Graph Theory Mini Course: Flow

By Maki & Tom

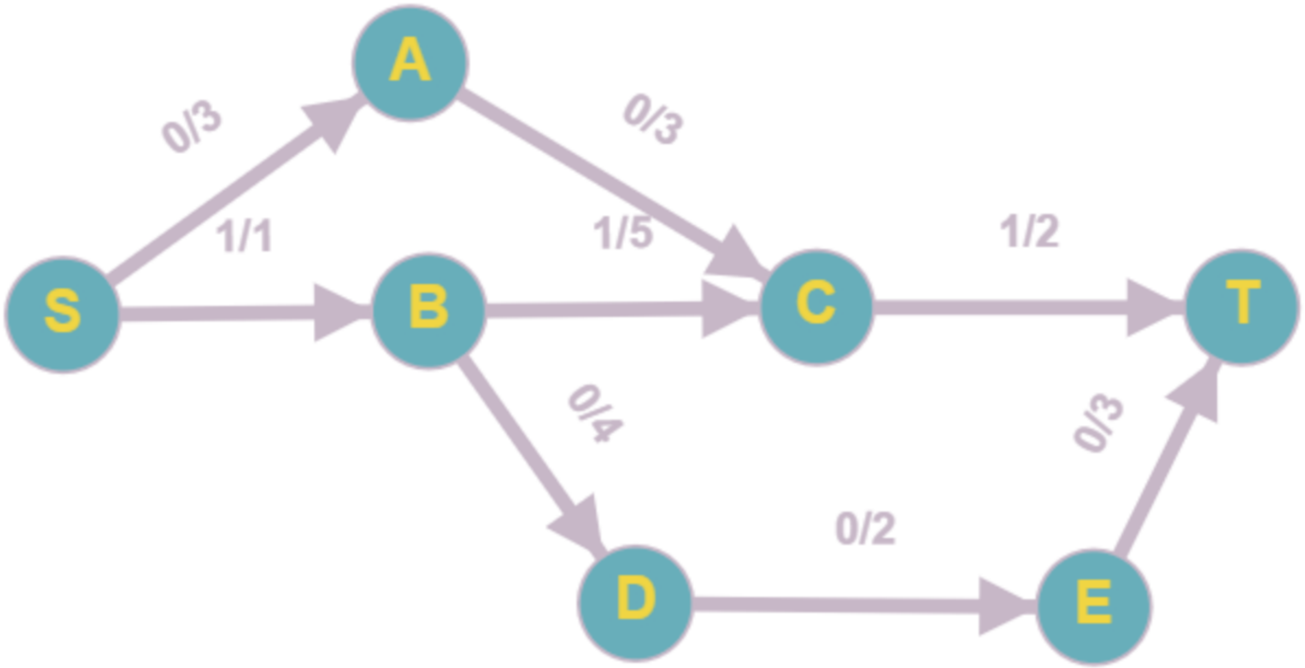
Tuesday 9 March 2021

Capacity and Flow

- Capacity C of an edge: amount of flow allowable along an edge
- A valid flow f on a digraph G identifies one vertex as the source s and another as the sink t which obeys:

1. <u>Capacity Rule:</u> For all edges, $f \leq C$
2. <u>Skew Symmetry:</u> For all vertices except s and t, indegree flow is equal to outdegree flow
3. <u>Flow Conservation:</u> The outdegree flow from s is equal to the indegree flow at t

f/C

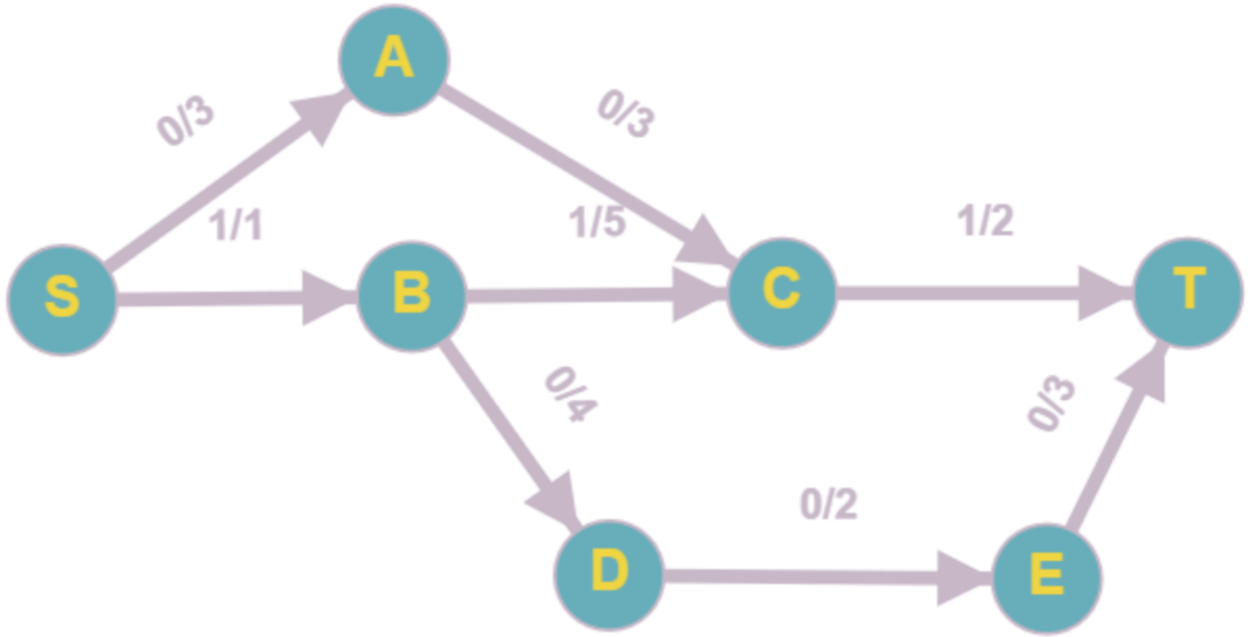


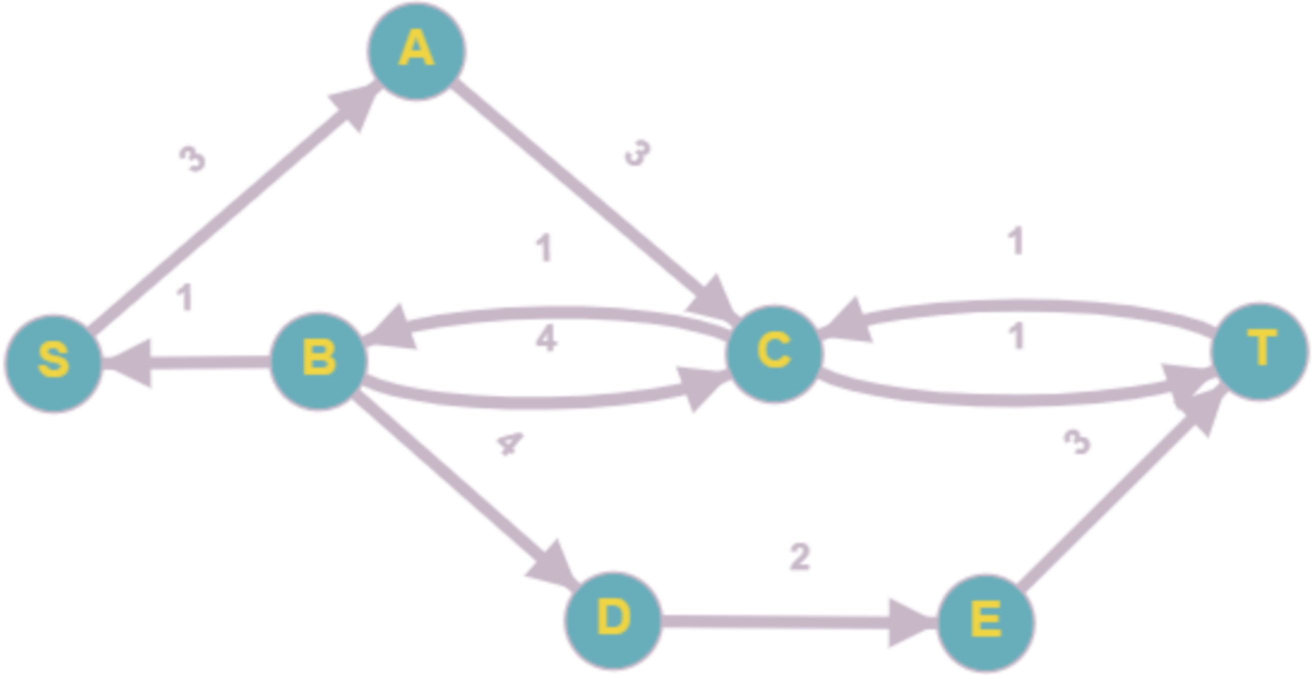
*a **"circulation"** is a graph which obeys 1 and 2, but there is no source or sink*

Residual Network

Given a directed graph G with edge capacities C (e.g. weights) and a flow f, there is a residual network G_f, which has:

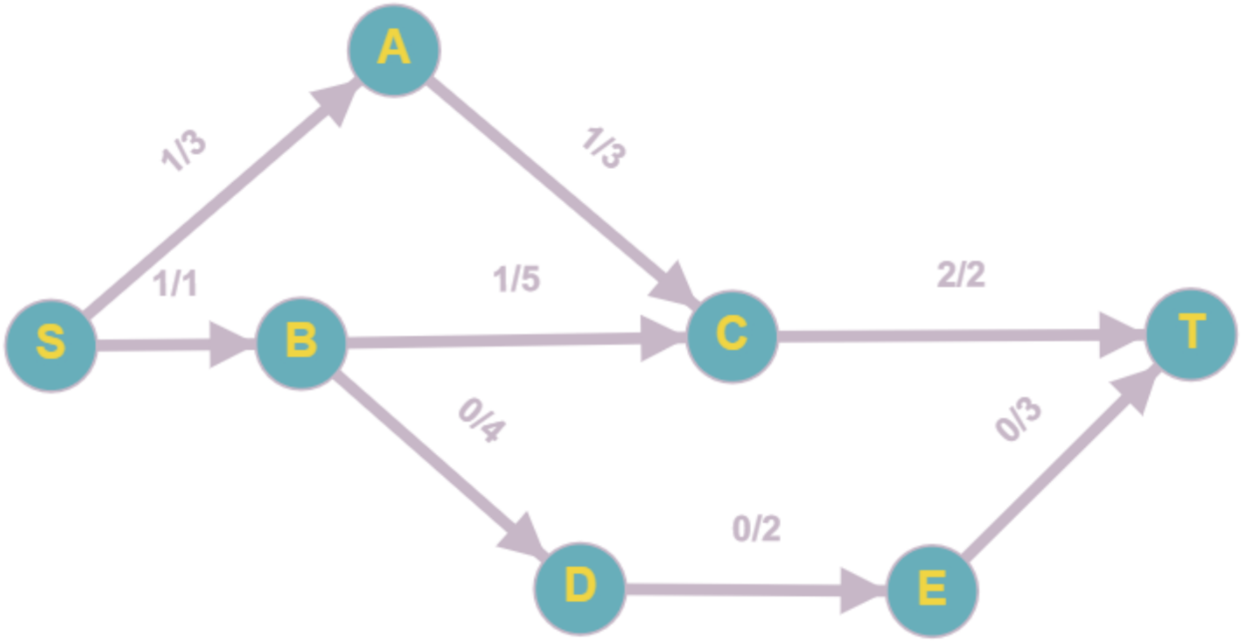
- $V(G) = V(G_f)$
- For all edges $(\mathrm{u}, \mathrm{v}) \in E(G)$ there are 1 or 2 edges in $E(G_f)$:
- If $f(u, v) < C(u, v)$, then $(\mathrm{u}, \mathrm{v}) \in E(G_f)$ with capacity $C_f(u, v) = C(u, v) - f(u, v)$
- If $f(u, v) > 0$, then $(\mathrm{v}, u) \in E(G_f)$ with capacity $C_f(v, u) = f(u, v)$

