{P})^{-1} d\xi \, d\bar{\xi} \quad \begin{cases} \xi' = \xi'(\xi) \quad \bar{\xi}' = \bar{\xi}'(\bar{\xi}) & \text{conformal coordinate trsf.} \\ P'(x) = P(x) e^{-E(x)}, \quad \bar{P}'(x) = \bar{P}(x) e^{-\bar{E}(x)} & \text{complex Weyl rescaling} \end{cases}$$

$x = (\xi, \bar{\xi})$

zweibeins $\quad ds^2 = e^a_\mu dx^\mu \eta_{ab} e^b_\nu dx^\nu \quad \eta_{ab} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \quad e_1^\mu \frac{\partial}{\partial x^\mu} = P \frac{\partial}{\partial \xi} \quad e_2^\mu \frac{\partial}{\partial x^\mu} = \bar{P} \frac{\partial}{\partial \bar{\xi}}$

conformal fields $\quad \phi'_{h,\bar{h}}(x') = \left(\frac{\partial \xi}{\partial \xi'}\right)^h \left(\frac{\partial \bar{\xi}}{\partial \bar{\xi}'}\right)^{\bar{h}} \phi_{h,\bar{h}}(x) \qquad P'(x') = P(x) e^{-E(x')} \frac{\partial \xi'}{\partial \xi}$ combined trsf.

weighted scalars $\quad \eta'^{s,w}(x') = e^{w E_R(x')} e^{-is E_I(x')} \eta^{s,w}(x)$

interpolation map $\quad \eta^{s,w} = P^h \bar{P}^{\bar{h}} \phi_{h,\bar{h}}$

$s = h - \bar{h}, \quad w = -(h + \bar{h}) \qquad h = \frac{s-w}{2}, \quad \bar{h} = -\frac{s+w}{2}$

Held, Rosados, Newman JMP 1970

Lorentz group & sphere

D'Hoker, Phong Rev. Mod. Phys. 1988

covariant derivative ∇ : $\Gamma^{\xi}_{\xi\xi} = -\partial \ln(P\bar{P})$ $\Gamma^{\bar{\xi}}_{\bar{\xi}\bar{\xi}} = -\bar{\partial} \ln(P\bar{P})$

$$\Gamma^{\xi'}_{\xi'\xi'}(x') = \Gamma^{\xi}_{\xi\xi}(x)\frac{\partial\xi}{\partial\xi'} + \frac{\partial\xi'}{\partial\xi}\frac{\partial^2\xi}{\partial\xi'\partial\xi'} + 2\partial' E_R(x')$$

introduce Weyl connection D: $W'(x') = \frac{\partial\xi}{\partial\xi'} W + 2\partial' E_R(x')$, $\bar{W}'(x') = \frac{\partial\bar{\xi}}{\partial\bar{\xi}'}\bar{W}(x) + 2\bar{\partial}' E_R(x')$

$$\underbrace{D\phi_{h,\bar{h}}}_{(h+1,\bar{h})} = [\nabla + hW]\phi_{h,\bar{h}}, \quad \underbrace{\bar{D}\phi_{h,\bar{h}}}_{(h,\bar{h}+1)} = [\bar{\nabla} + \bar{h}W]\phi_{h,\bar{h}}$$

$\partial, \nabla, D \equiv \partial_\xi, D_\xi, D_\xi$
$\bar{\partial}, \bar{\nabla}, \bar{D} = \partial_{\bar{\xi}}, D_{\bar{\xi}}, D_{\bar{\xi}}$

weighted scalars

$$\eth\, \eta^{s,w} = P^{h+1}\bar{P}^{\bar{h}}\nabla\phi_{h,\bar{h}} = P\bar{P}^{-s}\partial(\bar{P}^{s}\eta^{s,w}), \quad \bar{\eth}\,\eta^{s,w} = P^{h}\bar{P}^{\bar{h}+1}\bar{\nabla}\phi_{h,\bar{h}} = \bar{P}P^{s}\bar{\partial}(P^{-s}\eta^{s,w})$$

Weyl covariant

$$\underbrace{\mathcal{D}\eta^{s,w}}_{[s+1,\,w-1]} = P^{h+1}\bar{P}^{\bar{h}} D\phi_{h,\bar{h}} = \left(\eth + \left(\frac{s-w}{2}\right)PW\right)D\phi_{h,\bar{h}}, \quad \underbrace{\bar{\mathcal{D}}\eta^{s,w}}_{[s-1,\,w-1]} = P^{h}\bar{P}^{\bar{h}+1}\bar{D}\phi_{h,\bar{h}} = \left[\bar{\eth} - \left(\frac{w+s}{2}\right)\bar{P}\bar{W}\right]\phi_{h,\bar{h}}$$

$$[\mathcal{D}, \bar{\mathcal{D}}]\eta^{s,w} = -\frac{s}{2}R_s\eta^{s,w} - P\bar{P}\left(\frac{s-w}{2}\bar{\partial}W + \frac{s+w}{2}\partial\bar{W}\right)\eta^{s,w}$$

