

# 6d superconformal field theories I

(lecture 3/4)

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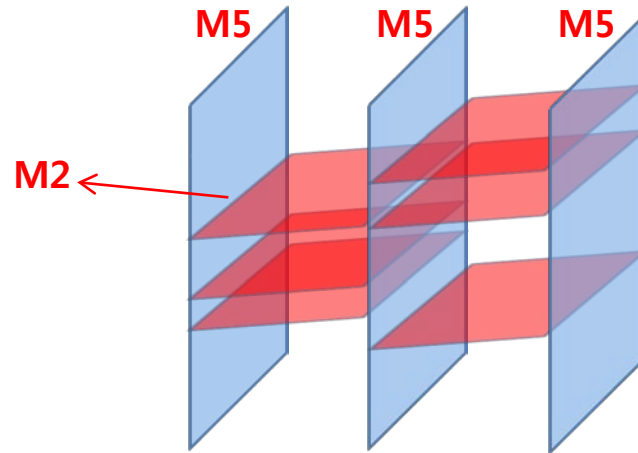
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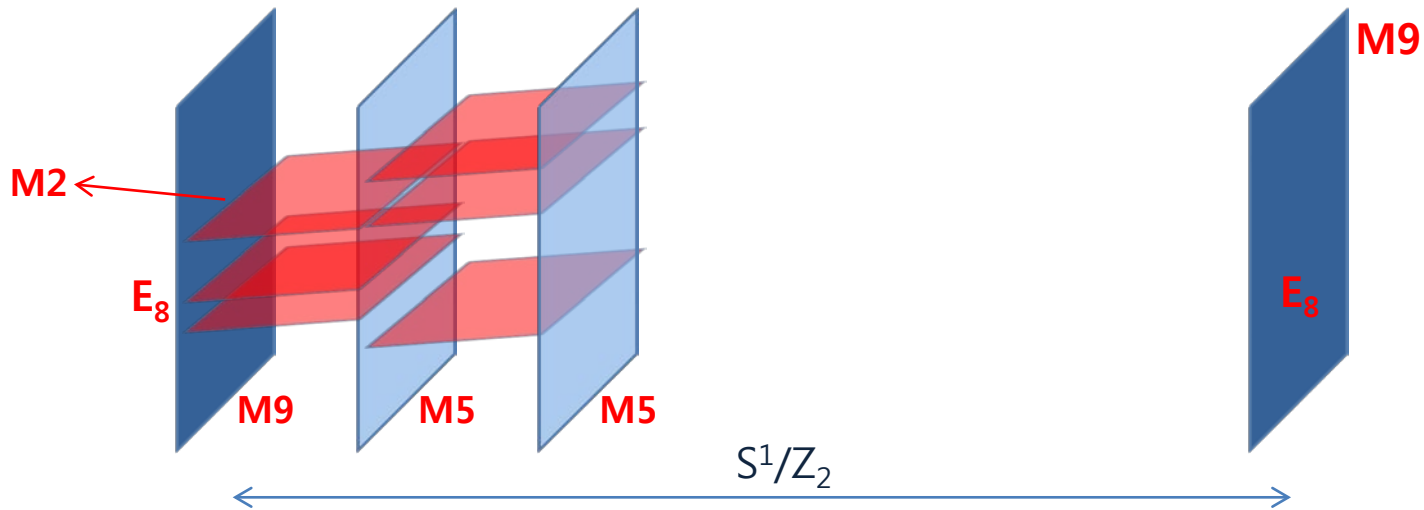
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# Two 6d SCFTs of our interest today

- (2,0) SCFT



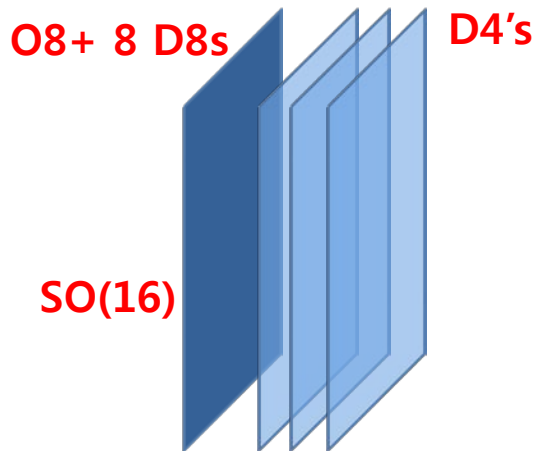
- (1,0) SCFT on M5-M9: “E-string theory” (non-perturbative sector of HE)



- important building block in the F-theory models

# Properties

- $N^3$  d.o.f.: in various observables (thermal free energy, anomalies, vacuum Casimir energy)
- Strings in the tensor branch, tensionless in CFT
- $S^1$  reduction: 5d SYM descriptions vs. strong-coupling 5d CFT
  - (2,0) theory: 5d MSYM w/ ADE gauge groups
  - E-string theory:  $S^1$  reduction yields **strongly coupled 5d SCFT** w/  $E_8$  (yesterday)
  - Reduce E-string theory on  $S^1$  w/  $E_8$  background Wilson line: HM to type I'