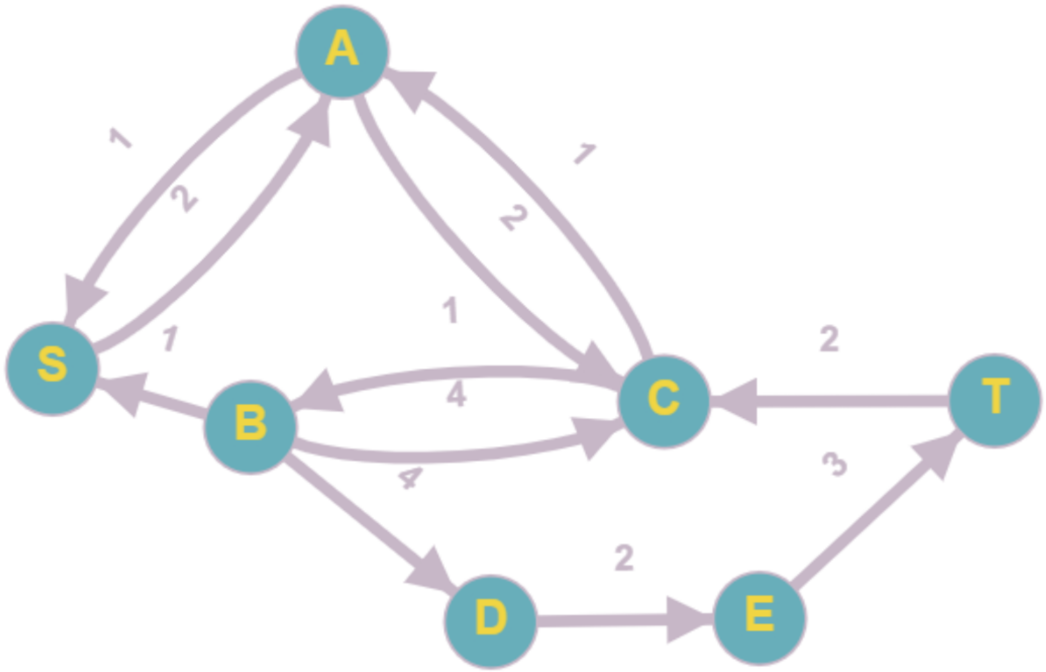




Increasing Flow by Augmenting Paths

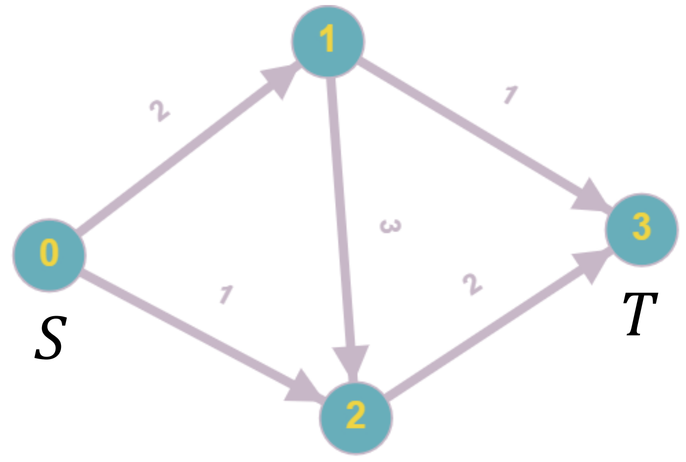
- An augmenting path p in a residual network G_f is any existing path from s to t, e.g.
$$p = (s, a, c, t)$$
- The minimum capacity of the path can be defined like so:
$$\delta(p) = \min\{C_f(s,a), C_f(a,c), C_f(c,t)\} = \min\{3,3,1\} = 1$$
- We can increase the flow by $\delta(p)$





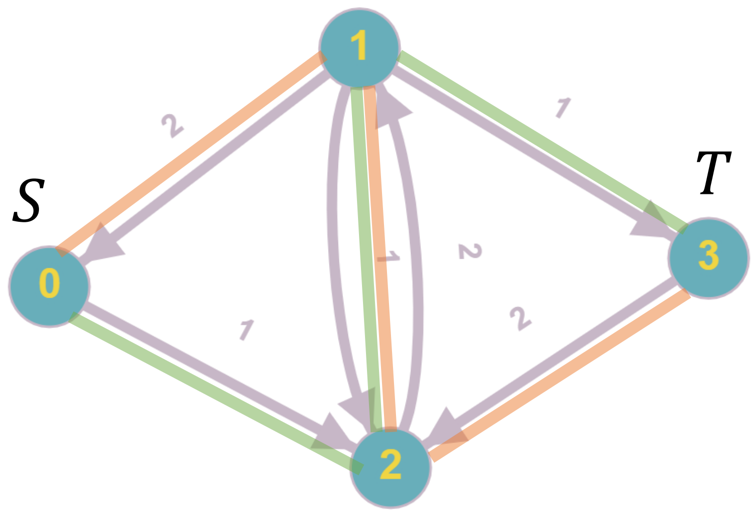
Max Flow

Ford-Fulkerson Algorithm (1956) for determining max flow

$G =$ 

1. Residual(G) for greedy flow

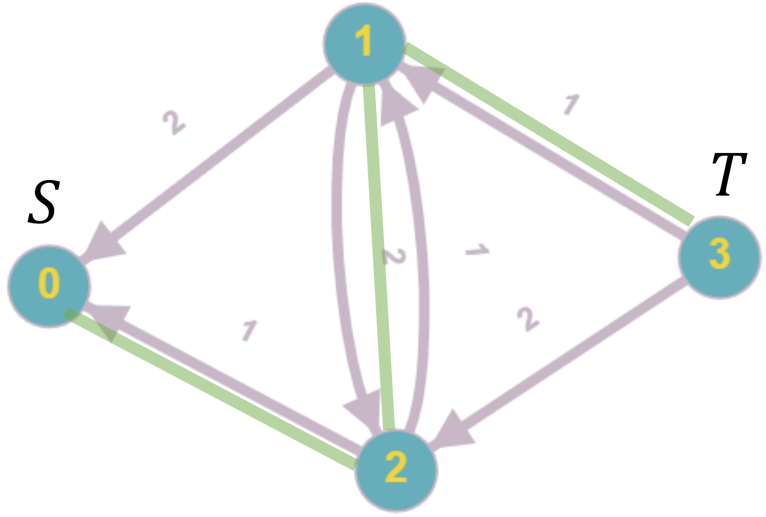
$P_{greedy\ flow} = (0,1,2,3)$



2. Find any paths from S to T

$P_{residual\ flow} = (0,2,1,3)$

3. Reverse and redistribute flow along such paths while respecting capacities



iterate until step 2 finds no paths

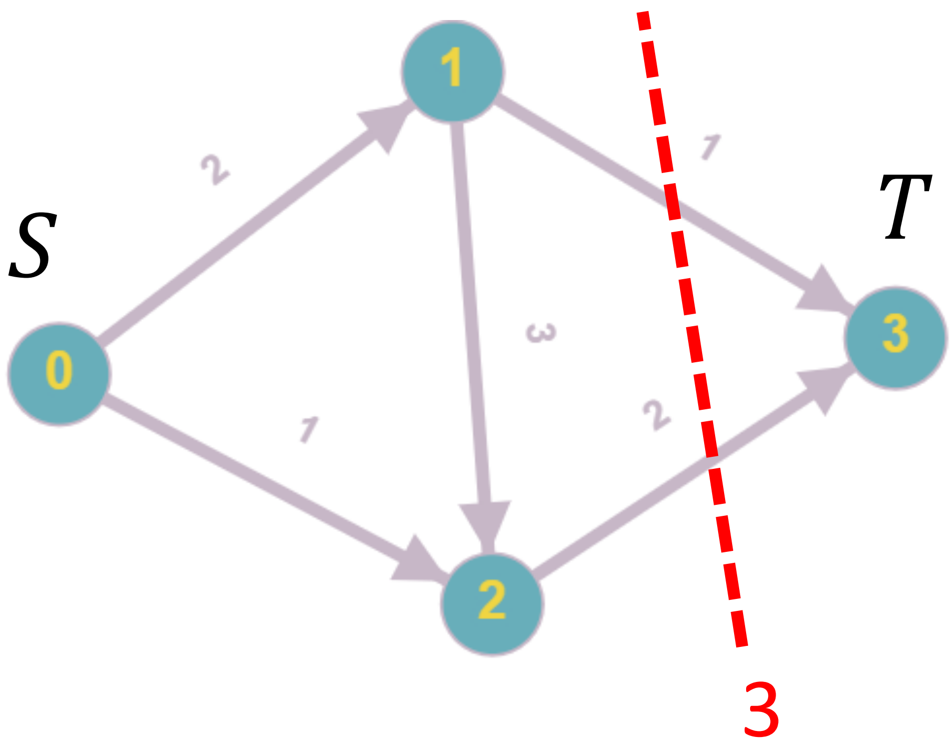
More efficient max flow algorithms

- Edmonds-Karp Algorithm: uses breadth-first search instead of depth-first search in Ford-Fulkerson Algorithm