R_s : scalar curvature

Ingredients

(super-)translation	$T : [0,1]$	$\tilde{T} : (-\frac{1}{2}, -\frac{1}{2})$	real
(super-) rotation	$y : [-1,1]$	$\tilde{y} : (-1, 0)$	$\bar{\mathcal{D}} y = 0 \Leftrightarrow \bar{D}\tilde{y} = 0$
	$\bar{y} : [1,1]$	$\tilde{\bar{y}} : (0,-1)$	$\mathcal{D}\bar{y} = 0 \Leftrightarrow D\tilde{\bar{y}} = 0$
(super-) momentum	$P : [0,-3]$	$\tilde{P} : (\frac{3}{2}, \frac{3}{2})$	real
(super-) angular momentum	$J : [-1,-3]$	$\tilde{J} : (1, 2)$	$J \sim J + \mathcal{D} \underset{[-2,-2]}{\mathcal{L}}$, $\tilde{J} \sim \tilde{J} + D \underset{(0,2)}{\tilde{\mathcal{L}}}$
	$\bar{J} : [1,-3]$	$\tilde{\bar{J}} : (2,1)$	$\bar{J} \sim \bar{J} + \bar{\mathcal{D}} \underset{[2,-2]}{\bar{\mathcal{L}}}$, $\tilde{\bar{J}} \sim \tilde{\bar{J}} + \bar{D} \underset{(2,0)}{\tilde{\bar{\mathcal{L}}}}$

In all relations, weights / dimensions are such that Weyl connection drops out!

$\mathcal{D} \to \eth$ $\quad D \to \partial$ $\quad$ simplest description in terms of conformal fields

bms_4 algebra $\quad [(\mathcal{Y}_1, \bar{\mathcal{Y}}_1, \mathcal{J}_1), (\mathcal{Y}_2, \bar{\mathcal{Y}}_2, \mathcal{J}_2)] = (\hat{\mathcal{Y}}, \hat{\bar{\mathcal{Y}}}, \hat{\mathcal{J}})$

$$\hat{\mathcal{Y}} = \mathcal{Y}_1 \eth \mathcal{Y}_2 - \mathcal{Y}_2 \eth \mathcal{Y}_1 \qquad \hat{\mathcal{J}} = \mathcal{Y}_1 \eth \mathcal{J}_2 - \tfrac{1}{2} \eth \mathcal{Y}_1 \mathcal{J}_2 - (1 \leftrightarrow 2) + c.c.$$

subalgebra $\mathfrak{g}$ $\quad (\mathcal{Y}, \bar{\mathcal{Y}}, 0) \qquad (\tilde{\mathcal{Y}}, \bar{\tilde{\mathcal{Y}}}, 0)$

(Lorentz, witt⊕witt)

representation of $\mathfrak{g}$ on $\eta^{s,w}$ $\qquad$ on $\phi_{h,\bar{h}}$

$$\mathcal{Y} \cdot \eta^{s,w} = \mathcal{Y} \eth \eta^{s,w} + \frac{s-w}{2} \eth \mathcal{Y} \, \eta^{s,w} \qquad \tilde{\mathcal{Y}} \cdot \phi_{h,\bar{h}} = \tilde{\mathcal{Y}} \eth \phi_{h,\bar{h}} + h \eth \tilde{\mathcal{Y}} \phi_{h,\bar{h}}$$

$$\bar{\mathcal{Y}} \cdot \eta^{s,w} = \bar{\mathcal{Y}} \bar{\eth} \eta^{s,w} - \frac{s+w}{2} \bar{\eth} \bar{\mathcal{Y}} \, \eta^{s,w} \qquad \bar{\tilde{\mathcal{Y}}} \cdot \phi_{h,\bar{h}} = \bar{\tilde{\mathcal{Y}}} \bar{\eth} \phi_{h,\bar{h}} + \bar{h} \bar{\eth} \bar{\tilde{\mathcal{Y}}} \phi_{h,\bar{h}}$$

$$\mathcal{E}_{\times} \alpha = (\mathcal{Y}, \bar{\mathcal{Y}}) \cdot \mathcal{T}^{[0,1]} \qquad \mathcal{E}_{\times} \alpha = (\tilde{\mathcal{Y}}, \bar{\tilde{\mathcal{Y}}}) \cdot \tilde{\mathcal{T}}^{(-1/2,-1/2)}$$

action of inf rotation on translations

$\mathfrak{bms}_4^*$ dual space $([\mathcal{J}], [\bar{\mathcal{J}}], \mathcal{P})$ $\quad ([\tilde{\mathcal{J}}], [\tilde{\bar{\mathcal{J}}}], \tilde{\mathcal{P}})$

$(0,0)$; $[0,-2]$

pairing $\quad \langle ([\mathcal{J}],[\bar{\mathcal{J}}],\mathcal{P}); (\mathcal{Y},\bar{\mathcal{Y}},\mathcal{T})\rangle = \int_S d\mu \,[\,\bar{\mathcal{J}}\mathcal{Y} + \mathcal{J}\bar{\mathcal{Y}} + \mathcal{P}\mathcal{T}\,]$, $d\mu(\zeta,\bar{\zeta}) = \frac{iC}{P\bar{P}}\, d\zeta \wedge d\bar{\zeta}$