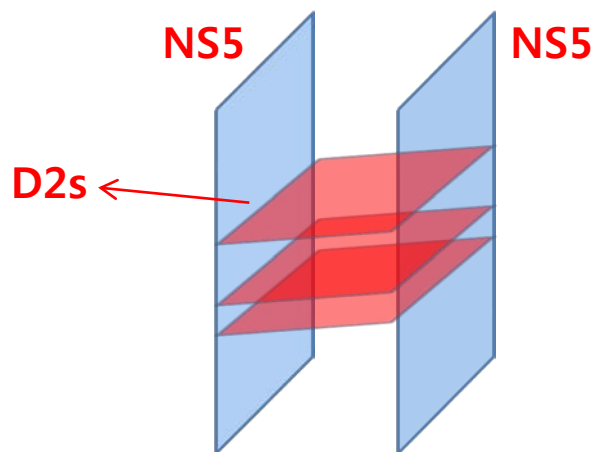
$$248 \rightarrow 120 + 128$$

$$RA_9 = (0, 0, 0, 0, 0, 0, 0, 2\pi)$$

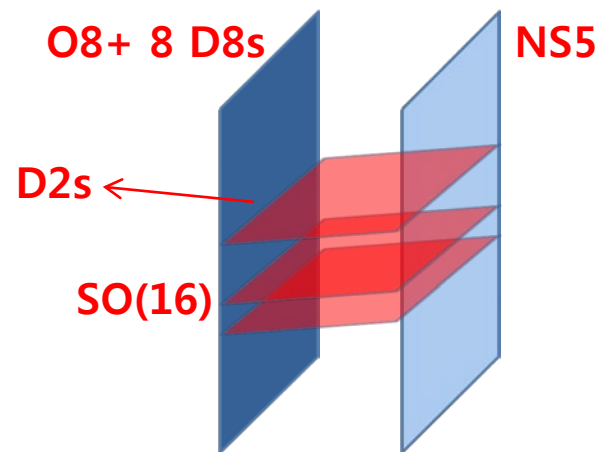
- 5d  $Sp(N)$  **super-Yang-Mills** w/  $N_f = 8$  fundamental + 1 antisymm. hypers

# 6d CFTs in tensor branch

- M-strings & E-strings: admits 2d QFT approaches



and more useful variant (later)



	0	1	2	3	4	5	6	7	8	9
NS5	•	•		•	•	•	•			
D8-O8	•	•		•	•	•	•	•	•	•
D2	•	•	•							

- Partition functions after  $S^1$  compactifications: e.g. elliptic genera
- If 5d SYM indeed sees 6d physics,

$$Z^{\mathbb{R}^4 \times T^2}(q, v) = Z_{\text{NS5}}(q) \sum_{n=0}^{\infty} Z_n^{T^2}(q) e^{-n\lambda} \stackrel{?}{=} Z^{\mathbb{R}^4 \times S^1}(q, v) = Z_{\text{pert}}(v) \sum_{k=0}^{\infty} q^k Z_k(v)$$

# Instanton partition functions: (2,0)

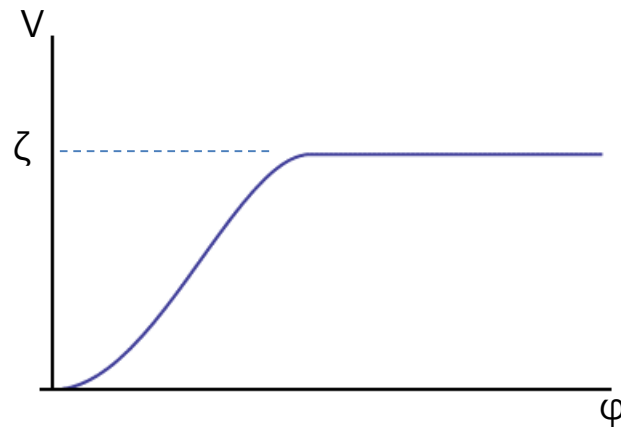
- Witten index:  $Z_{\text{Nek}} = \text{Tr} \left[ (-1)^F q^k e^{-\beta\{Q, Q^\dagger\}} e^{-\epsilon_+ (2J_R^3 + J_1 + J_2)} e^{-\epsilon_- (J_1 - J_2)} e^{-\text{tr}(v\Pi)} \right]$  (flavor fugacities)

- $SU(2)_F$  in  $SO(5)$  chemical potential plays a role similar to 5d  $N=1^*$  mass  $e^{2mJ_F}$
- index of 1d  $N=(4,4)$  mechanics:  $U(k)$  or  $Sp(k)$  gauge groups

$$L_{QM} = \frac{1}{g_{QM}^2} \text{tr}_{k,N} \left[ \frac{1}{2} (D_t \varphi^I)^2 + \frac{1}{2} (D_t a_m)^2 + \frac{1}{4} [\varphi^I, \varphi^J]^2 + \frac{1}{2} [a_m, \varphi^I]^2 \right. \\ \left. + D_t q_{\dot{\alpha}} D_t \bar{q}^{\dot{\alpha}} - (\varphi^I \bar{q}^{\dot{\alpha}} - \bar{q}^{\dot{\alpha}} v^I) (q_{\dot{\alpha}} \varphi^I - v^I q_{\dot{\alpha}}) - D_{\dot{\beta}}^{\dot{\alpha}} D_{\dot{\alpha}}^{\dot{\beta}} + \dots \right] \\ D_{\dot{\beta}}^{\dot{\alpha}} = \bar{q}^{\dot{\alpha}} q_{\dot{\beta}} - \frac{1}{2} \zeta^A (\tau^A)^{\dot{\alpha}}_{\dot{\beta}} + \frac{1}{4} (\bar{\sigma}^{mn})^{\dot{\alpha}}_{\dot{\beta}} [a_m, a_n] - \frac{1}{2} \delta_{\dot{\beta}}^{\dot{\alpha}} (\text{trace})$$