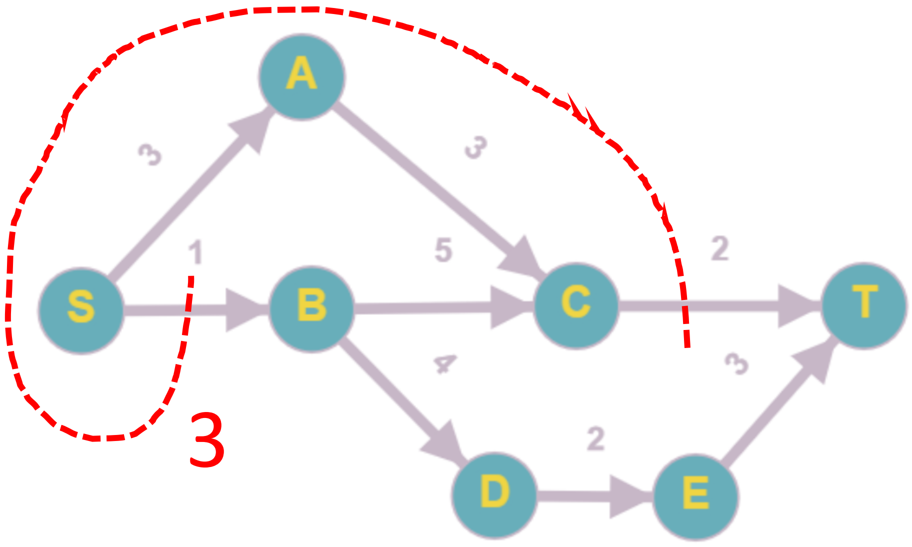
- Dinitz Algorithm: encourages augmenting paths to take shorter paths

Max-flow min-cut theorem

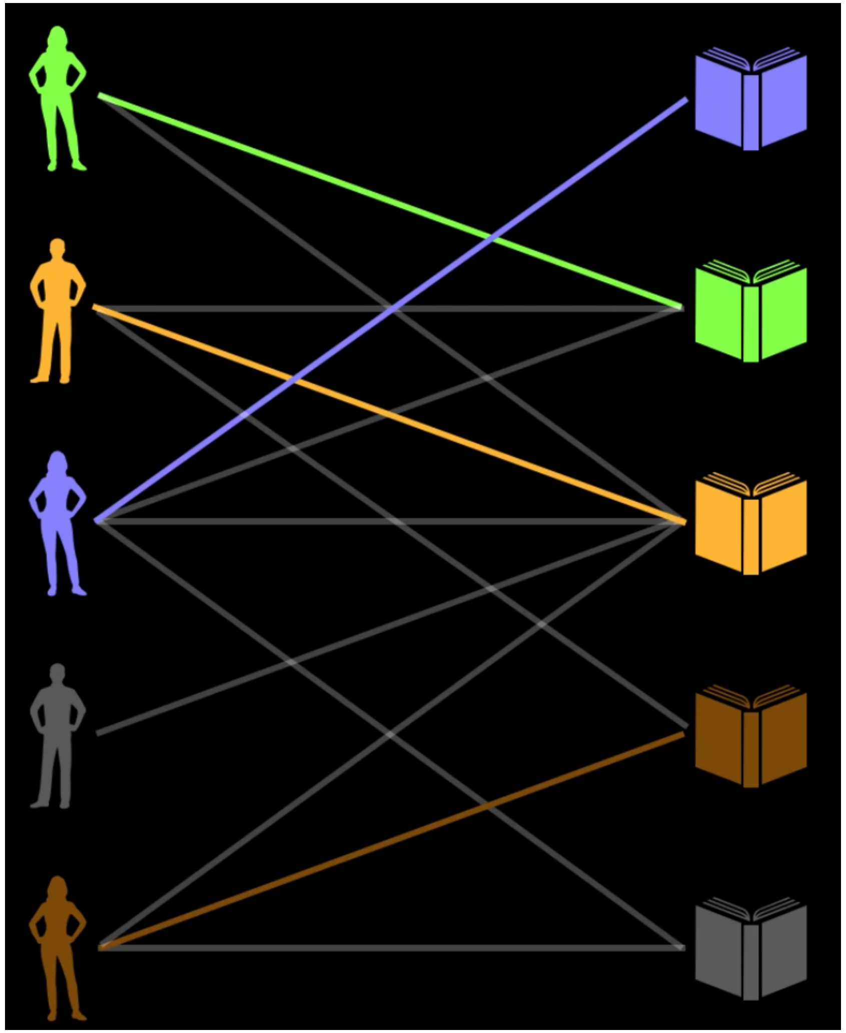
Theorem: In any digraph with a source S and sink T, the value of the maximum flow from S to T is equal to the capacity of the minimum cut.

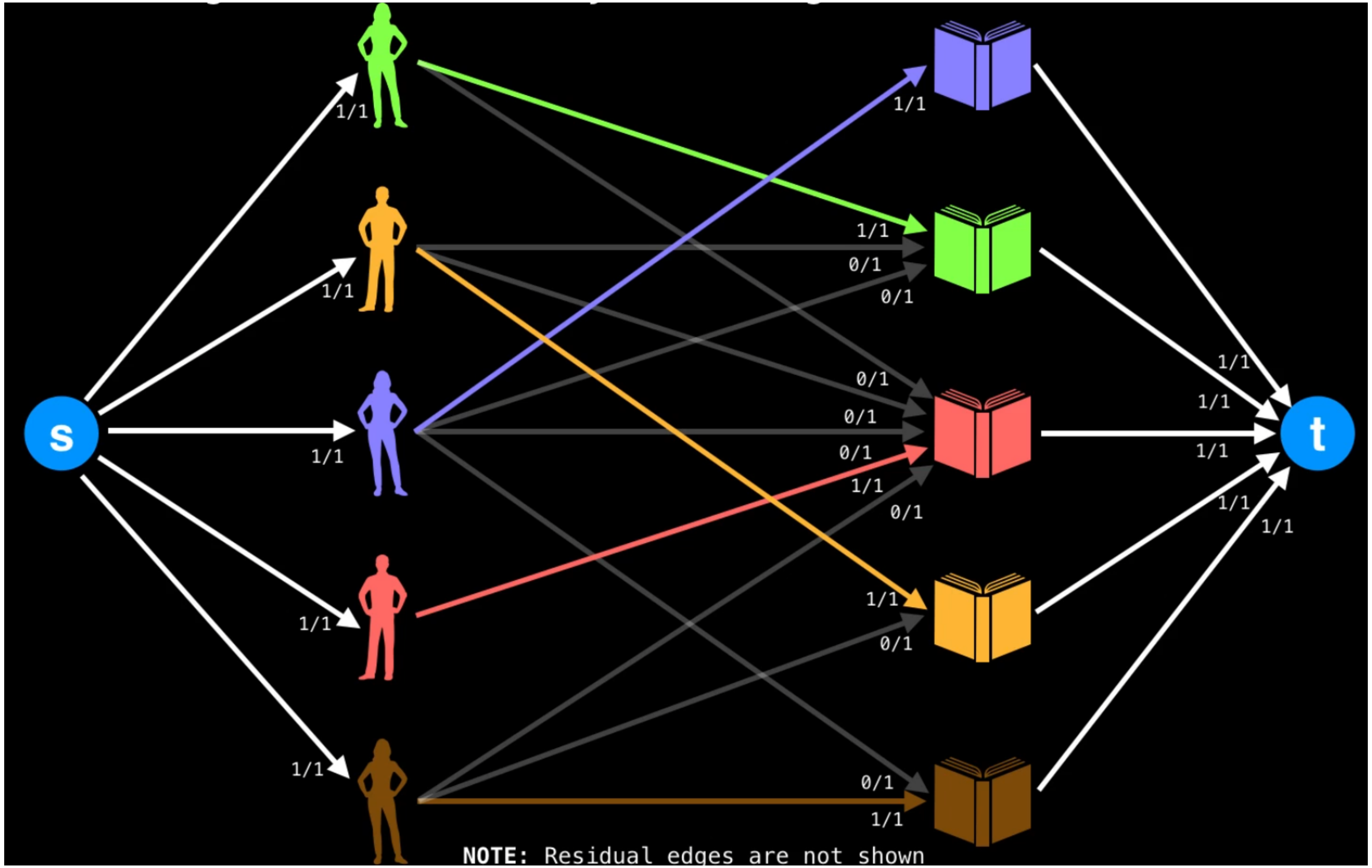


- Maximum flow: largest possible flow from two vertices designated as source S and sink T
- Minimum cut: the cut with smallest possible capacity