$$\langle ([\tilde{\mathcal{J}}],[\tilde{\bar{\mathcal{J}}}],\tilde{\mathcal{P}}), (\tilde{\mathcal{Y}},\tilde{\bar{\mathcal{Y}}},\tilde{\mathcal{T}})\rangle = \int_S d\tilde{\mu}\,[\,\tilde{\bar{\mathcal{J}}}\tilde{\mathcal{Y}} + \tilde{\mathcal{J}}\tilde{\bar{\mathcal{Y}}} + \tilde{\mathcal{P}}\tilde{\mathcal{T}}\,] \qquad d\tilde{\mu} = iC\, d\zeta \wedge d\bar{\zeta}$$

assumption : pairing annihilates total $\eth, \bar{\eth}$ $(\partial, \bar{\partial})$ derivatives
non-degenerate $\longrightarrow$ integrations by parts

$$\mathrm{ad}^*_{(\mathcal{Y},\bar{\mathcal{Y}},\mathcal{T})}\mathcal{J} = \bar{\mathcal{Y}}\bar{\eth}\mathcal{J} + 2\bar{\eth}\bar{\mathcal{Y}}\mathcal{J} + \underbrace{\eth(\mathcal{Y}\mathcal{J})}_{=\mathrm{ad}^*_{\bar{\mathcal{Y}}}\mathcal{J}\sim 0} + \underbrace{\tfrac{1}{2}\mathcal{T}\bar{\eth}\mathcal{P} + \tfrac{3}{2}\bar{\eth}\mathcal{T}\mathcal{P}}_{\alpha \times P} ,$$

$$\mathrm{ad}^*_{(\mathcal{Y},\bar{\mathcal{Y}},\mathcal{T})}\mathcal{P} = \underbrace{\mathcal{Y}\eth\mathcal{P} + \tfrac{3}{2}\eth\mathcal{Y}\,\mathcal{P} + \text{c.c.}}_{\mathcal{E}^*_X P}$$

work out formulas for the group ✓

Realization on the sphere

stereographic coord. $\quad \xi = \cot\frac{\theta}{2}\, e^{-i\phi} \qquad ds^2 = -2(P_\xi \bar{P}_\xi)\, d\xi\, d\bar{\xi} \qquad P_\xi = \frac{1}{R\sqrt{2}}(1+\xi\bar{\xi})$

globally well-defined conf. coord. trsf $\quad \xi' = \frac{a\xi+b}{c\xi+d}, \quad ad-bc=1, \quad a,b,c,d \in \mathbb{C} \qquad \frac{\partial\xi}{\partial\xi'} = (c\xi+d)^2$

compensating Weyl trsf. $\quad e^{E_R(x')} = \frac{1+\xi\bar{\xi}}{|a\xi+b|^2+|c\xi+d|^2} \qquad e^{iE(x')} = \frac{\bar{c}\bar{\xi}+\bar{d}}{c\xi+d} \qquad w:$ boost weight

Pairing $\quad \langle K^{s,-w-2}, \eta^{s,w} \rangle = \frac{1}{4\pi R^2}\int_{S^2} \frac{i\,d\xi \wedge d\bar{\xi}}{P_\xi \bar{P}_\xi}\, \overline{K^{s,-w-2}}\, \eta^{s,w} \qquad C = (4\pi R^2)^{-1}$

assumptions ✓

$$\frac{1}{4\pi R^2}\int_{S^2} \frac{i\,d\xi \wedge d\bar{\xi}}{P_\xi \bar{P}_\xi} = \frac{1}{4\pi}\int_0^\pi d\theta \sin\theta \int_0^{2\pi} d\phi = 1$$

adjoint repres. group

$$\gamma'(x') = e^{\Xi_R(x')} e^{i\Xi_I(x')} \gamma(x)$$

$$\beta'(x') = e^{\Xi_R(x')} \left(\beta - (\gamma\, \partial\alpha - \tfrac{1}{2} \partial\alpha\, \partial\gamma + c.c.\right)(x)$$

coadjoint repres. group

$$J'(x') = e^{-3\Xi_R(x')} e^{i\,\Xi_I(x')} \left(J + \tfrac{1}{2} J \partial P + \tfrac{3}{2} \partial J P \right)(x)$$

$$P'(x') = e^{-3\Xi_R(x')} P(x)$$

in terms of conf. fields

$$\tilde{\gamma}'(\xi') = (c\xi + d)^{-2} \tilde{\gamma}(\xi)$$

$$\tilde{\beta}'(x') = (c\xi+d)^{-1} (\bar{c}\bar{\xi}+\bar{d})^{-1} \left(\tilde{\beta} - \tilde{\gamma}\, \partial\tilde{\alpha} - \tfrac{1}{2} \tilde{\alpha}\, \partial\tilde{\gamma} + c.c.\right)(x)$$

$$\tilde{J}'(x') = (c\xi+d)^{2} (\bar{c}\bar{\xi}+\bar{d})^{4} \left(\tilde{J}(x) + \left(\tfrac{1}{2} \tilde{J}\, \partial\tilde{P} + \tfrac{3}{2} \partial\tilde{J}\, \tilde{P}\right)(x)\right)$$

$$\tilde{P}'(x') = (c\xi+d)^{3} (\bar{c}\bar{\xi}+\bar{d})^{3} \tilde{P}(x)$$

Expansions : spin weighted spherical harmonics $\ _sZ_{jm}$ unnormalized

$\ _sY_{jm}$ normalized

conformal Killing eq. on S^2 $\quad \bar{\eth} \mathcal{Y}^{[-1,1]} = 0 = \eth \bar{\mathcal{Y}}^{[1,-1]}$