- $U(N)$ : [Nekrasov] 2002 [H.-C.Kim, Koh, SK, K. Lee, S. Lee] 2011
- decoupled sector (D0 escaping D4 into  $R^5$ )
- no  $Z_{\text{extra}}$ : introduce FI deformation

$$V \leftarrow |\phi^i q_{\dot{\alpha}}|^2 + \left( (\tau^I)^{\dot{\beta}}_{\dot{\alpha}} q_{\dot{\alpha}} \bar{q}^{\dot{\beta}} - \zeta^I \right)^2$$



- $SO(2N)$ :  $Z_{\text{extra}}$  from  $k$  D0 moving on  $R^5/Z_2$  [Y. Hwang, J. Kim, SK] work in progress
- $E_n$ : little to say systematically, but some statements later

# The answers

- The results are given in terms of residue sums:
- U(N): nonzero JK-Res classified using N-colored Young diagrams w/ k boxes

[Flume, Poghossian] [Bruzzo, Fucito, Morales, Tanzini] 2002, [Hwang, J. Kim, SK, Park] 2014

$$Z_k = \sum_{\sum_i |Y_i|=k} \prod_{i,j=1}^N \prod_{s \in Y_i} \frac{\sinh \frac{E_{ij}+m-\epsilon_+}{2} \sinh \frac{E_{ij}-m-\epsilon_+}{2}}{\sinh \frac{E_{ij}}{2} \sinh \frac{E_{ij}-2\epsilon_+}{2}}$$

$$E_{ij} = \alpha_i - \alpha_j - \epsilon_1 h_i(s) + \epsilon_2 (v_j(s) + 1)$$

$h_i(s)$ : distance from box "s" to the right end of i'th Young diagram

$v_j(s)$ : distance from box "s" to the lower end of the j'th Young diagram

- SO(2N): very complicated, if one lists all nonzero JK-Res.

- Question: Are they really 6d partition functions at all?
- Consider U(N) in detail. Separate the analysis to U(1) and SU(N). [H.-C.Kim, SK, Koh, K. Lee, S. Lee] 2011, [Haghighat, Iqbal, Kozcaz, Lockhart, Vafa] 2013

(Similar results for SO(2N) are partly available [Y. Hwang, J. Kim, SK] work in progress)

# The KK momenta: U(1)

- This is the 5d/6d version of a fundamental problem in M-theory:
  - “bulk” problem: 11d SUGRA is free in IR, making one multiplet of KK fields at each k.
  - Do D0’s make unique threshold bounds? [Yi] [Sethi, Stern] (1997)

$$L_{\text{bulk}} = \frac{1}{g_{QM}^2} \text{tr}_k \left[ \frac{1}{2} (D_t X^M)^2 + \frac{1}{4} [X^I, X^J]^2 + \dots \right] \quad X^M = (a_m, \varphi^I)$$

- Similar question holds for 5d U(1) instantons, since 6d Abelian (2,0) is free.

- U(1) partition function [Iqbal, Kozcaz, Shabbir] [Awata, Kanno] [H.-C. Kim, SK, Koh, K. Lee, S. Lee]

$$Z[q] = PE \left[ I_-(\epsilon_{1,2}, m) \frac{q}{1-q} \right] = 1 + q I_- + q^2 \left( \frac{I_-^2 + I_-(\cdot^2)}{2} + I_- \right) + q^3 \left( \frac{I_-^3 + 3I_- I_-(\cdot^2) + I_-(\cdot^3)}{6} + I_-^2 + I_- \right) + \dots$$

$$I_-(e^{-\epsilon_{1,2}}, e^m) = \frac{\sinh \frac{m+\epsilon_-}{2} \sinh \frac{m-\epsilon_-}{2}}{\sinh \frac{\epsilon_1}{2} \sinh \frac{\epsilon_2}{2}} \quad PE[f(x)] = \exp \left[ \sum_{n=1}^{\infty} \frac{1}{n} f(x^n) \right]$$

- Single particle index:

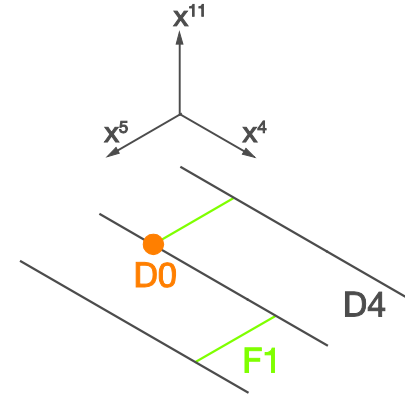
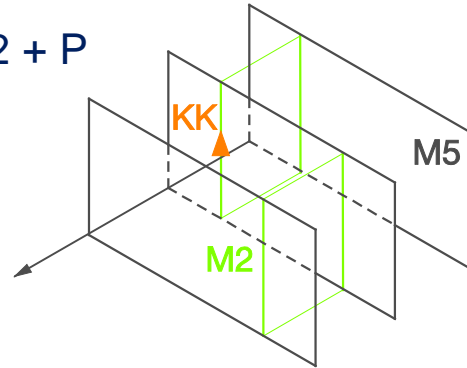
one 5d massive tensor multiplet for every k.