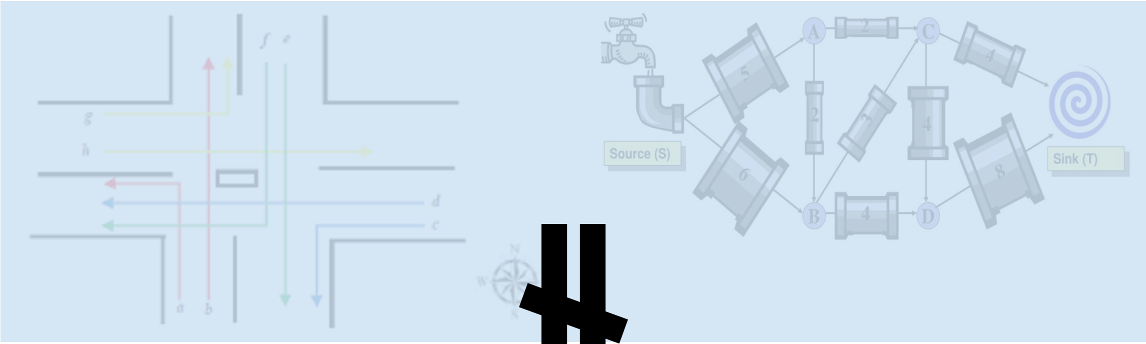
Bipartite Matching using Max Flow

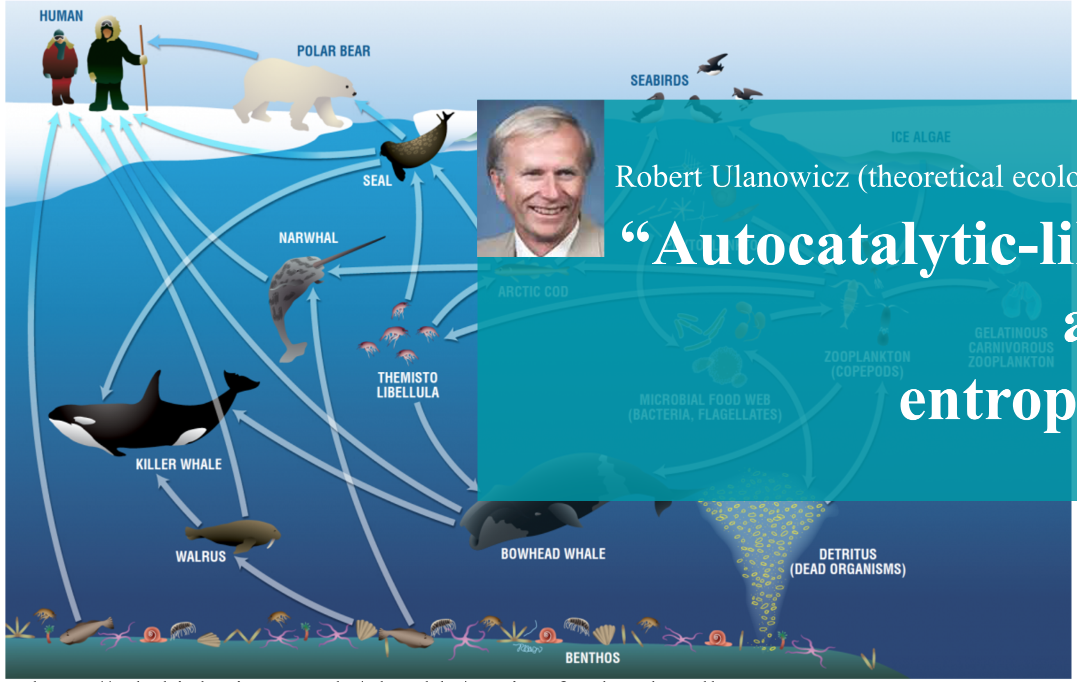




From William Fiset: https://www.youtube.com/watch?v=GhjwOiJ4SqU&list=PLDV1Zeh2NRsDj3NzHbbFIC58etjZhiGcG&index=3&ab_channel=WilliamFiset

Applying Flow to Ecology





https://askabiologist.asu.edu/plosable/marine-food-web-collapse

Robert Ulanowicz (theoretical ecologist)

“Autocatalytic-like self organization and entropic decay”

The Royal Canadian Geographical Society

Applying Flow to Ecology

A tool for exploring **movement and flow** across a range of **scales and settings.**

Conservation Biology

Review

Circuit-theory applications to connectivity science and conservation

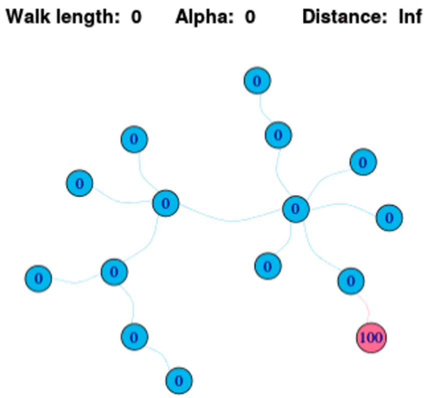
Brett G. Dickson,[1,2*] Christine M. Albano,[1] Ranjan Anantharaman,[3] Paul Beier,[4] Joe Fargione,[5] Tabitha A. Graves,[6] Miranda E. Gray,[1] Kimberly R. Hall,[5] Josh J. Lawler,[7] Paul B. Leonard,[8] Caitlin E. Littlefield,[7] Meredith L. McClure,[1] John Novembre,[9] Carrie A. Schloss,[10] Nathan H. Schumaker,[11] Viral B. Shah,[3] and David M. Theobald[1]

Key points

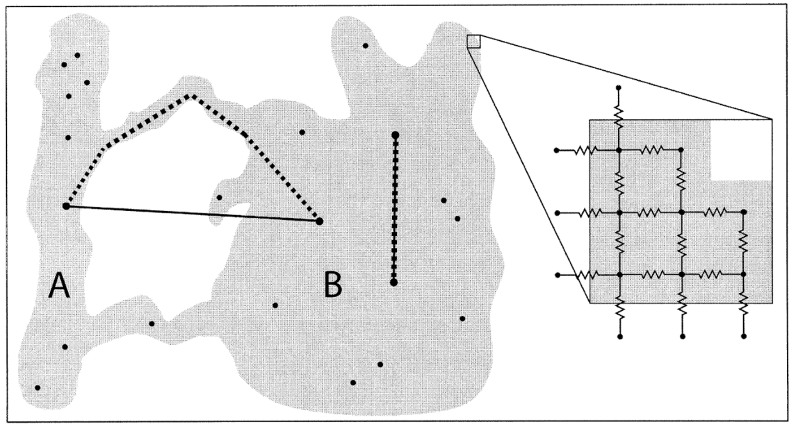
1. Preservation of biodiversity and ecosystem function
2. Ecosystems do not progress to maximum efficiency
3. Achieve a balance between the mutually exclusive attributes of efficiency and reliability.
4. Purely theoretical approach to understand

Resistance

Isolation By Resistance (IBR): predicts a positive relationship between genetic differentiation and the resistance distance, a graph theoretic distance metric based in circuit theory.



Resistance distances from circuit theory are directly proportional to the movements of Markovian random walkers on graphs, "commute times," or the time it takes a random walker to travel from one point to another and back again.



McRae 2006
Isolation By Distance

$$F_{ST}/(1 - F_{ST}) \approx \hat{R}_{xy}/16.$$

(linearized Genetic differentiation) (effective resistance)

$$\hat{M}_{xy} \approx \frac{1}{16}\left(\frac{1}{F_{ST}} - 1\right),$$

Fst/R_x

(1/resistance = # of migrants)

$$\hat{M}_{xy} \approx \hat{G}_{xy},$$

(effective conductance)

CONCEPTS & SYNTHESIS
EMPHASIZING NEW IDEAS TO STIMULATE RESEARCH IN ECOLOGY

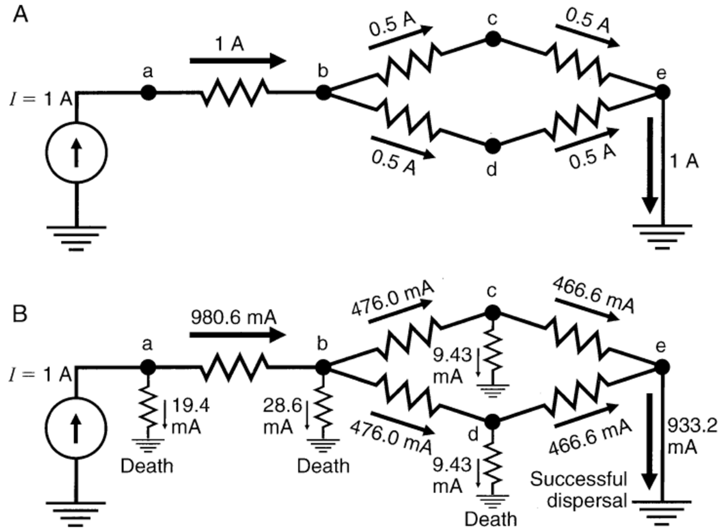
Ecology, 89(10), 2008, pp. 2712–2724

USING CIRCUIT THEORY TO MODEL CONNECTIVITY IN ECOLOGY, EVOLUTION, AND CONSERVATION

Brad H. McRae,[1,5] Brett G. Dickson,[2] Timothy H. Keitt,[3] and Viral B. Shah[4]

[1]National Center for Ecological Analysis and Synthesis, Santa Barbara, California 93101 USA
[2]Center for Environmental Sciences and Education, Northern Arizona University, Flagstaff, Arizona 86011 USA
[3]Section of Integrative Biology, University of Texas at Austin, Austin, Texas 78712 USA
[4]Department of Computer Science, University of California, Santa Barbara, California 93106 USA

- **Voltage**: probability of random walkers reaching a given destination.
- **Current Density** (flow): Landscape corridors
- **Effective resistance:** A measure of isolation between nodes
- **Pinch Points:** points where movement may be most constrained or "bottlenecked"
- **Effective Conductance**: A measure of connectivity