$$\mathcal{Y}_m = -R\sqrt{2}\ {}_{-1}Z_{1,m} \qquad m = -1,0,1 \qquad \mathcal{Y} = \sum_{m=-1}^{1} y_m \mathcal{Y}_m$$

$$T_{jm} = {}_0Z_{j,m} \qquad T = \sum_{j,|m|\le j} t_{j,m} T_{j,m}\,, \qquad \bar{t}_{jm} = (-1)^m t_{j,-m}$$

dual basis $\quad \mathcal{Y}^m_* = \dfrac{-6}{R\sqrt{2}\,(1+m)!\,(1-m)!}\ {}_{-1}Z_{1,m} \qquad \mathcal{J} = \sum_{m=-1}^{1} j_m \mathcal{Y}^m_*$

$$T^{j,m}_* = \frac{(2j+1)!\,(2j)!}{j!\,j!\,(j+m)!\,(j-m)!}\ {}_0Z_{j,m} \qquad \mathcal{P} = \sum_{j,|m|\le j} p_{j,m} T^{j,m}_*\,, \qquad \bar{p}_{j,m} = (-1)^m p_{j,-m}$$

NB: conformal fields : $\tilde{\mathcal{Y}}_m = \mathcal{Y}_m P_S = \zeta^{1-m} \Rightarrow [\tilde{\mathcal{Y}}_m, \tilde{\mathcal{Y}}_n] = (m-n)\,\tilde{\mathcal{Y}}_{m+n}$

$\longrightarrow$ all other structure constants can be worked out explicitly (ugly)

Remark (i) Penrose & Rindler Vol I, section 4.15

In the study of spin-weighted spherical harmonics it is useful to contemplate the following array:

j \ s	…	$-\frac{5}{2}$	-2	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	…
0							1						
$\frac{1}{2}$						2		2					
1		$\overleftarrow{\eth'}$			3		3		3		$\overrightarrow{\eth}$		
$\frac{3}{2}$				4		4		4		4			
2			5		5		5		5		5		
$\frac{5}{2}$	⋰	6		6		6		6		6		6	⋱
⋮													

(4.15.60)

The numbers in this triangular array (which extends indefinitely downwards) represent the complex *dimensions* of the various spaces of spin-weighted spherical harmonics, as discussed in (4.15.43) *et seq.* Each of these spaces is characterized by its values of s and j, as shown. The dimension *zero* is assigned wherever a blank space appears in the array. The operator $\eth$ carries us a step of one s-unit to the right and $\eth'$ one s-unit to the left. (From our earlier discussion, the j-value is not affected by $\eth$ or $\eth'$.) Whenever such a step carries us off the array, the result of the operator $\eth$ or $\eth'$ is zero. Note that the dimension remains constant whenever it does not drop to, or increase from, zero.

$w \geq |s|$ $\qquad \eth^{w+s+1}\eta^{s,w}$ $\qquad \bar\eth^{w+s+1}\eta^{s,w}$

$[w+1, s-1]$ $\qquad [-w-1, -s-1]$

definite boost weight

$\bar\eth\eta = 0 \iff \eth^3\eta = 0$

$\eth\bar\eta = 0 \iff \bar\eth^3\bar\eta = 0$ $\qquad$ same solutions

dual situation $w \leq -|s|-2$

$\eth^{s-w-1}\kappa^{w+1,s-1}$ $\qquad \bar\eth^{-s-w-1}\kappa^{-w-1,-s-1}$

$[s,w]$ $\qquad [s,w]$ $\qquad$ definite boost weight

$\bar{\mathcal{J}} \sim \bar{\mathcal{J}} + \bar\eth\bar{\mathcal{I}} \iff \tilde{\bar{\mathcal{J}}} \sim \tilde{\bar{\mathcal{J}}} + \eth^3 \mathcal{M}$

$[2,-2]$ $\qquad [-2,0]$

same equivalence classes

Remark (ii) reduction to Poincaré

$\eth^2\mathcal{J} = 0 = \bar\eth^2\mathcal{J}$ $\qquad \mathcal{P} \sim \mathcal{P} + \eth^2\mathcal{N} + \bar\eth^2\bar{\mathcal{N}}$

$[-2,1]$ $\qquad [2,-1]$

Remark (iii)

$$\tilde{\mathcal{Y}}_m \cdot \phi_{h,\bar{h}} = \xi^{-m}\left(\xi \partial \phi_{h,\bar{h}} + h(1-m)\,\phi_{h,\bar{h}}\right)$$
$$\bar{\tilde{\mathcal{Y}}}_m \cdot \phi_{h,\bar{h}} = \bar{\xi}^{-m}\left(\bar{\xi}\bar{\partial}\,\phi_{h,\bar{h}} + \bar{h}(1-m)\,\phi_{h,\bar{h}}\right)$$

Goldberg et al. JMP 1967 $\quad {}_sY_{j,m} \quad j \leq L \quad \longleftrightarrow \quad {}_sZ^{L}_{m_1,m_2} = (1+\xi\bar{\xi})^{-L}\,\xi^{L-s-m_1}\,\bar{\xi}^{L+s-m_2}$

invertible $\qquad 0 \leq m_1 \leq L-s, \quad 0 \leq m_2 \leq L+s$

overcomplete set of functions, look like expansions on the punctured plane

when transforming to associated conformal fields structure constants look like those on the punctured plane, up to corrections.