	$SU(2)_{1L}$	$SU(2)_{1R}$	$SU(2)_{2L}$	$SU(2)_{2R}$
$B_2$	3	1	1	1
$\phi_I$	1	1	2	2
	1	1	1	1
$\lambda$	2	1	2	1
	2	1	1	2



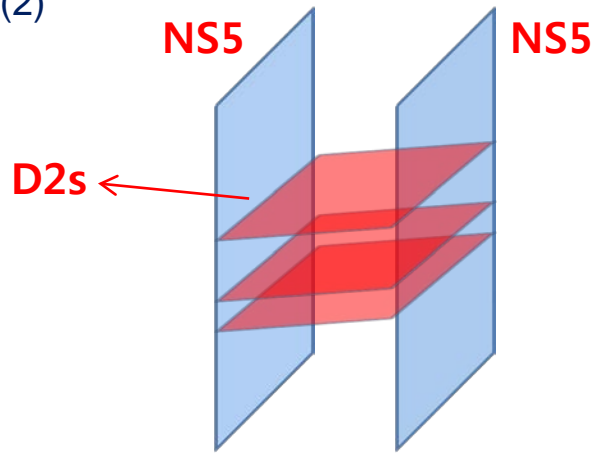
# SU(N): "M-strings"

- bound states of D0+F1: ~ open M2 + P

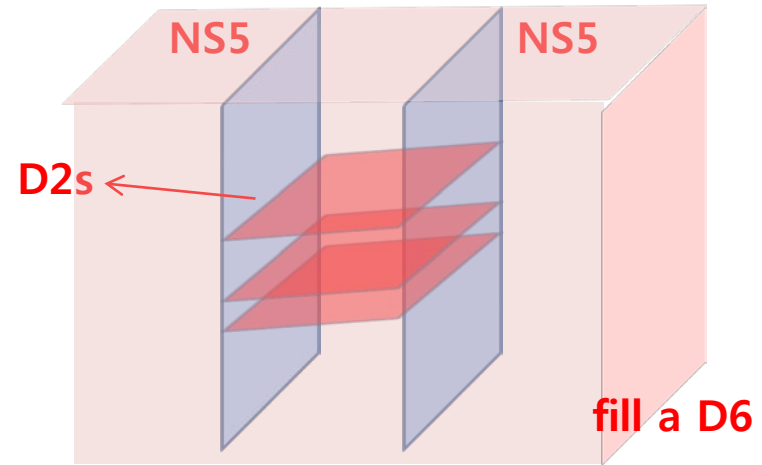


- Seeing 6d physics ~ seeing 2d QFT on open M2's.

- Branes: e.g. SU(2)



2d N=(4,4) gauge theories



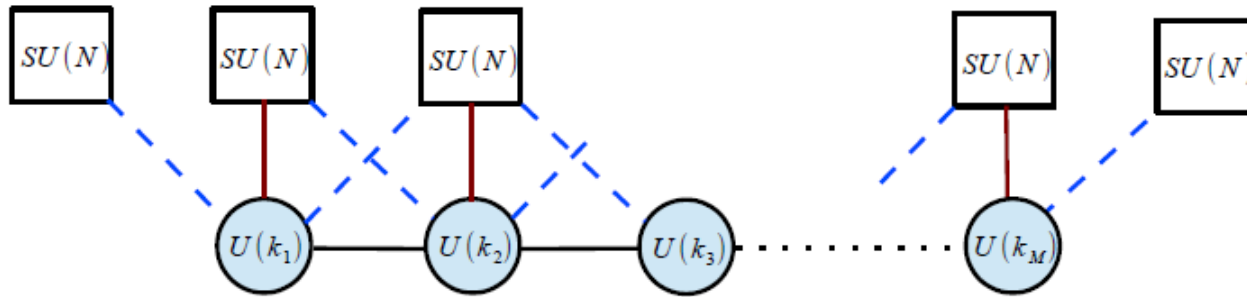
2d N=(0,4) gauge theories

[2d IR limit ~ strong coupling limit ~ decompactifying]

	0	1	2	3	4	5	6	7	8	9
NS5	•	•	•	•	•	•	-	-	-	-
D6	•	•	•	•	•	•	•	-	-	-
D2	•	•	-	-	-	-	•	-	-	-

# SU(N) instantons vs. M-strings

- 2d N=(0,4) quiver: [Haghighat, Iqbal, Kozcaz, Lockhart, Vafa] [HKLV] 2013



vector :  $A_\mu, \lambda_{A\dot{\alpha}}$

hyper :  $\varphi_{\dot{\alpha}}, \psi_A$

twisted hyper :  $\varphi_A, \psi_{\dot{\alpha}}$

Fermi :  $\Psi$

- M-strings' elliptic genera: [Benini, Eager, Hori, Tachikawa]

$$Z = \frac{1}{(2\pi i)^r} \oint \frac{1}{|W(G)|} Z_{1\text{-loop}} \quad Z_{\text{vec}} = \prod_{I=1}^r \frac{2\pi\eta^2 du_I}{i} \cdot \prod_{\alpha \in \text{root}} \frac{\theta_1(\alpha \cdot u)}{i\eta} \quad Z_{\text{chiral}} = \frac{i\eta}{\theta_1(Q \cdot u + \rho \cdot z)}, \quad Z_{\text{Fermi}} = \frac{\theta_1(Q \cdot u + \rho \cdot z)}{i\eta}$$

- Up to high orders in  $q$  & Coulomb VEVs,

$$Z^{\mathbb{R}^4 \times T^2}(q, v) = Z_{\text{NS5}}(q) \sum_{n=0}^{\infty} Z_n^{T^2}(q) e^{-n\lambda} \stackrel{?}{=} Z^{\mathbb{R}^4 \times S^1}(q, v) = Z_{\text{pert}}(v) \sum_{k=0}^{\infty} q^k Z_k(v)$$

- So, the 5d MSYM partition function is indeed 6d partition function.