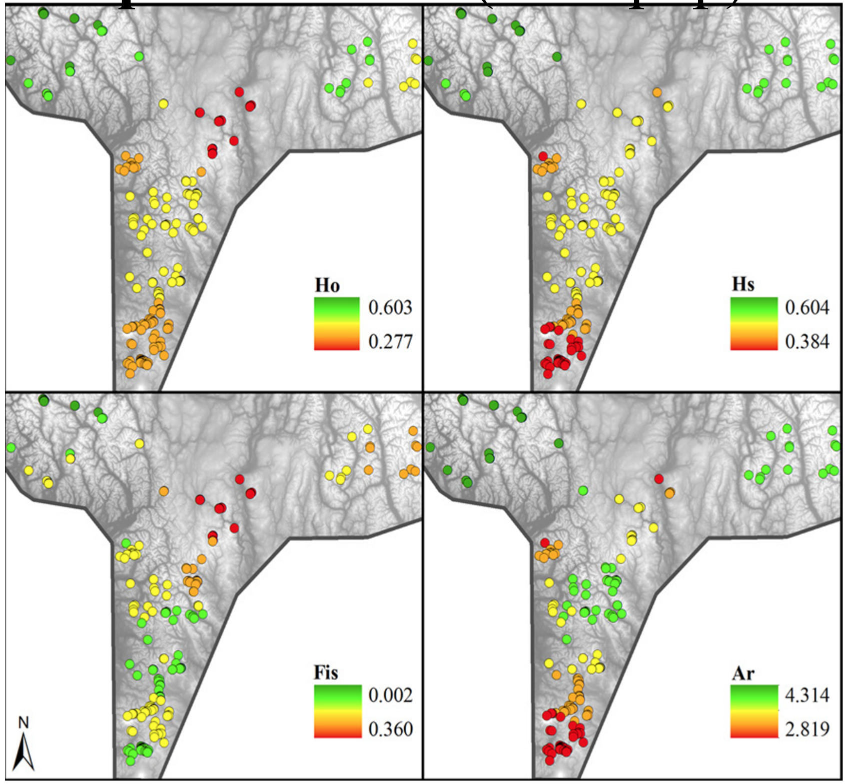


Resistence

Q. How landscapes shape gene flow? How climate change impact on landscape genetics?

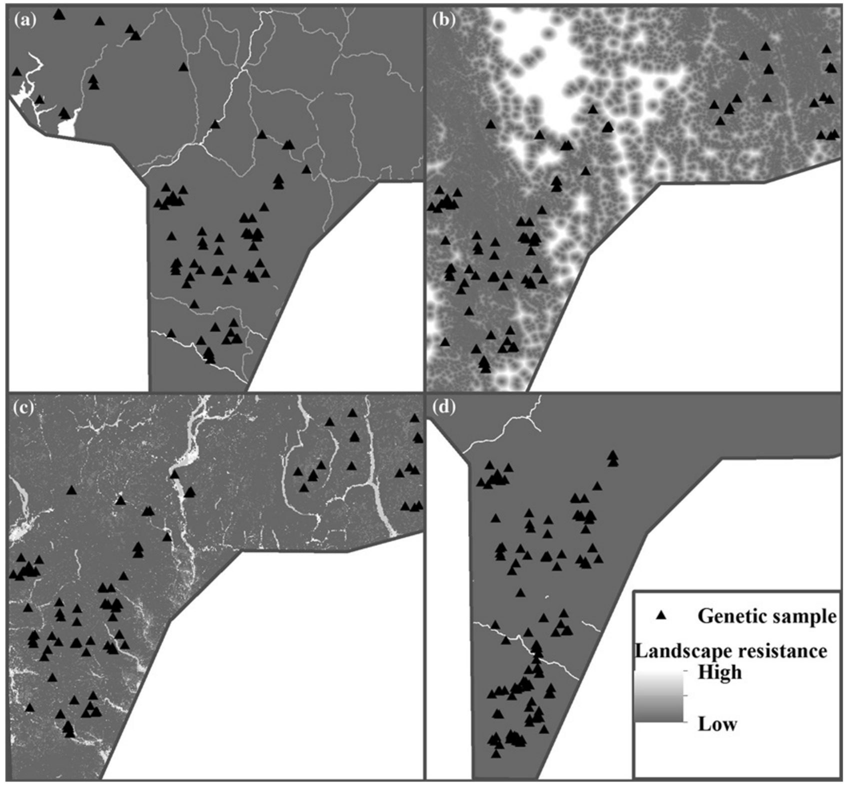
A. Dominant landscape variables limiting gene flow varied across the study area

Genetic Diversity in each sample locations (Genepop)



Parks et al. 2015

Quantify resistance distance between sample locations (Circuitscape 3.5.8)

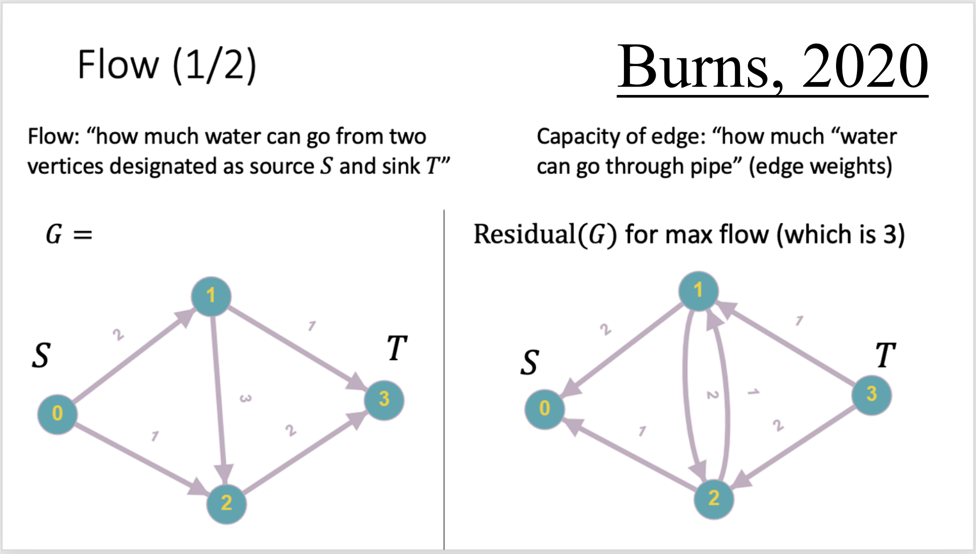


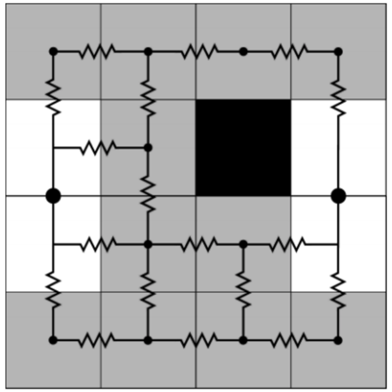
Parks et al. 2015

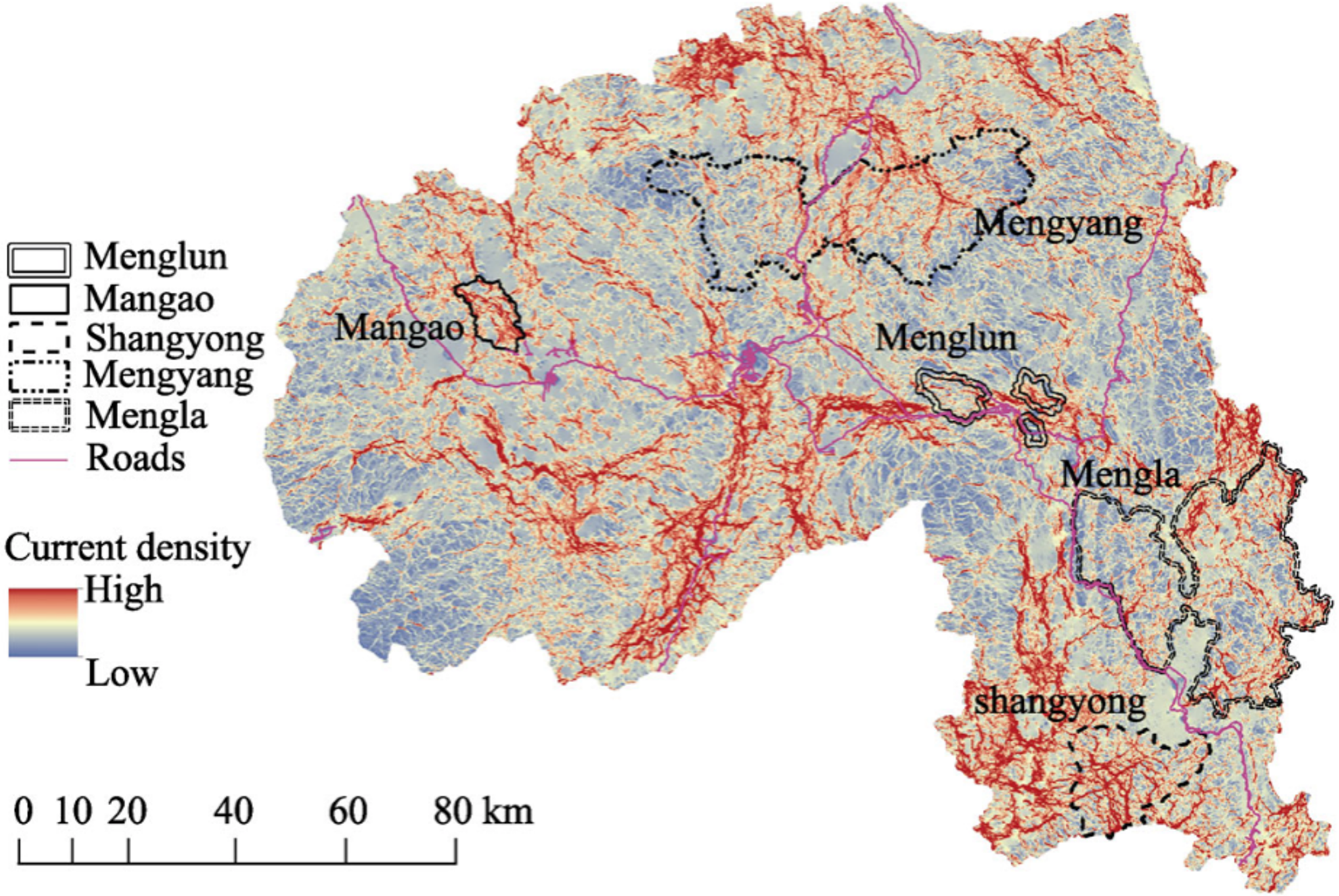
*multiple pathways through intervening populations over many generations

Flow for movement ecology

Flow plus Resistance



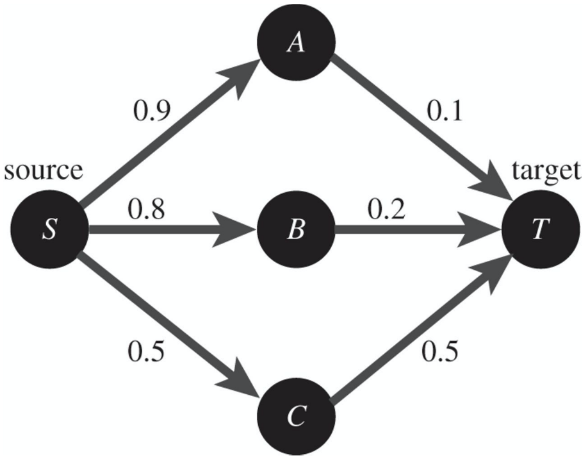


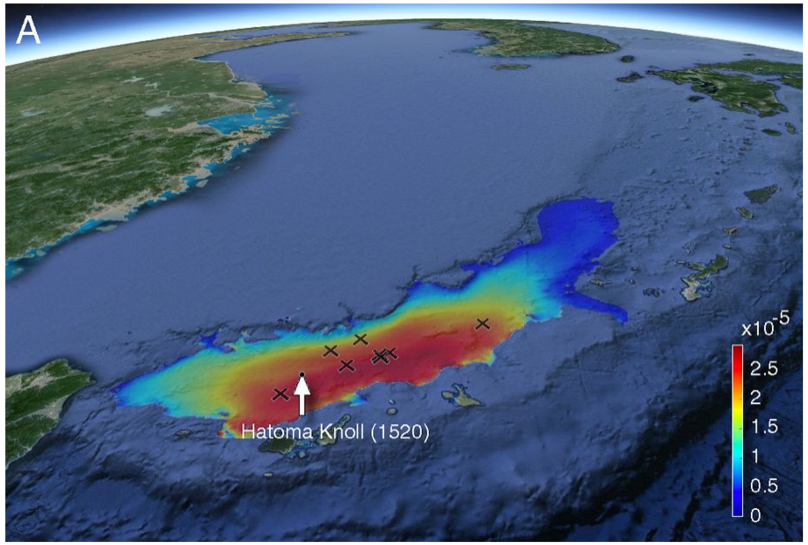


Yin *et al.*, 2019

Landscape resistance between nature reserves in China

Flow for movement ecology (Probabilistic pathways)

