SKILL PILL: FOURIER TRANSFORMS

1 Creating square signal

Add $\sin((2k+1)x)/(2k+1)$, $k=0,1,\dots$

keep track of frequency, amplitude.

$\rightarrow$ it is another way to keep track of fct

$\rightarrow$ historically how Fourier Tr. came about

2 Properties of FT

- $\hat{f}(\xi) = \int_{-\infty}^{+\infty} e^{-i2\pi x\xi} f(x)\,dx$ — if $\int |f(x)|dx < \infty$ — $\rightarrow \sqrt{\frac{1}{2\pi}} \int_{-\infty}^{+\infty} e^{ix\xi} f(x)\,dx$ $\rightarrow \sqrt{2\pi} \int e^{i2\pi x\xi} f(x)\,dx$
- $f(x) = \int_{-\infty}^{\infty} e^{2\pi i\xi x} \hat{f}(\xi)\,d\xi$ $\rightarrow$ inverse
- Other notations: $\mathcal{F}$
- PROPERTIES:

$\rightarrow$ LINEARITY : $h(x) = af(x) + bg(x) \rightarrow \hat{h}(\xi) = a\hat{f}(\xi) + b\hat{g}(\xi)$

$\rightarrow$ Prove it

$\rightarrow$ Translation: $h(x) = f(x-x_0) \longrightarrow \hat{h}(\xi) = e^{-i2\pi x_0\xi}\hat{f}(\xi)$

$\rightarrow$ frequency shifting is the same

$\rightarrow$ Scaling : $h(x) = f(ax) \rightarrow \hat{h}(\xi) = \frac{1}{|a|}\hat{f}(\xi/a)$ $(a\neq 0)$

$\rightarrow$ They prove it

$\hat{h}(\xi) = \int_{-\infty}^{\infty} e^{i2\pi x\xi} f(ax)\,dx = \int_{-\infty}^{+\infty} e^{i2\pi x'\xi/a} f(x')\,dx'/a = \dots = \hat{f}$ ($x'=ax$, $dx'=a\,dx$)

$\rightarrow$ $a=-1$ time reversal: $h(x) = f(-x)$ $\hat{h}(\xi) = \hat{f}(-\xi)$

$\rightarrow$ CONJugation: $h(x) = \overline{f(x)} \rightarrow \hat{h}(\xi) = \overline{\hat{f}(-\xi)}$

$\rightarrow$ Integral at $\xi=0$ $\hat{f}(0) = \int_{-\infty} f(x)\,dx$

$\rightarrow$ Differentiation : $h(x) = f'(x)$ $\hat{h}(\xi) = (2\pi i\xi)\hat{f}(\xi)$

$h(x) = f^{(n)}(x)$ $\hat{h}(\xi) = (2\pi i\xi)^n \hat{f}(\xi)$

$\rightarrow$ convolution $(f * g)(x) = \int_{-\infty} f(y)\, g(x-y)\,dy$

$h(x) = (f * g)(x) \rightarrow \hat{h}(\xi) = \hat{f}(\xi)\,\hat{g}(\xi)$

link to cross-correlation

$\rightarrow$ $\mathcal{F}^0 = \mathbb{1}$ $\mathcal{F}^1 = \mathcal{F}$ $\mathcal{F}^2 = P$ $\mathcal{F}^3 = \mathcal{F}^{-1}$ $\mathcal{F}^4 = \mathbb{1}$

3 List of important functions

4 Laplace Transform

$F(s) = \int_0^{\infty} e^{-st} f(t)\,dt$

s complex

5 FOURIER SERIES

- $s(x)$ function on $[x_0, x_0+P]$

$$s(x) = \frac{A_0}{2} \sum_{n=1}^{\infty} A_n \sin\left(\frac{2\pi nx}{P} + \phi_n\right)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi nx}{P}\right) + b_n \sin\left(\frac{2\pi nx}{P}\right)$$

$$= \sum_{n=-\infty}^{\infty} c_n e^{i\frac{2\pi nx}{P}}$$

$$\begin{cases} a_n = \frac{2}{P}\int_{x_0}^{x_0+P} s(x) \cos\left(\frac{2\pi nx}{P}\right)dx \\ b_n \quad \sin \end{cases} \qquad \left\{ c_n = \frac{1}{P}\int_{x_0}^{x_0+P} s(x)\, e^{-\frac{i2\pi nx}{P}}\,dx \right.$$