Realization on punctured plane

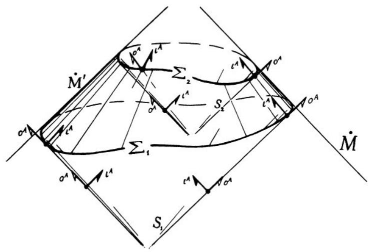
- Weyl trsf $e^{-\Phi(\xi,\bar\xi)} = \frac{\sqrt{2}}{1+\xi\bar\xi}$ $\xi = R^{-1} z$ $ds^2 = -2\, dz\, d\bar z$

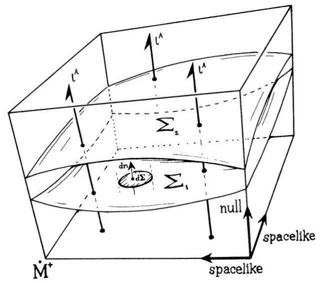
- 2-punctures: remove points at origin & infinity $\mathbb{C}_0$

- on the level of the algebra, look at the algebra of all infinitesimal local conformal trsf.

Not the Lie algebra of globally well-defined trsf.

For asymptotically flat spaces, $\dot M$ is in fact a null hypersurface [7]. The structure of $\dot M$ is essentially the same as for Minkowski space (Figure 4). We shall omit the three points I^-, I^0, I^+ here. Then $\dot M$ consists of two portions, each of which is topologically a "cylinder" $S^2 \times E^1$. We are concerned, here, only with the future portion $\dot M^+$, and by judicious choice of conformal factor Ω, we can ensure that the geometry of $\dot M^+$ is as simple as possible. In fact, by taking one generator of $\dot M^+$ "back to infinity" we can open out the cylinder into a space with Euclidean three-space topology. Furthermore, it turns out that we can also make this three-space metrically *flat* (Figure 6). This will simplify matters considerably.





Penrose 1967 AMS

$P=1 \Rightarrow$ weighted scalars = conformal fields

$$e^{\Phi(x')} = \frac{\partial z'}{\partial z} \qquad e^{\Phi_R(x')} = \left(\frac{\partial z'}{\partial z}\frac{\partial \bar z'}{\partial \bar z}\right)^{1/2}, \quad e^{i\Phi_I(x')} = \left(\frac{\partial z'/\partial z}{\partial \bar z'/\partial \bar z}\right)^{1/2}$$

gravity : sphere $\to$ plane

CFT : plane $\to$ sphere

Coulomb gas ?

Expansions

$$\phi_{h,\bar h}(z,\bar z) = \sum_{k,l} a_{k,l}\ \tilde{\underset{h,\bar h}{z}}_{k,l}\,, \qquad \tilde{\underset{h,\bar h}{z}}_{k,l} = z^{-h-k}\ \bar z^{-\bar h-l}$$

$h,\bar h \in \mathbb{N} \Rightarrow k,l \in \mathbb{Z}$

$h,\bar h \in \frac{\mathbb{N}}{2} \Rightarrow k,l \in \frac{1}{2}+\mathbb{Z}$

(NS)

Pairing $\quad \widetilde{\langle \psi_{-\bar h+1,-h+1}, \phi_{h,\bar h} \rangle} = \mathrm{Res}_z\, \mathrm{Res}_{\bar z} \left[\overline{\psi_{-\bar h+1,-h+1}}\ \phi_{h,\bar h} \right]$

assumptions ✓

$\mathrm{Res}_z(\partial\phi) = 0 = \mathrm{Res}_{\bar z}(\bar\partial\phi)$

adjoint repr. group $\quad \tilde y'(z') = \left(\frac{\partial z}{\partial z'}\right)^{-1} \tilde y(z)$

$$\tilde\beta'(x') = \left(\frac{\partial z}{\partial z'}\right)^{-1/2} \left(\frac{\partial \bar z}{\partial \bar z'}\right)^{-1/2} \left(\tilde\beta - (\tilde y\,\partial\tilde\alpha - \tfrac{1}{2}\tilde\alpha\,\partial\tilde y + c.c.)\right)(x)$$

coadjoint repr group $\quad J'(x') = \left(\frac{\partial z}{\partial z'}\right)^{1} \left(\frac{\partial \bar z}{\partial \bar z'}\right)^{2} \left(J + \tfrac{1}{2}\tilde F\,\partial\tilde F + \tfrac{3}{2}\,\partial\tilde\partial\tilde F\right)(x)$

$$\tilde F'(x) = \left(\frac{\partial z}{\partial z'}\right)^{3/2} \left(\frac{\partial \bar z}{\partial \bar z'}\right)^{3/2} \tilde F(x)$$

to be used for conformal mapping.