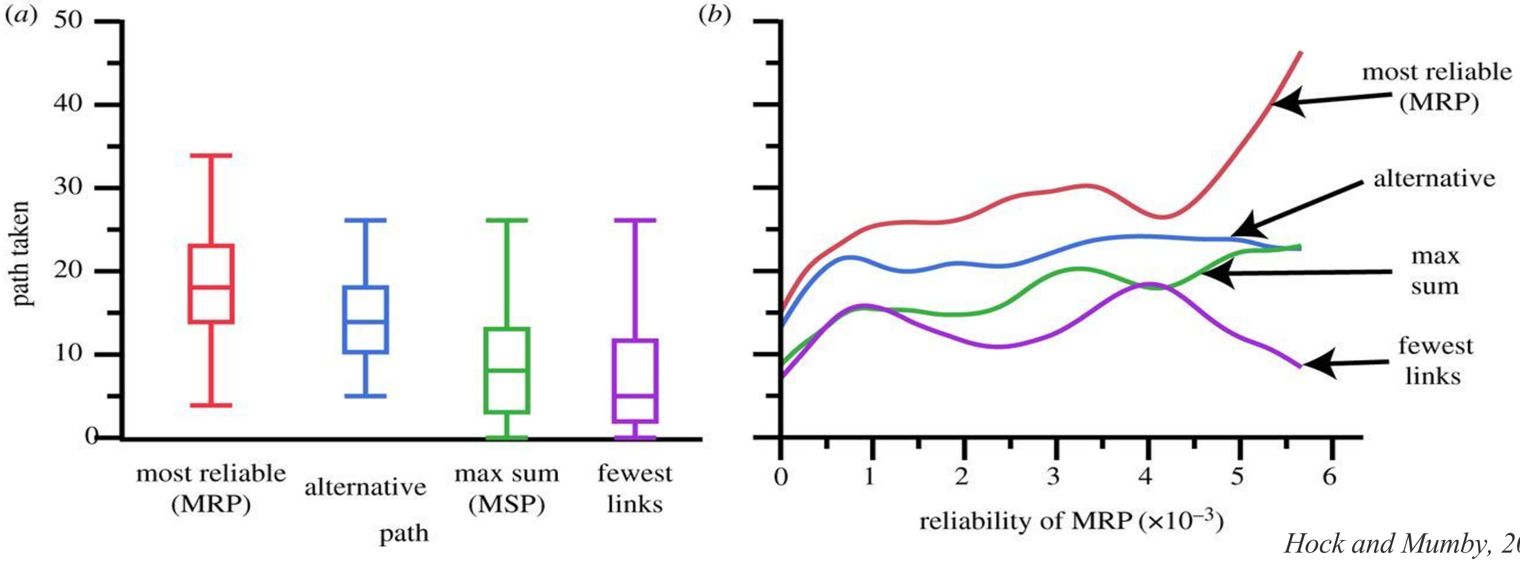




<u>**Identifying the most reliable path through a probabilistic network.**</u>

Simple addition of links would not **identify a difference among paths**.

Multiplication reveals that will be the most reliable path (SCT) through which ***T* could become occupied from *S*.**



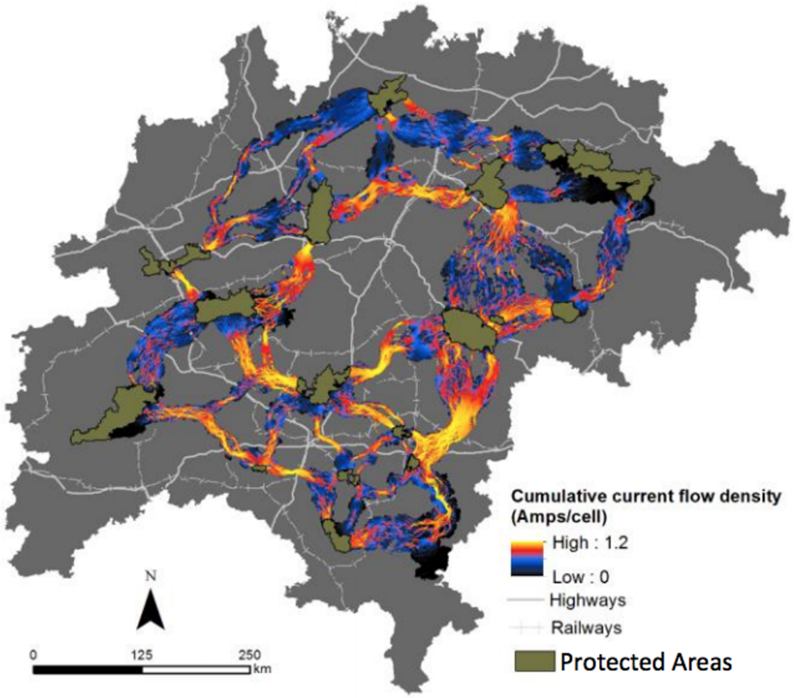
Hock and Mumby, 2015
Quantifying the reliability of dispersal paths in connectivity networks.

Ecological connectivity

Improved insights into landscape dynamics, animal movement, and habitat-use studies and through the development of new software tools for data analysis and visualization.

Circuitscape(software)
(https://circuitscape.org/)

- revealing important areas for connectivity over continuous landscape gradients
- identify key places for mitigating or restoring connectivity
- coupling between near real-time data on landscape dynamics,



Now for exercises/demos!

https://colab.research.google.com/drive/1xOrgsGXXG_MPzugCWa_Hz3C2PEvBSwtc?usp=sharing

Bonus content
(if there's time)

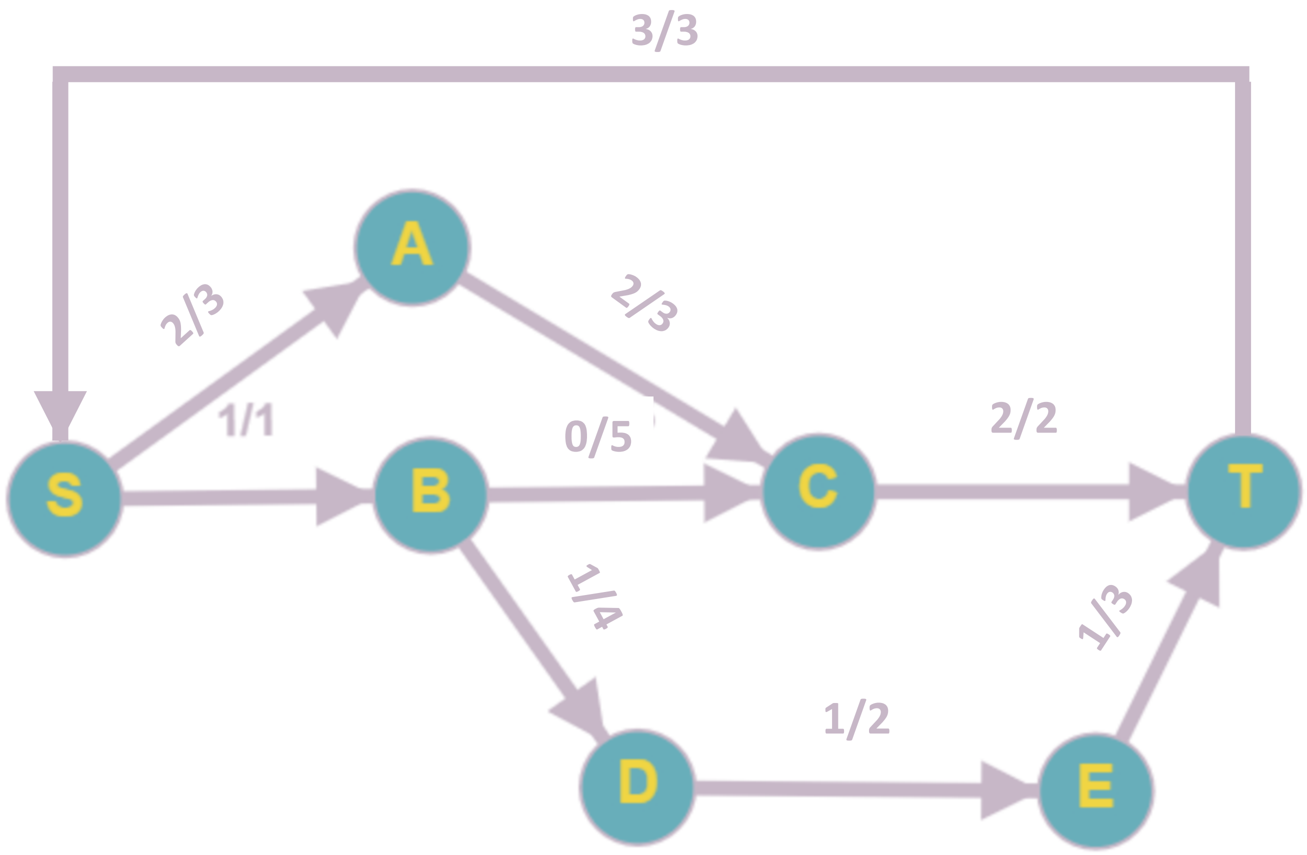
(H-)Circulations

Circulations are flows where no source or sink are specified and obey:

1. Capacity Rule: For all edges, $f \leq C$
2. Skew Symmetry: For all vertices, indegree flow is equal to outdegree flow

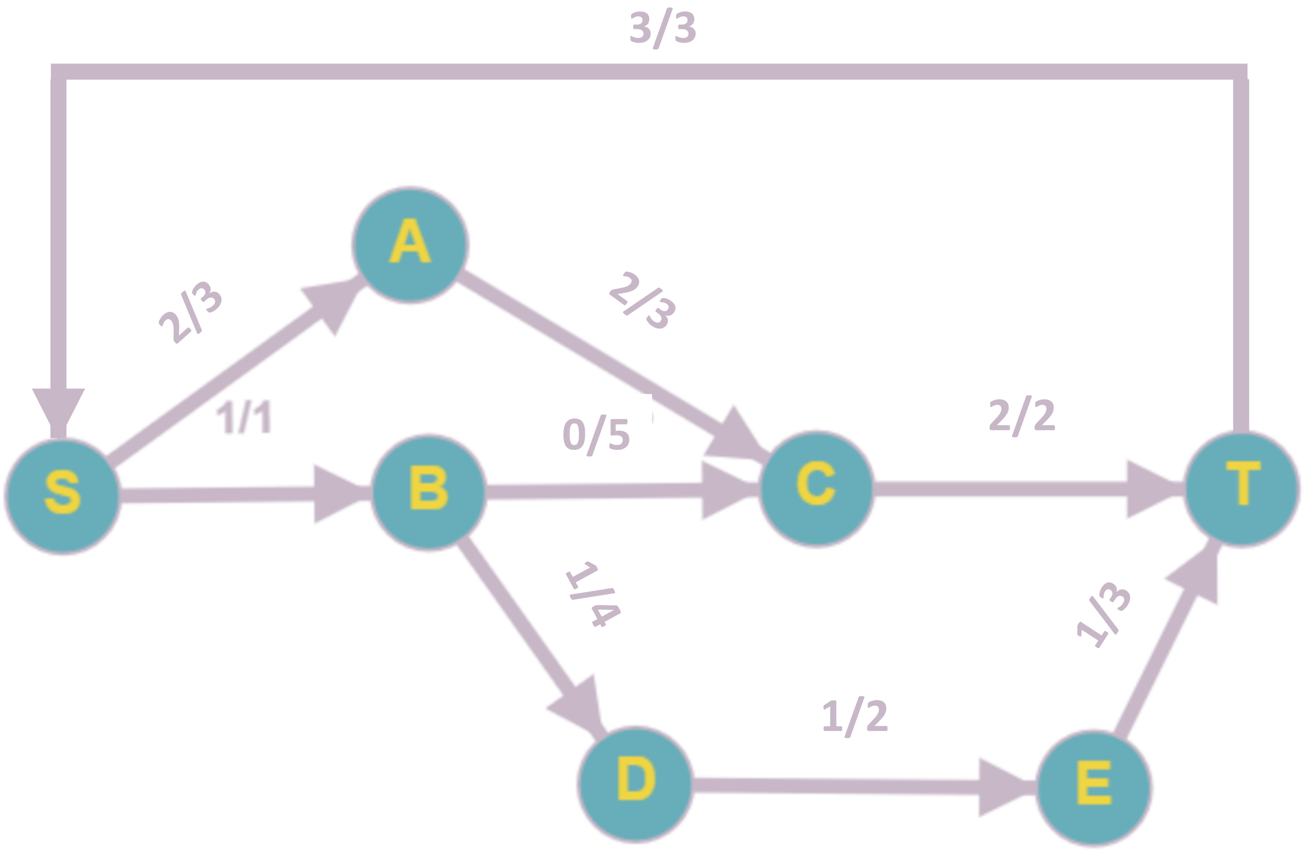
- H-circulations take flow values from an abelian semigroup (e.g. $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ …)
- If a H-circulation is nowhere zero, it is called a H*-flow*

e.g. add (t, s) to a flow



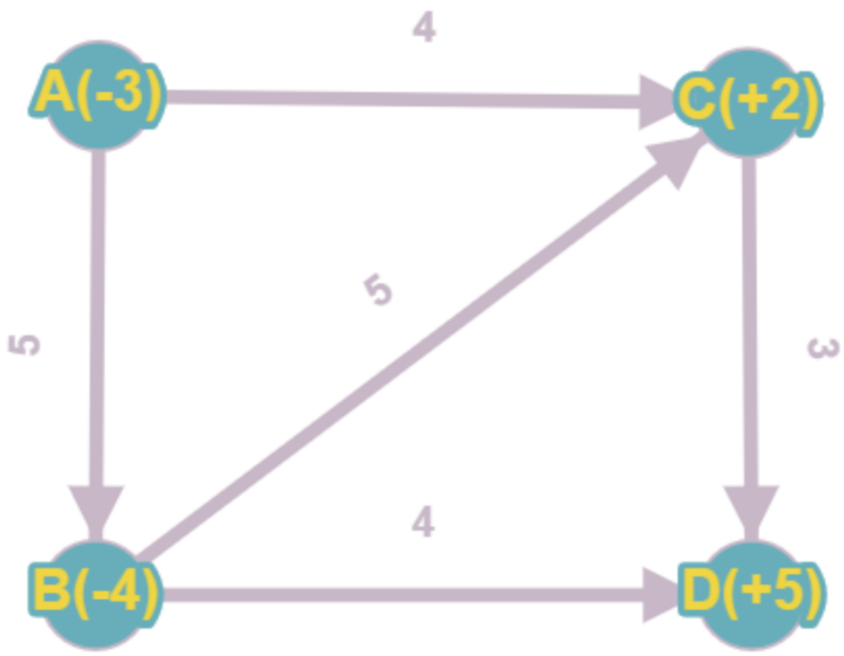
Facts of circulations

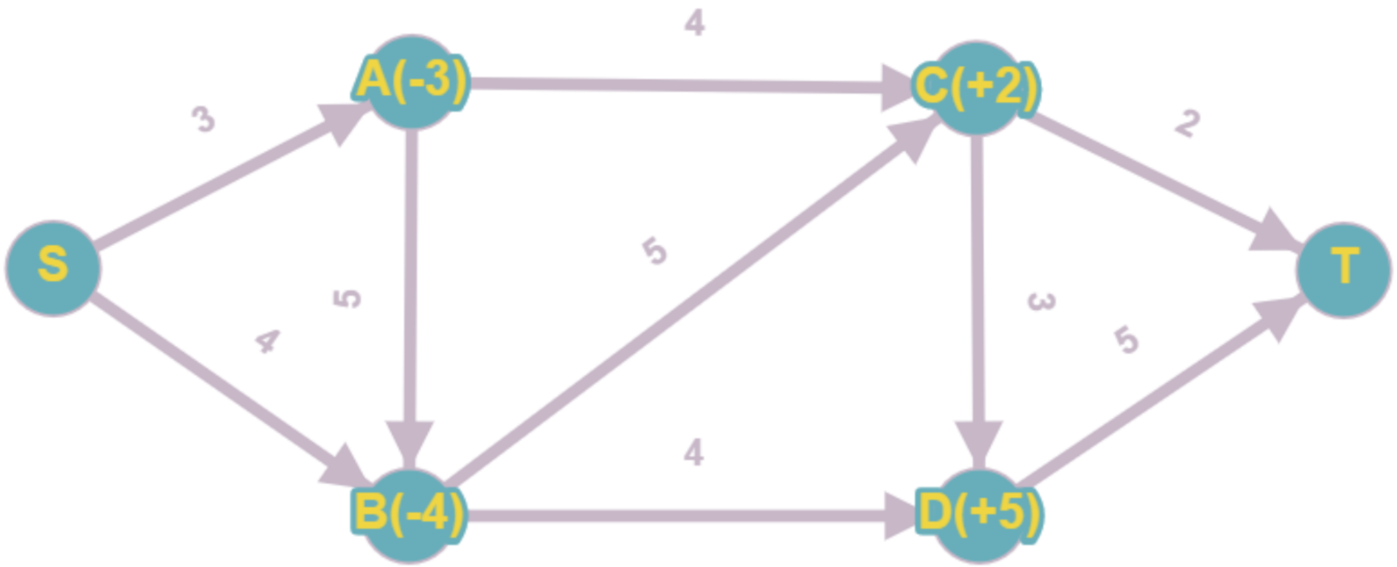
1. Any sub-graph a of circulation has zero net flow / any cut across a circulation has zero net flow.
2. For every multigraph G there exists a polynomial P such that, for any finite abelian group H, the number of H-flows on G is $P(|H|-1)$. P is known as the flow polynomial of G.

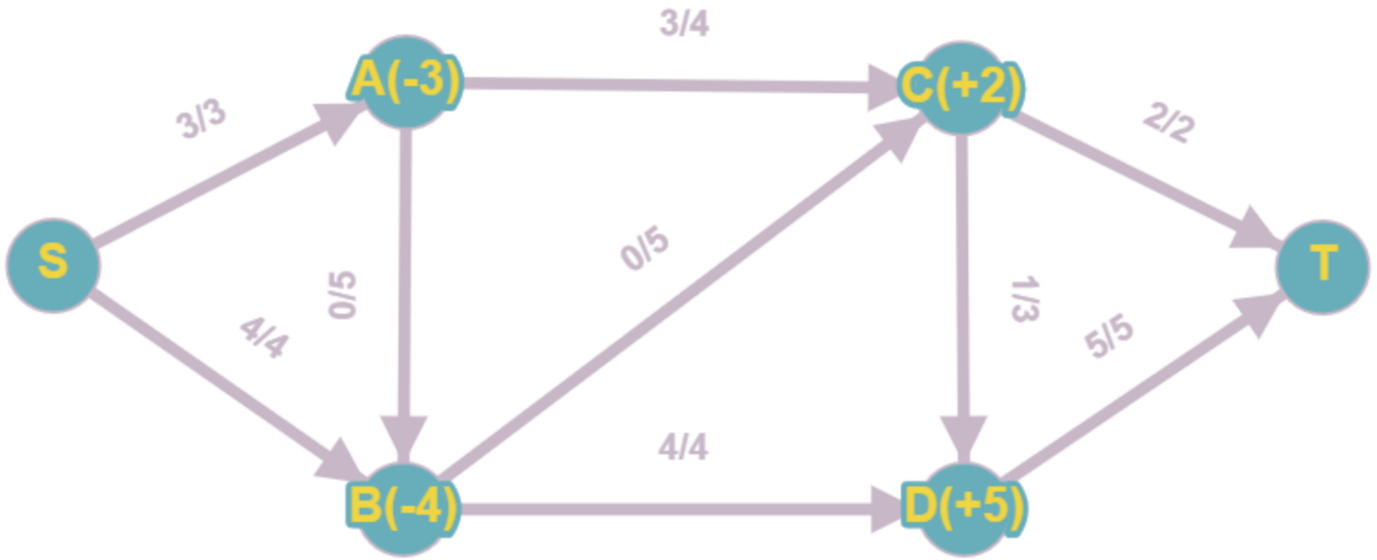


Circulations with demands

Circulations with demands are circulations where vertices act as sources or sinks (demands)