- $\infty \rightarrow N$ approximation
- EX: square function

$$s(x) = \begin{cases} 1 & x \leq \pi \\ -1 & x > \pi \end{cases} \quad \text{on } [0, 2\pi]$$

$$a_n = \frac{2}{2\pi}\int_0^{2\pi} s(x)\cos\left(\frac{2\pi nx}{2\pi}\right)dx = \frac{1}{\pi}\left(\int_0^{\pi}\cos(nx)\,dx - \int_{\pi}\cos(nx)\,dx\right)$$

$$= \frac{1}{\pi}\left(\left[\frac{\sin nx}{n}\right]_0^{\pi} - \left[\frac{\sin nx}{n}\right]_{\pi}^{2\pi}\right) = \frac{1}{\pi}(0) = 0$$

$$b_n = \frac{1}{\pi}\left(\left[\frac{-\cos nx}{n}\right]_0^{\pi} - \left[\frac{-\cos nx}{n}\right]_{\pi}^{2\pi}\right) = \frac{1}{n\pi}(-\cos n\pi + 1 + 1 - \cos n\pi)$$

$$= \frac{2}{n\pi}(1 - \cos n\pi) \begin{cases} \xrightarrow{n\text{ even}} 0 \\ \xrightarrow{n\text{ odd}} \frac{4}{n\pi} \end{cases}$$

$$\Rightarrow s(x) = \sum_{k=1} \frac{4}{\pi} \sin((2k+1)x)$$

6 Uncertainty principle

Assume $\psi(x)$: $\int_{-\infty}^{\infty} |\psi(x)|^2 dx = 1$ $\rightarrow$ wavefunction

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7 Diff eq.

Solving heat equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ $\quad]-\infty, \infty[, \; t \geq 0$

$$\int_{-\infty}^{+\infty} |u(x,t)|^2 \, dx < \infty \quad \forall t$$

- Initially $u(x,0) = f(x)$
- Fourier in x direct°: $\cancel{\widehat{\partial_t u}(k,t) = \partial_t \hat{u}}$

$$\mathcal{F}(\partial_t u(x,t)) = \partial_t \mathcal{F}(u)(k,t)$$

$$\mathcal{F}(\partial_x u(x,t)) = ik \, \mathcal{F}(u)(k,t)$$

$$\mathcal{F}(\partial_{x^2} u(x,t)) = -k^2 \mathcal{F}(u)(k,t)$$

$$\longrightarrow \partial_t \mathcal{F}(u) = -k^2 \mathcal{F}(u)$$

$$\text{w/} \quad \mathcal{F}(u)(k,0) = \mathcal{F}(f)$$

$\longrightarrow$ ODE

$$\longrightarrow \mathcal{F}(u)(k,t) = e^{-k^2 t} \mathcal{F}(u)(k,0) = e^{-k^2 t} \mathcal{F}(f)(k)$$

$\longrightarrow$ convolut° thm: $\quad \to \mathcal{F}(g) = e^{-k^2 t}$

$$\mathcal{F}(u)(k,t) = \mathcal{F}(g)(k) \mathcal{F}(f)(k)$$

$$\longrightarrow g(x,t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-k^2 t} e^{-ikx} \, dk = \frac{1}{\sqrt{4\pi t}} e^{-x^2/4t}$$

$$\Longrightarrow u(x,t) = \int_{-\infty}^{+\infty} g(x-\xi) f(\xi) \, d\xi$$

$$= \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{+\infty} e^{-(x-\xi)^2/4t} f(\xi) \, d\xi$$

$$f(x) = \delta(x) \longrightarrow u(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-x^2/4t}$$

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