[Often, the 1d (rather than 2d) expressions are more useful to study the CFT in symmetric phase.]

## Application: the superconformal index

- Put the theory on  $S^5 \times \mathbb{R}$ : energy  $E$  ;  $SO(6)$   $j_1, j_2, j_3$  ;  $SO(5)_R$   $R_1, R_2$
- Choose a pair of  $Q, S (= Q^+)$

$$Q_{(j_1, j_2, j_3)}^{(R_1, R_2)} \rightarrow Q_{-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}}^{(\frac{1}{2}, \frac{1}{2})} : \text{ BPS bound } E = 2R_1 + 2R_2 + j_1 + j_2 + j_3$$

[Bhattacharya, Bhattacharyya, Minwalla, Raju] 2008

- Index on  $S^5 \times S^1$ : counts local BPS operators on  $\mathbb{R}^6$

$$I(\beta, m, \epsilon_1, \epsilon_2) = \text{Tr} \left[ (-1)^F e^{-\beta' \{Q, S\}} e^{-\beta(E - \frac{R_1 + R_2}{2})} e^{\beta m (R_1 - R_2)} e^{-\gamma_1(j_1 - j_3)} e^{-\gamma_2(j_2 - j_3)} \right]$$

$$e^{-\gamma_1(j_1 - j_3)} e^{-\gamma_2(j_2 - j_3)} = e^{-\beta(a j_1 + b j_2 + c j_3)} \quad a + b + c = 0$$

# The superconformal index: result

- **Result:** (even technically, some issues on general saddle points are open)

$$\lambda = r\phi \quad (W: \text{Weyl group, } r: \text{rank})$$

Each factor takes the form of Nekrasov partition function on  $R^4 \times S^1$

$$Z(\beta, m, a_i) = \frac{1}{|W|} \int_{-\infty}^{\infty} \left[ \prod_{i=1}^r d\lambda_i \right] \exp \left[ -\frac{2\pi^2 \text{tr} \lambda^2}{\beta(1+a)(1+b)(1+c)} \right] Z_{\text{pert}}^{(1)} Z_{\text{inst}}^{(1)} \cdot Z_{\text{pert}}^{(2)} Z_{\text{inst}}^{(2)} \cdot Z_{\text{pert}}^{(3)} Z_{\text{inst}}^{(3)}$$

$$\begin{aligned} 1 & : (\epsilon_1, \epsilon_2) = \left( \frac{2\pi i(b-a)}{1+a}, \frac{2\pi i(c-a)}{1+a} \right), \quad m_0 = \frac{2\pi i(m + \frac{3}{2}(1+a))}{1+a}, \quad \mu = \frac{2\pi\phi}{1+a}, \quad q = e^{-\frac{4\pi^2}{\beta(1+a)}} \\ 2 & : (\epsilon_1, \epsilon_2) = \left( \frac{2\pi i(c-b)}{1+b}, \frac{2\pi i(a-b)}{1+b} \right), \quad m_0 = \frac{2\pi i(m + \frac{3}{2}(1+b))}{1+b}, \quad \mu = \frac{2\pi\phi}{1+b}, \quad q = e^{-\frac{4\pi^2}{\beta(1+b)}} \\ 3 & : (\epsilon_1, \epsilon_2) = \left( \frac{2\pi i(a-c)}{1+c}, \frac{2\pi i(b-c)}{1+c} \right), \quad m_0 = \frac{2\pi i(m + \frac{3}{2}(1+c))}{1+c}, \quad \mu = \frac{2\pi\phi}{1+c}, \quad q = e^{-\frac{4\pi^2}{\beta(1+c)}} \end{aligned}$$

- proposed from non-renormalizable QFT, but can phrase it in an intrinsic manner.
- All three  $Z_{\text{Nekrasov}}$  factors are replaced by intrinsic  $Z_{R^4 \times T^2}$  of 6d QFT
- “tr” or “integral” over matrix  $\sim$  sum over the tensor branch moduli
- The integral is difficult to carry out in full generality
- But for (2,0) theory, can do so in a simple case.

## The unrefined index

- 16 (maximal) SUSY at  $m = \frac{1}{2}$  or  $-\frac{1}{2}$  &  $a = b = c = 0$ ;

$$\text{tr}[(-1)^F e^{-\beta(E-R_1)}] \quad Q_{(j_1, j_2, j_3)}^{(+\frac{1}{2}, R_2)}$$

- Can work with simpler setting  $m = \frac{1}{2} - c$ : measure commutes w/ 4 SUSY
- Simplifications of three  $Z[\mathbb{R}^4 \times T^2]$ 's at this point: extra SUSY, cancelation.

$$Z(\beta) = \frac{1}{|W|} \int d\lambda e^{-\frac{2\pi^2 \text{tr}(\lambda^2)}{\beta}} \prod_{\alpha} 2 \sinh^2(\pi\alpha(\lambda)) \cdot \eta(2\pi i/\beta)^{-N}$$