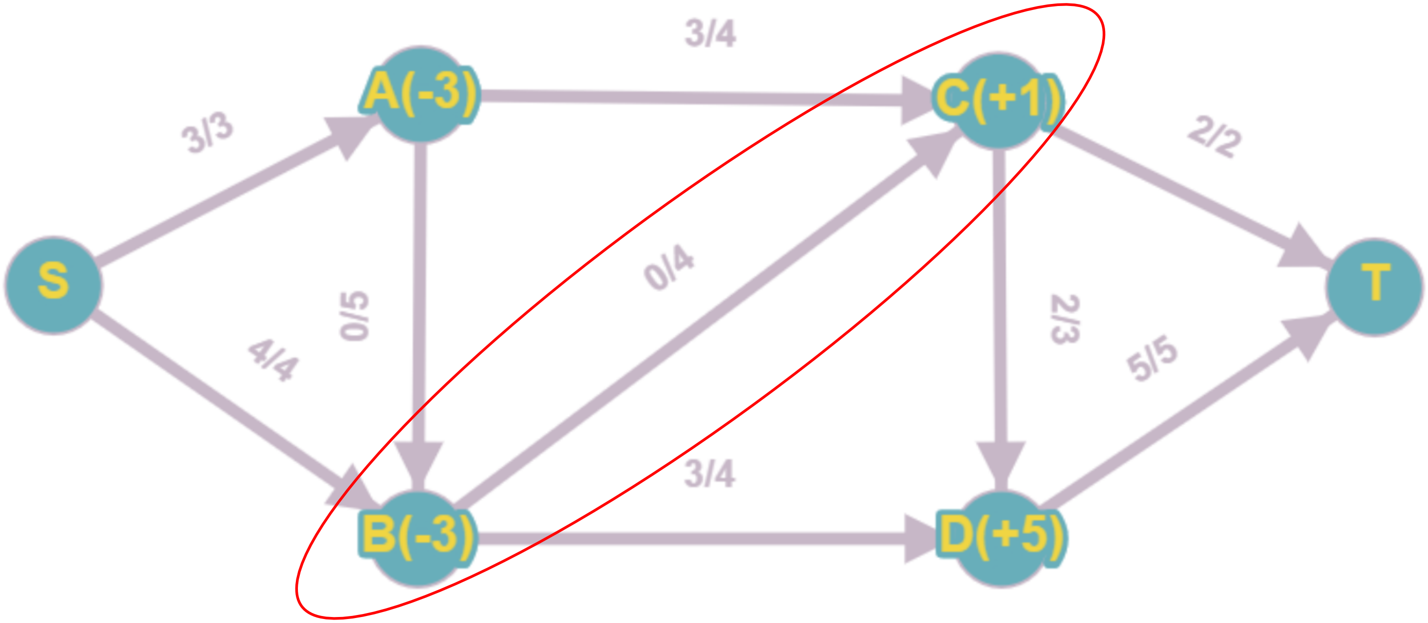


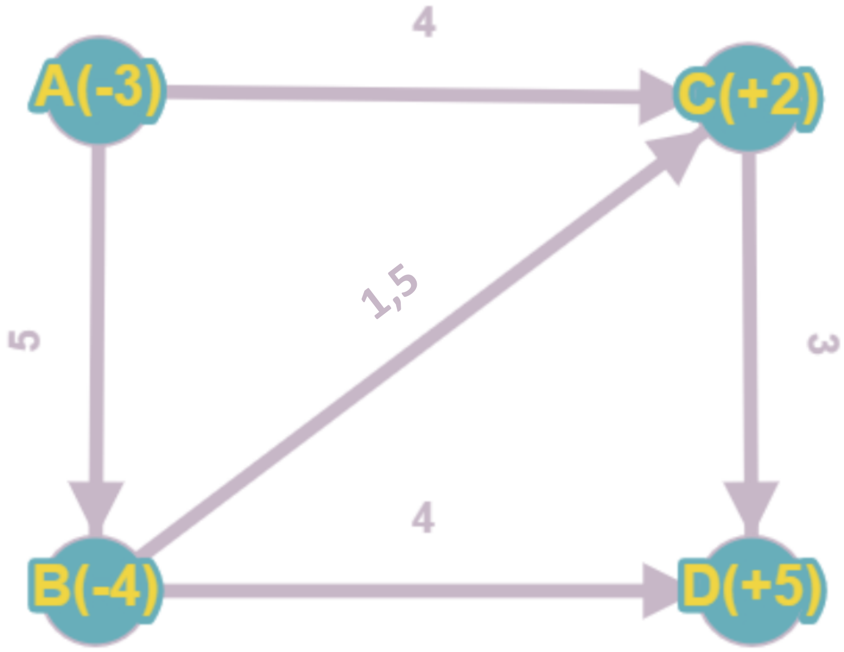


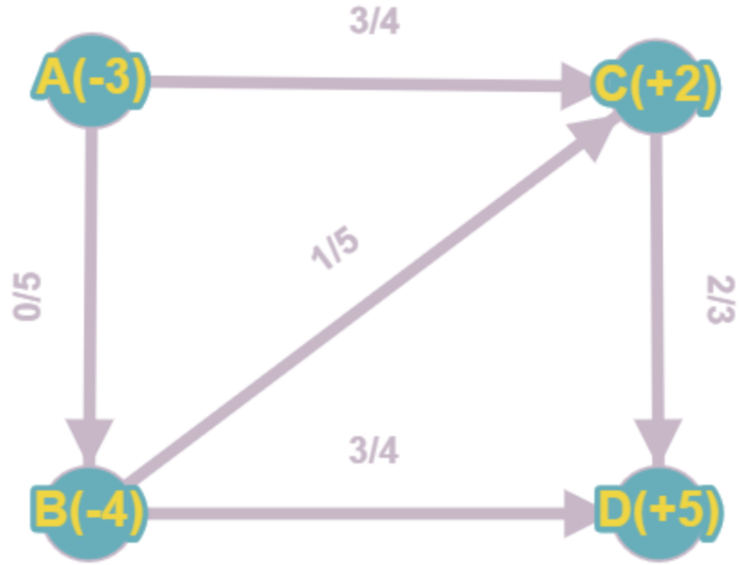


Circulations with demands and lower flow bounds

Circulations with demands and lower flow bounds require a minimum amount of flow along some edge(s)







k-flows and a graph's "flow number"

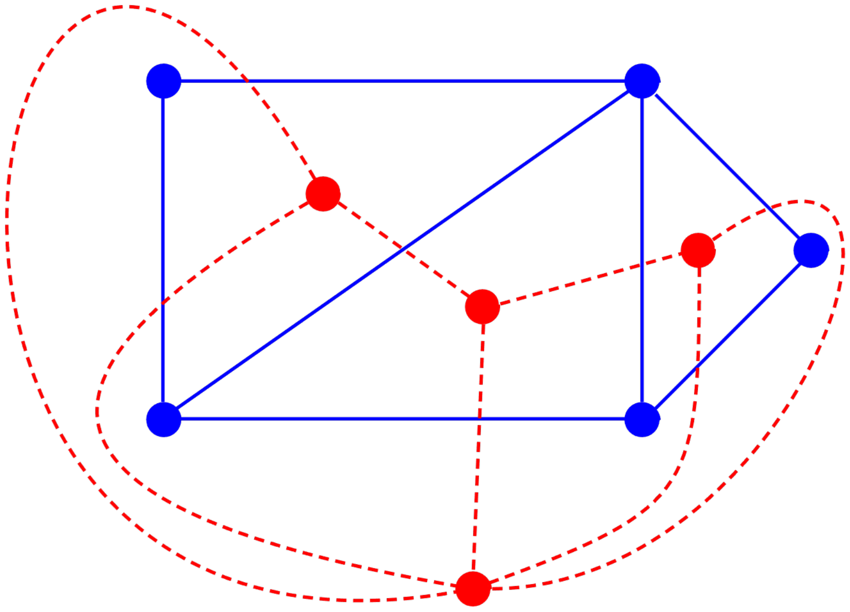
- k-flows are $\mathbb{Z}$-flows where $0 < f(e) < k$ for all edges $e \in E$ in a graph G
- All k-flows are $k + 1$-flows
- The lowest value of k is the flow number of a graph $\varphi(G)$

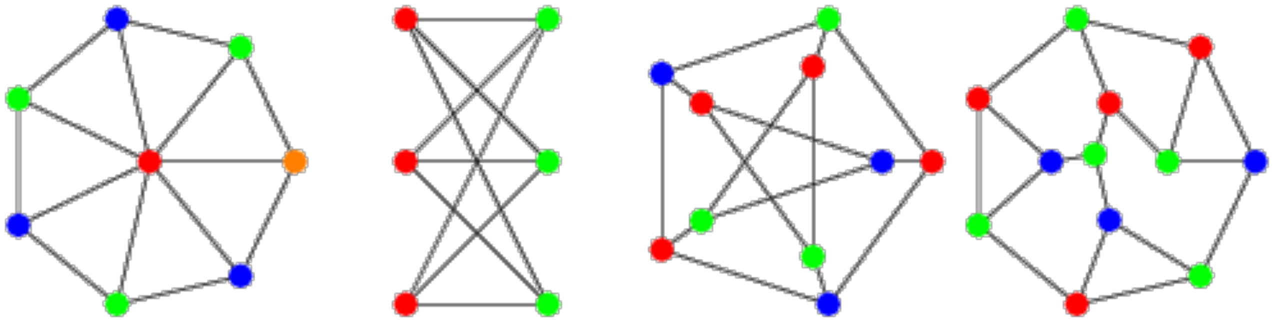
Some facts of k-flows:

1. A graph has a 2-flow iff all its degrees are even.
2. A cubic graph (graph with all degrees=3) has a 3-flow iff it is bipartite.
3. For all even $n > 4$, $\varphi(K^n) = 3$
4. Every 4-edge-connected graph has a 4-flow.
5. A graph has a 4-flow iff it is the union of two even subgraphs.
6. A cubic graph has a 4-flow iff it is 3-edge-colourable.

Flow-colouring duality

Every k-flow on a plane multigraph gives rise to a k-vertex-colouring of its dual





Tutte's Flow Conjectures

True for planar graphs (unproven for non-planar):

1. Every bridgeless multigraph has a 5-flow. (1954)

2. Every bridgeless multigraph not containing the Peterson graph as a minor has a 4-flow. (1966)

3. Every multigraph without a cut consisting of exactly 1 or exactly 3 edges has a 3-flow. (1972)

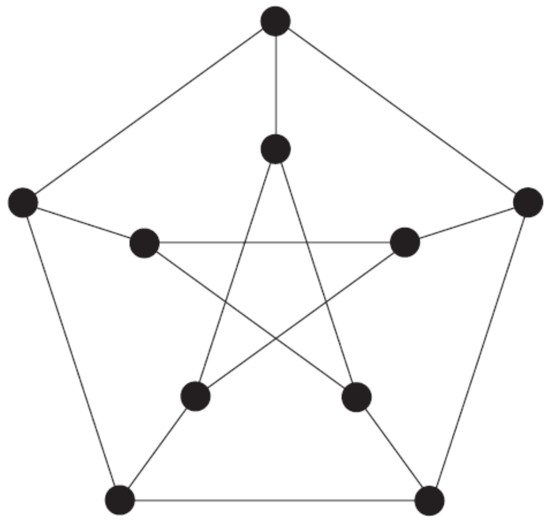


Fig. 6.6.1. The Petersen graph

Proven by Seymour in 1981:

Every bridgeless graph has a 6-flow.