Expansions $\quad \langle {}_{-\bar h+1,-h+1}\tilde z_{k',l'},\ {}_{h,\bar h}\tilde z_{k,l} \rangle = \delta^0_{l'+k}\,\delta^0_{k'+l}$

$$\tilde y_m = z^{1-m}, \qquad \tilde J_{k,l} = z^{1/2-k}\,\bar z^{1/2-l} \qquad m,\ \tfrac12+k,\ \tfrac12+l \in \mathbb{Z}$$

$$\tilde y_*^{m} = z^{-1}\,\bar z^{-2+m} \qquad \tilde J_*^{k,l} = z^{-3/2+l}\,\bar z^{-3/2+k}$$

$$\tilde y_m \cdot {}_{h,\bar h}\tilde z_{k,l} = -(hm+k)\,{}_{h,\bar h}\tilde z_{k+m,l}, \qquad \bar{\tilde y}_m \cdot {}_{h,\bar h}\tilde z_{k,l} = -(\bar h m+l)\,{}_{h,\bar h}\tilde z_{k,l+m}$$

Structure constants $\quad [\tilde y_m, \tilde y_n] = (m-n)\,\tilde y_{m+n} \qquad [\tilde y_m, \tilde J_{k,l}] = (\tfrac12 m-k)\,\tilde J_{k+m,l}$

$$[\bar{\tilde y}_m, \tilde J_{k,l}] = (\tfrac12 m-l)\,\tilde J_{k,l+m}$$

$$[\tilde y_m, \bar{\tilde y}_n] = 0 = [\tilde J_{k,l}, \tilde J_{r,s}]$$

coadjoint repr. algebra

$$\mathrm{ad}^*_{\tilde{y}_m}\tilde{y}^n_* = (-2m+n)\,\tilde{y}^{n-m}_*\ ,\qquad \mathrm{ad}^*_{\tilde{y}_m}\tilde{J}^{k,l}_* = \left(-\tfrac{3}{2}m+k\right)\tilde{J}^{k-m,l}_*\ ,$$

$$\mathrm{ad}^*_{\tilde{J}_{k,l}}\tilde{J}^{r,s}_* = \left(\frac{r-3k}{2}\right)\delta^s_l\, y^{r-k}_* + \left(\frac{s-3l}{2}\right)\delta^r_k\,\tilde{y}^{s-l}_*$$

Realization on cylinder

$$z = e^{-i\frac{2\pi}{L_1}w}\ ,\quad w = w_1 + i w_2\ ,\quad w_1 \sim w_1 + L_1\ ,\quad \phi^C_{h,\bar h}(w,\bar w) = \left(-i\frac{2\pi}{L_1}z\right)^h\left(i\frac{2\pi}{L_1}\bar z\right)^{\bar h}\phi_{h,\bar h}(z,\bar z)$$

use formulas for the group to map generators $\quad y^C_m = i\left(\frac{2\pi}{L_1}\right)^{-1} e^{i\frac{2\pi}{L_1}mw}\ \ldots$

same structure constants, obtained from ad^* still provide a representation but pairing issues ....

(Torus: $w_2^T \sim w_2^T + L_2$ $\quad e^{i\frac{2\pi}{L_1}(w_1+iw_2)} \to e^{i\frac{2\pi}{L_1}w_1^T}\, e^{i\frac{2\pi}{L_2}w_2^T}$

$$w_2 = +i\frac{L_2}{L_1}w_2^T\ )$$

Identification in non-radiative asymptotically flat spacetimes at $\mathscr{I}^+$

Back to S^2 & GR: BMS metric $\Longleftrightarrow$ $\boxed{\text{NP first order}}$

Solution space, free data at $\mathscr{I}^+$: $\boxed{\psi^0_2 + \bar\psi^0_2, \quad \psi^0_1, \quad \sigma^0}$ undetermined u-dependence

$\dot\sigma^0$ news

evolution equations. $\partial_u \psi^0_2 = \eth \psi^0_3 + \sigma^0 \psi^0_4$, $\quad \partial_u \psi^0_1 = \eth \psi^0_2 + 2\sigma^0 \psi^0_3$

constraints $\psi^0_2 - \bar\psi^0_2 = \bar\eth^2 \sigma^0 - \eth^2 \bar\sigma^0 + \dot\sigma^0 \bar\sigma^0 - \sigma^0 \dot{\bar\sigma}^0$

$\psi^0_3 = -\eth \dot{\bar\sigma}^0, \quad \psi^0_4 = -\ddot{\bar\sigma}^0$

additional data to construct solutions

$$\psi_0 = \sum_{n \geq 0} \psi^n_0 (\xi, \bar\xi, u_0)\, r^{-5-n}$$

Transformation of (relevant) free data at $\mathcal{J}^+$ $\quad s = (\mathcal{Y}, \bar{\mathcal{Y}}, T)$, $\quad f = T + \frac{1}{2} u (\eth \mathcal{Y} + \bar{\eth} \bar{\mathcal{Y}})$

$$\delta_s \sigma^\circ = \left[f \partial_u + \mathcal{Y} \eth + \bar{\mathcal{Y}} \bar{\eth} + \frac{3}{2} \eth \mathcal{Y} - \frac{1}{2} \bar{\eth} \bar{\mathcal{Y}} \right] \sigma^\circ - \eth^2 f$$

$$\delta_s \psi^\circ_2 = \left[\quad '' \quad '' \quad '' \quad + \frac{3}{2} \eth \mathcal{Y} + \frac{3}{2} \bar{\eth} \bar{\mathcal{Y}} \right] \psi^\circ_2 + 2 \eth f \, \psi^\circ_3$$