# The spectrum: W-algebra

- U(N):

$$Z^{U(N)} = e^{\beta \left( \frac{N(N^2-1)}{6} + \frac{N}{24} \right)} \prod_{n=0}^{\infty} \prod_{s=1}^N \frac{1}{1 - e^{-\beta(n+s)}}$$

- SO(2N):

$$Z^{SO(2N)} = e^{\beta \left( \frac{c_2|G|}{6} + \frac{N}{24} \right)} \prod_{n=0}^{\infty} \left[ \frac{1}{1 - e^{-\beta(n+N)}} \prod_{s=1}^{N-1} \frac{1}{1 - e^{-\beta(n+2s)}} \right]$$

- Large N limits tested against SUGRA indices on AdS7 x S4 (or x S4/Z2)

- General form: (for all ADE)

$$Z^{ADE} = e^{\beta \left( \frac{c_2|G|}{6} + \frac{r}{24} \right)} \prod_{n=0}^{\infty} \prod_{\text{Casimir op.}} \frac{1}{1 - e^{-\beta(n+d)}} \quad \text{d: orders of the Casimir invariants}$$

- We only computed perturbative part of  $E_n$ , but is compatible with this full form.
- These turn out to be the so-called “vacuum characters” of the  $W_G$  algebra.
- subset of local operators isomorphic to W-algebra, playing important roles in the mini-bootstrap: [Beem, Rastelli, Van Rees] 2014
- generalized with 6d surface operators [Bullimore, Hee-Cheol Kim] 2014

# The Casimir energy

- The prefactor takes the form of “vacuum energy.”  $e^{-\beta\epsilon_0} \equiv e^{\beta\left(\frac{c^2|G|}{6} + \frac{r}{24}\right)}$

- Consider a simple example of free QFT on  $S^n \times \mathbb{R}$ .

$$\epsilon_0 \equiv \text{tr} \left[ (-1)^F \frac{E}{2} \right] = \sum_{\text{bosonic modes}} \frac{E}{2} - \sum_{\text{fermionic modes}} \frac{E}{2}$$

- regularize/renormalize the infinite sum: symmetries of the problem constrain it.

$$\epsilon_0 = \lim_{\beta' \rightarrow 0} \text{tr} \left[ (-1)^F \frac{E}{2} e^{-\beta' E} \right]$$

- In the index, these are constrained by different symmetries: Maximal SUSY

$$(\epsilon_0)_{\text{index}} \equiv \text{tr} \left[ (-1)^F \frac{E - R_1}{2} \right] = \sum_{\text{bosonic modes}} \frac{E - R_1}{2} - \sum_{\text{fermionic modes}} \frac{E - R_1}{2} \quad (\epsilon_0)_{\text{index}} = \lim_{\beta' \rightarrow 0} \text{tr} \left[ (-1)^F \frac{E - R_1}{2} e^{-\beta' (E - R_1)} \right]$$

- “supersymmetric Casimir energy”: morally, it is also a kind of measure of d.o.f.

- Indeed, it sees  $N^3$ .

- Question: SUSY Casimir energy from gravity dual?

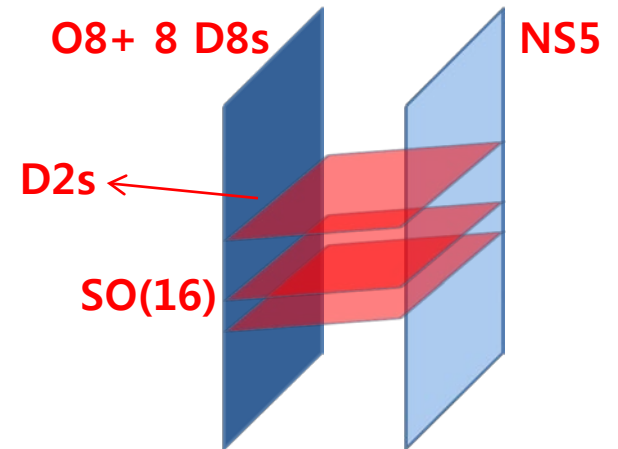
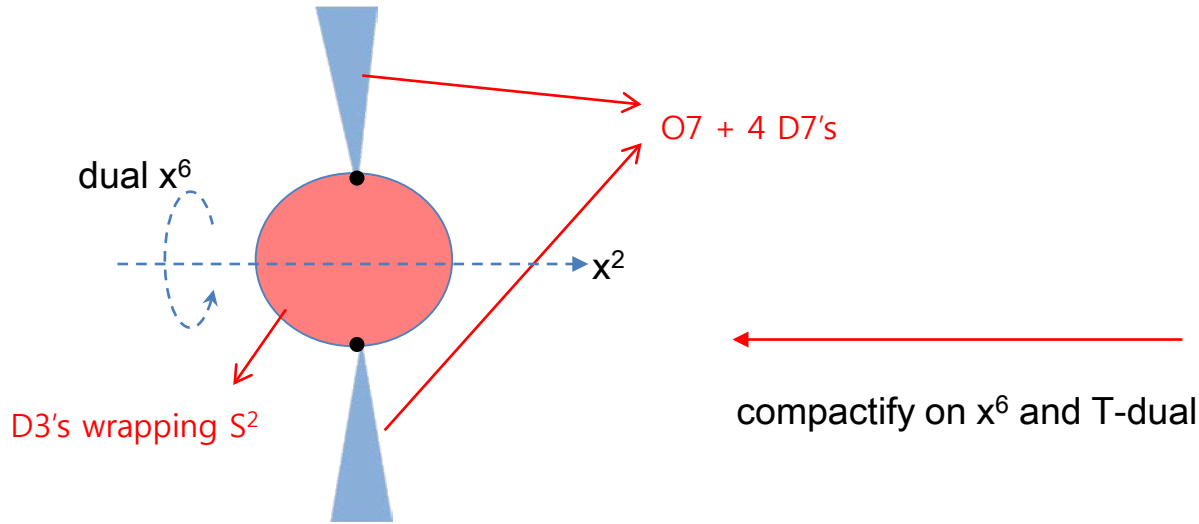
- “SUSY Casimir energy” = chiral anomalies in  $\Omega$ -background

