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Differential equations

OIST

What is a differential equation?

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An equation containing *derivatives*

The *order* of a differential equation is the order of the highest derivative in the equation.

1st order: $$\frac{dy}{dx} + xy^2 = 1$$

2nd order: $$\frac{d^2x}{dt^2} = -\frac{k}{m}x$$ (Simple harmonic motion)

nth order:

$$a_0 y + a_1 \frac{dy}{dx} + a_2 \frac{d^2y}{dx^2} + \ldots + a_{n-1}\frac{d^{n-1}y}{dx^{n-1}} + a_n \frac{d^n y}{dx^n} = b$$

What is a differential equation?

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Can classify differential equations as:

linear or non-linear

Non-linear equations contain products of the function and its derivatives, or powers higher than one, i.e:

$$y\frac{dy}{dx} \qquad \left(\frac{dy}{dx}\right)^4 \qquad xy^3$$

Ordinary Differential Equations

$\longrightarrow$ contain *ordinary* derivatives

Partial Differential Equations

$\longrightarrow$ contain *partial derivatives*

Euler Method

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Numerical method that only uses information about derivatives at the beginning of the interval

$$\frac{dy}{dx} = f(x, y)$$

$$y_{n+1} = y_n + hf\left(x_n, y_n\right)$$

$$x_n \to x_{n+1} = x_n + h$$

Usually not super accurate or stable

Lets implement this!

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Simple first order ODE

$$\frac{dy}{dt} = -by$$

Exact solution:

$$y = y(0)\exp(-bt)$$

Runge-Kutta

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Runge-Kutta methods use a trial step at the midpoint of the interval

2nd order

$$k_1 = hf(x_n, y_n)$$

$$k_2 = hf(x_n + \frac{h}{2}, y_n + \frac{k_1}{2})$$

$$y_{n+1} = y_n + k_2$$

Runge-Kutta

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4th order

$$k_1 = hf(x_n, y_n)$$

$$k_2 = hf(x_n + \frac{h}{2}, y_n + \frac{k_1}{2})$$

$$k_3 = hf(x_n + \frac{h}{2}, y_n + \frac{k_2}{2})$$

$$k_4 = hf(x_n + h, y_n + k_3)$$

$$y_{n+1} = y_n + \frac{k_1}{6} + \frac{k_2}{3} + \frac{k_3}{3} + \frac{k_4}{6} + O\left(h^5\right)$$

Matlab built-in ODE solvers

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Matlab provides efficient ODE integrators,

such as ode45

4th and 5th order Runge-Kutta method

> help ode45

Default Relative Error: 10^{-3}
If $|y_{n+1} - y_n| >$ relative error,
decreases timestep and repeats.

Examples

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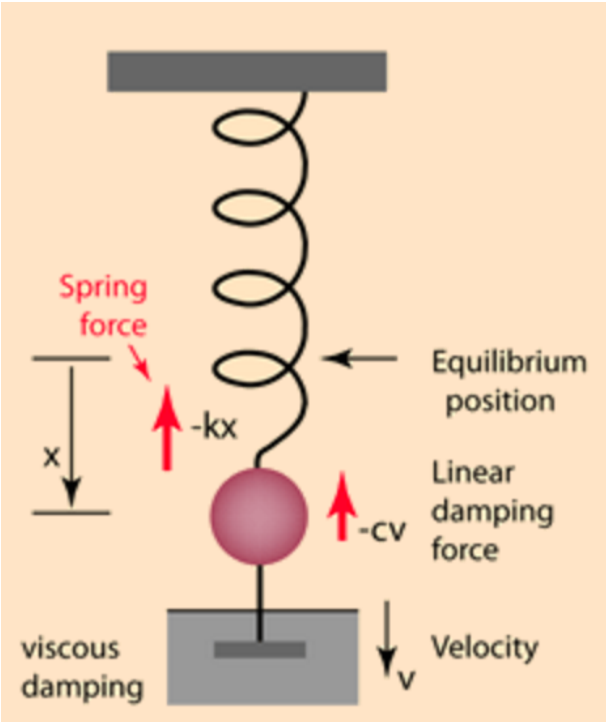
Damped simple harmonic oscillator



$$\frac{dx^2}{dt^2} + \alpha \frac{dx}{dt} + \omega^2 x = 0$$

Take: $y_1 = x \quad y_2 = \frac{dx}{dt}$

SOLVE

$$\frac{dy_2}{dt} = -\alpha y_2 - \omega^2 y_1$$

$$\frac{dy_1}{dt} = y_2$$

Examples

Predator-Prey model

Prey population x

Predator population y

SOLVE

$$\frac{dx}{dt} = Ax - Bxy$$

$$\frac{dy}{dt} = -Cy + Dxy$$

A - Growth rate of prey
B - Rate at which predators destroy prey
C - Rate at which predators die off
D - Growth rate of predators

Example

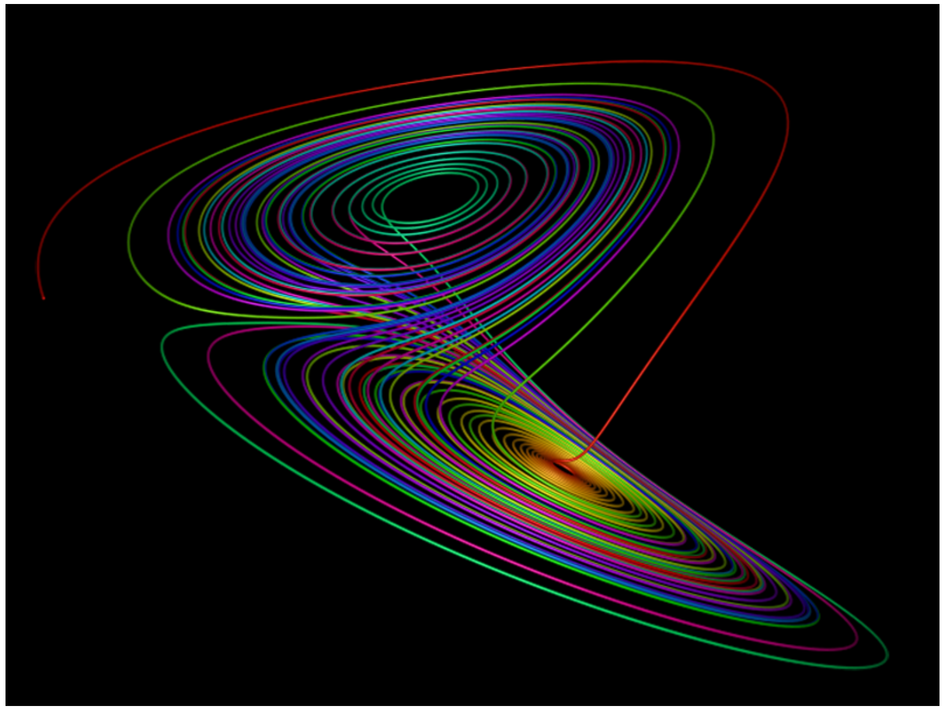
Lorenz Equations

SOLVE

$$\frac{dx}{dt} = \sigma(y - x)$$

$$\frac{dy}{dt} = x(\rho - z) - y$$

$$\frac{dz}{dt} = xy - \beta z$$



For certain parameter values and initial conditions, chaotic solutions exist, such as the Lorentz attractor.

Try: $\rho = 28$, $\sigma = 10$ and $\beta = 8/3$.

Further built-in examples

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> help odeexamples
> odeexamples('ode')

→ Run examples
→ View code

Have fun!