(constraints to be imposed)

$$\delta_s \psi^\circ_1 = \left[\quad '' \quad '' \quad '' \quad + 2 \eth \mathcal{Y} + \bar{\eth} \bar{\mathcal{Y}} \right] \psi^\circ_1 + 3 \eth f \, \psi^\circ_2$$

broken current algebra

$$J_s = \frac{i}{\ell^2} \left[(P_s \bar{P}_s)^{-1} J^u_s \, d\xi \wedge d\bar{\xi} + P_s^{-1} J_s^{\bar{\xi}} \, du \wedge d\xi - \bar{P}_s J_s^{\xi} \, du \wedge d\bar{\xi} \right]$$

$$\delta_{s_1} J_{s_2} + \boxed{\Theta_{s_2}(\delta_{s_1} X)} \approx - J_{[s_1, s_2]} + d L_{s_1, s_2}$$

non-conservation

$s_1 : (\mathcal{Y}, \bar{\mathcal{Y}}, T) = (0, 0, 1)$

$$d J_s + \Theta_s(\delta_{(0,0,1)} X) \approx 0$$

$$\Theta_s(\delta X) \sim \dot{\sigma}^\circ, \dot{\bar{\sigma}}^\circ$$ vanishes in the absence of news

time components

$$J^u_s = -\frac{1}{8\pi G}\Big\{ \big[\overbrace{\psi_2^0 + \bar\psi_2^0}^{BH} + \overbrace{\sigma^0\dot{\bar\sigma}^0 + \bar\sigma^0\dot\sigma^0}^{\text{gravitons}}\big] f + \big[\psi_1^0 + \sigma^0\eth\bar\sigma^0 + \tfrac{1}{2}\eth(\sigma^0\bar\sigma^0)\big]\bar{\mathcal{Y}} + \big[\bar\psi_1^0 + \bar\sigma^0\bar\eth\sigma^0 + \tfrac{1}{2}\bar\eth(\sigma^0\bar\sigma^0)\big]\mathcal{Y}\Big\}$$

$$\Theta^u_s(\delta\chi) = \frac{1}{8\pi G}\left[\dot{\bar\sigma}^0\delta\sigma^0 + \dot\sigma^0\delta\bar\sigma^0\right] f$$

charges $\quad Q_s = \int_{S^2,\, u=cte} \frac{i}{R^2}\frac{d\zeta\, d\bar\zeta}{P_s\bar P_s} J^u_s \qquad \Theta_s[\delta\chi] = \int_{S^2,\, u=cte} \frac{i}{R^2}\frac{d\zeta\, d\bar\zeta}{P_s\bar P_s}\Theta^u_s[\delta\chi]$

algebra $\quad \delta_{s_1} Q_{s_2} + \Theta_{s_2}[\delta_{s_1}\chi] = -Q_{[s_1,s_2]}$

(non-)conservation of BMS_4 charges $\quad \frac{d}{du}Q_s = -\underbrace{\int_{S^2,\, u=cte}\frac{i}{R^2\, 8\pi G}\frac{d\zeta\, d\bar\zeta}{P_s\bar P_s}\left[\dot{\bar\sigma}^0\delta_s\sigma^0 + \dot\sigma^0\delta_s\bar\sigma^0\right]}_{\text{fluxes}}$

G.B. & C.Troessaert JHEP (2011)
JHEP (2013)

generalizes Bondi mass loss

$\boxed{\text{non-radiative space-times (no news)}}$ $\sigma^0 = \sigma^0(\xi, \bar{\xi}, \cancel{u})$ $\left(\Rightarrow \dot{\sigma}^0 = 0 = \psi_3^0 = \psi_4^0, \quad O_s[\delta X] = 0 \right)$

compare "abstract" construction of $\mathfrak{bms}_4^*$

identification at $u = 0$ $\quad P = -\frac{1}{2G}(\psi_2^0 + \bar{\psi}_2^0)$ $\qquad \bar{J} = -\frac{1}{2G}\Big(\underbrace{\psi_1^0 + \sigma^0 \eth \bar{\sigma}^0 + \frac{1}{2}\eth(\sigma^0 \bar{\sigma}^0)}_{\psi^0_{1\bar{J}}} \Big)$

super-momentum
= Bondi mass aspect

~~super~~- angular momentum
= Bondi angular momentum aspect

(pre) moment map: $\mathfrak{F}$. algebra of non-radiative free data

$\mathfrak{bms}_4$ representation δ_s, $[\delta_{s_1}, \delta_{s_2}] = \delta_{[s_1, s_2]}$

$$\mu : \mathfrak{F} \longrightarrow \mathfrak{bms}_4^*$$

$$\mu\left(-\frac{1}{2G}(\psi_2^0 + \bar{\psi}_2^0)\right) = P, \qquad \mu\left(-\frac{1}{2G}\psi^0_{1\bar{J}}\right) = [\bar{J}], \qquad \mu \circ \delta_s = \mathrm{ad}_s^* \circ \mu$$

transformation laws at $u=0$ $\qquad \delta_s(\psi_2^0+\bar\psi_2^0) = (\mathcal{Y}\eth + \bar{\mathcal{Y}}\bar\eth + \frac{3}{2}\eth\mathcal{Y} + \frac{3}{2}\bar\eth\bar{\mathcal{Y}})(\psi_2^0+\bar\psi_2^0)$ ✓