[Bobev, Bullimore, Hee-Cheol Kim] 2015

# The (1,0) CFT for E-strings

- This CFT is important for many reasons.
- Either from M5-M9, or by F-theory on  $B_4 = O(-1) \rightarrow P^1$ .

	0	1	2	3	4	5	6	7	8	9
NS5	•	•		•	•	•	•			
D8-O8	•	•		•	•	•	•	•	•	•
D2	•	•	•							



- Its compactification on  $S^1$  or  $T^2$  yield the 5d/4d Minahan-Nemeschansky CFTs
- non-perturbative sector of  $E_8 \times E_8$  heterotic strings (E-string  $\sim$  fractional heterotic string)
- Plays key roles of recent F-theory “atomic” classification: this is the “glue”

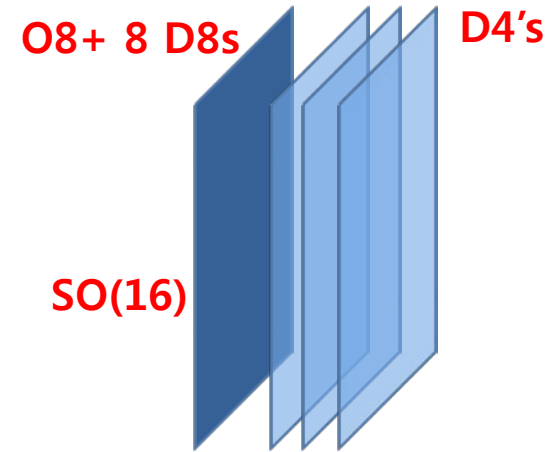
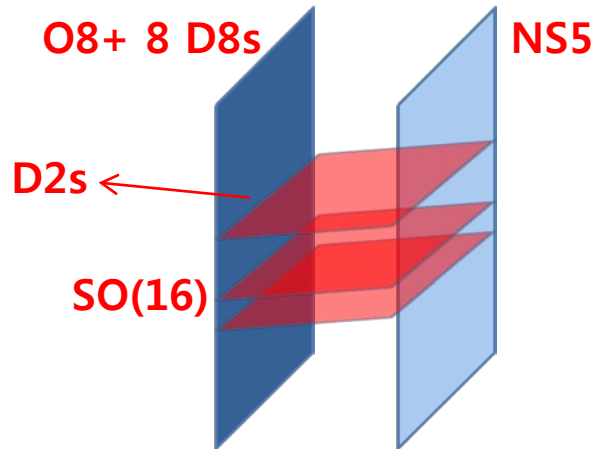


# Descriptions of E-strings

- Wilson line S1 reduction: 5d SYM & instantons

$$248 \rightarrow 120 + 128 \quad RA_9 = (0, 0, 0, 0, 0, 0, 0, 2\pi)$$

- 2d gauge theories for the E-strings



- Both can be used to compute  $Z[\mathbb{R}^4 \times T^2]$ :
  - Seiberg-Witten data of the Minahan-Nemeschansky SCFT
  - Can be used in curved space partition function (not studied in detail yet)

# Instanton partition function

- Results: One obtains  $Z = Z_{\text{QFT}} Z_{\text{extra}}$ , where  $Z_{\text{extra}}$  consists of two factors
  - 1<sup>st</sup> decoupled factor from D0's + 8 D8's + O8: makes the spectrum of 10d E8 SYM
  - 2<sup>nd</sup> decoupled factor from D0's in bulk: SUGRA continuum modes
  - $Z_{\text{extra}}$  is the index for the D0 + “ $N_f = 8$  D8” + O8 system

$$Z_{N_f=8} = \text{PE} \left[ \frac{(t+t^3)(u+u^{-1}+v+v^{-1})}{2(1-tu)(1-t/u)(1-tv)(1-t/v)} \frac{q^2}{1-q^2} \right] \rightarrow \text{contribution from the continuum of 11d SUGRA on } \mathbb{R}^{8+1} \times \mathbb{R}^+ \times S^1$$

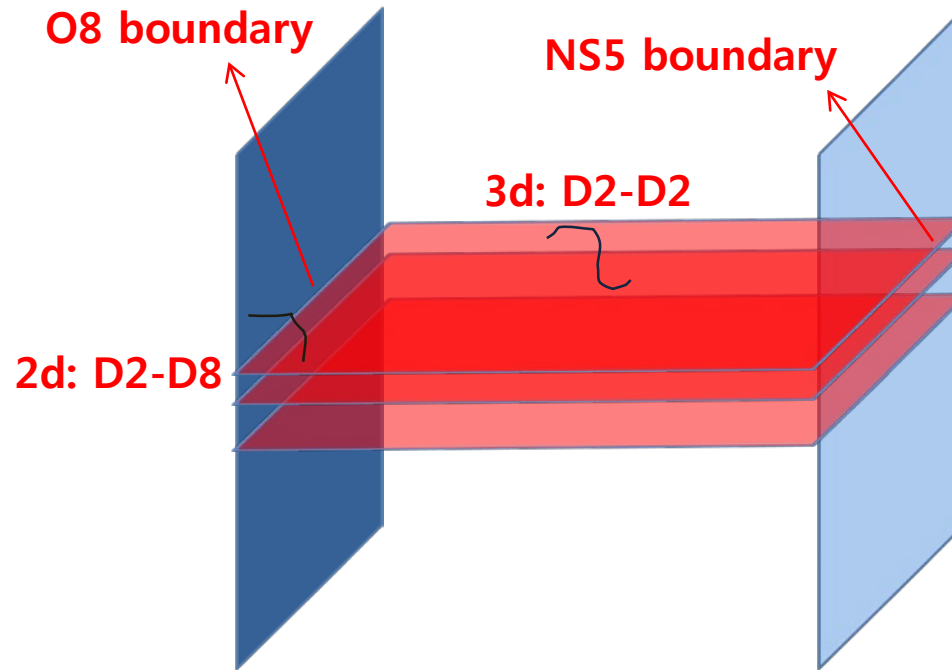
$$\left[ - \frac{t^2}{(1-tu)(1-t/u)(1-tv)(1-t/v)} \left( \chi(y_i)_{120}^{SO(16)} \frac{q^2}{1-q^2} + \chi(y_i)_{128}^{SO(16)} \frac{q}{1-q^2} \right) \right]$$

Combines with 9d SO(16) SYM on D8-O8 to be the index of circle-compactified 10d SYM on M9, with  $E_8$  gauge group

- Even the decoupled sector is interesting: supports “type I’ – HE duality”
- The QFT part is the partition function: Again, is this a 6d observable?

# E-string gauge theories in 2d

- The 2d QFT:



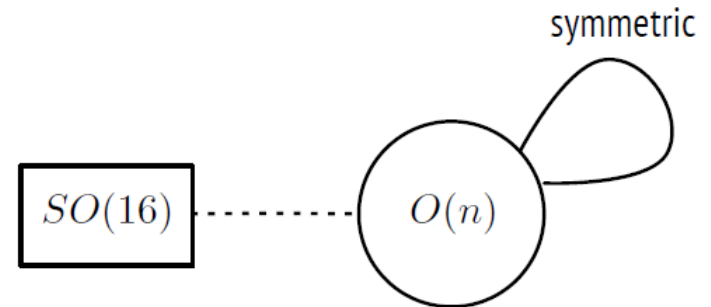
- Field content:

vector :  $O(n)$  antisymmetric  $(A_\mu, \lambda_+^{\dot{\alpha}A})$

hyper :  $O(n)$  symmetric  $(\varphi_{\alpha\dot{\beta}}, \lambda_-^{\alpha A})$

hyper + Fermi :  $O(n)$  fundamental  $(q_{\dot{\alpha}}, \psi_-^A), \psi_+^A$

Fermi :  $SO(16) \times O(n)$  bifundamental  $\Psi_l$



# The E-string partition functions

- The elliptic genus: similar contour integral formula + discrete  $O(n)$  holonomies
- Example: one E-string,  $O(1) \sim Z_2$  gauge symmetry
- 4 discrete holonomies on two circle factors of  $T^2$ .

$$(1, 1), (1, -1), (-1, 1), (-1, -1)$$

- Result:

$$Z_1 = \sum_{n=1}^4 \frac{Z_1^{(n)}}{2} = -\frac{\eta^2}{\theta_1(\epsilon_1)\theta_1(\epsilon_2)} \cdot \frac{\Theta(q, m_l)}{\eta^8}$$

$$\Theta(\tau, m_l) = \frac{1}{2} \sum_{n=1}^4 \prod_{l=1}^8 \theta_n(\tau, m_l)$$

partition function of  $E_8$  root lattice:  
 $E_8$  symmetry enhancement

- Agrees with expectations: one  $E_8$  current algebra on left-mover, “half of heterotic string”

[Ganor, Hanany] [Klemm, Mayr, Vafa] (95-96)

- Checked that 5d instanton partition function is a 6d observable

$$Z^{\mathbb{R}^4 \times T^2}(q, v, y) = Z_{\text{NS5}}(q) \sum_{n=0}^{\infty} Z_n^{T^2}(q, y_{1 \sim 7}, y_8 q^{-2}) e^{-n\lambda} = Z^{\mathbb{R}^4 \times S^1}(q, v, y) = Z_{\text{pert}}(v) \sum_{k=0}^{\infty} q^k Z_k(v, y)$$

# Conclusions and remarks

- Studied 6d CFTs' solitons & curved space partition functions
- Explores many features of 6d SCFTs, M-theory: chiral algebra,  $N^3$ , dualities, ...
  
- The approach using 2d GLSM: new 2d CFTs, w/ (0,4) SUSY
- The 2d QFTs for self-dual strings, for various 6d (1,0) CFTs, show very novel features: implications on both GLSM, instanton calculus, etc. (tomorrow)
- D3 + 7-branes wrapped on spheres (also with 7-brane “punctures”)