$$\delta_s\psi_1^0 = [\mathcal{Y}\eth + \bar{\mathcal{Y}}\bar\eth + 2\eth\mathcal{Y} + \bar\eth\bar{\mathcal{Y}}]\psi_1^0 + \frac{1}{2}\mathcal{T}\eth(\psi_2^0+\bar\psi_2^0 + \cancel{\bar\eth^2\sigma^0} - \cancel{\eth^2\bar\sigma^0})$$
$$+\frac{3}{2}\eth\mathcal{T}(\psi_2^0+\bar\psi_2^0 + \cancel{\bar\eth^2\sigma^0} - \cancel{\eth^2\bar\sigma^0})$$

$$\delta_s\psi^0_{1\mathcal{J}} = [\mathcal{Y}\eth + \bar{\mathcal{Y}}\bar\eth + 2\eth\mathcal{Y} + \bar\eth\bar{\mathcal{Y}}]\psi^0_{1\mathcal{J}} + \frac{1}{2}\mathcal{T}\eth(\psi_2^0+\bar\psi_2^0) + \frac{3}{2}\eth\mathcal{T}(\psi_2^0+\bar\psi_2^0)$$
$$+\frac{1}{2}\bar\eth(\mathcal{T}\bar\eth\eth\sigma^0 - \bar\eth\mathcal{T}\eth\sigma^0 + 3\eth\mathcal{T}\bar\eth\sigma^0 - 3\bar\eth\eth\mathcal{T}\sigma^0 - \frac{3}{2}\mathcal{T}\sigma^0) \boxed{-\frac{1}{2}\eth^3(\mathcal{T}\bar\sigma^0)}$$

$\boxed{\text{trivial !}}$

Remark: electric case $\bar\eth^2\sigma^0_e = \eth^2\bar\sigma^0_e \iff \sigma^0_e = \eth^2\chi_e$

$$\delta_s\chi_e = [\mathcal{Y}\eth + \bar{\mathcal{Y}}\bar\eth - \frac{1}{2}\eth\mathcal{Y} - \frac{1}{2}\bar\eth\bar{\mathcal{Y}}] - \mathcal{T}$$

Newman Penrose JMP 1966

Strominger et al. 2015-

Compère et al. 2016

simplified pre-momentum map $\mu' : \mathcal{F}_e \longrightarrow \mathfrak{bms}_4^*$

$$\mu'[-\tfrac{1}{2G}(\psi_2^0+\bar\psi_2^0)] = \mathcal{T}, \quad \mu'[-\tfrac{1}{2G}\psi_1^0] = [\mathcal{Y}], \quad \mu'\circ\delta_s = \mathrm{ad}^*_s\circ\mu'$$

Perspectives

1) on punctured plane $\eth^3 \tilde{\mathcal{Y}} \neq 0$

$\exists$ super-rotations & super-angular momentum $\quad K_{s_1,s_2} = \mathrm{Res}_z \mathrm{Res}_{\bar{z}} [r^0 f_1 \eth^3 Y_2 - (1,2) + \text{c.c.}]$

$\exists$ field-dependent central extension & associated Souriau cocycle

GB & Troessaert JHEP 2016 GB JHEP 2017 mapping from plane to cylinder to make r^0 non-zero for Kerr

2) coadjoint repres. of generalized BMS_4 Campiglia & Laddha Phys. Rev. 2014

$\mathrm{Diff}(S^2) \ltimes C^\infty(S^2)$ on S^2 drop $\eth \bar{\mathcal{Y}} = 0 = \bar{\eth} \mathcal{Y}$ $\quad \eth^3 \mathcal{Y} = 0 = \bar{\eth}^3 \bar{\mathcal{Y}}$

and also equivalence relations

$$\bar{\mathcal{J}} \sim \bar{\mathcal{J}} + \bar{\eth} \tilde{\mathcal{J}} \,, \quad \bar{\mathcal{J}} \sim \bar{\mathcal{J}} + \eth^3 \mathcal{M}$$

[related groups Donnelly et al. 2020

simply expand everything in spin-weighted spherical harmonics

3) Classify coadjoint orbits

symplectic manifolds can be quantized $\longrightarrow$ relation to UIRREPS of BMS_4

Write geometric actions Alekseev, Faddeev, Shatashvili J. Geom. Phys. 1988, Nucl. Phys. 1989

$$S = \int \mathrm{Tr}\left((g^{-1})_\mu g \; g^{-1} J^\mu g \right) d^4x \longrightarrow S = \int \left(\langle v_0^*, g^{-1} \frac{dg}{dt} \rangle - H \right) dt$$

no Killing metric needed. correct global symmetries

for 3d gravity $\Leftrightarrow$ to actions constructed from

CS $\to$ WZW Elitzur et al. Nucl. Phys. 1989

GB, Gonzalez, Salgado CQG 2018 Coussaert, Henneaux, Van Driel CQG 1995

Effective actions for Goldstone bosons

4) Complete pre-momentum map to bona fide one connection to spatial infinity Henneaux & Troessaert JHEP 2018

Torre CQG 1986

Oliveri & Speziale 2019

Wieland 2020

5) Study interactions of this group theory sector with radiative DoF at $\mathcal{J}^+$

Ashtekar & Streubel Proc. Roy. Soc. 1981

Ashtekar (1984)

6) Krichever-Novikov algebras for more than 2 punctures ?