5d superconformal field theories

(lecture 2/4)

Seok Kim

(Seoul National University)

10th Asian Winter School, OIST

13 Jan 2016

References:

[some early works on 5d SCFTs]

Witten, 9603150

Seiberg, 9608111

Morrison, Seiberg, 9609070

Intriligator, Morrison, Seiberg, 9702198

[instanton partition functions]

Nekrasov, 0206161

Nekrasov, Okounkov, 0306238

[Witten index of 1d gauge theories]

Chiung Hwang, Joonho Kim, SK, Jaemo Park, 1406.6793

Cordova, Shao, 1406.7853

Hori, Heeyeon Kim, Piljin Yi, 1407.2567

[5d superconformal index and $Z[S^5]$]

Hee-Cheol Kim, Sung-Soo Kim, Kimyeong Lee, 1206.6781

Jafferis, Pufu, 1207.4359

5d SCFTs & SYM

- Consider 5d SCFT w/ Yang-Mills relevant deformation.
- SYM w/ simple gauge group G.

$$\mathcal{L} \leftarrow -\frac{1}{4g_{YM}^2}\text{tr}\left(F_{\mu\nu}F^{\mu\nu}\right) + (\text{other relevant deformations})$$

- vector supermultiplet: $A_\mu\ ,\quad \lambda^A\ ,\quad \phi$
- hypermultiplet in R representation of G:

$$\varphi_A = (\varphi, \tilde{\varphi}^\dagger)\ ,\quad \Psi = (\psi_\alpha, \bar{\chi}_{\dot\alpha}) : 1 \text{ Dirac spinor}$$

- Coulomb branch: VEV for vector multiplet scalar
- Easier to study physics. In particular, can constraint 5d SCFTs.

$$\mathcal{F}(\phi) = \text{cubic or less polynomials in } r\ \phi\text{'s}$$

$$S_{\text{Coulomb}} \sim \boxed{\partial_i\partial_j\mathcal{F}}\left(F^i_{\mu\nu}F^{j\mu\nu} + \partial_\mu\phi^i\partial^\mu\phi^j\right) + \boxed{\partial_i\partial_j\partial_k\mathcal{F}}A^i\wedge F^j\wedge F^k$$

5d Chern-Simons level: should be constant & quantized

coupling matrix: should be positive

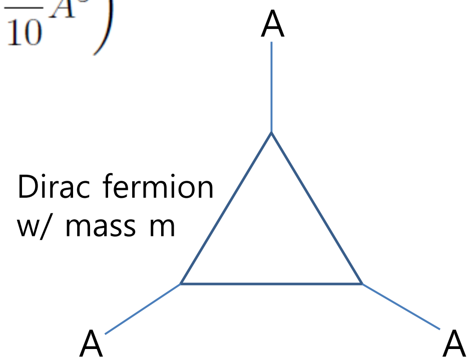
- Physics is still interesting, especially in its soliton sector.
- Can study some CFT observables.

$$Z_{S^4\times S^1}[x=e^{-\epsilon_+}, y=e^{-\epsilon_-}, m_i, q] = \int [d\alpha] Z_{\text{Nek}}^{\mathbb{R}^4\times S^1}(q,\epsilon_{1,2},m_i,\alpha) Z_{\text{Nek}}^{\mathbb{R}^4\times S^1}(q^{-1},\epsilon_{1,2},m_i,\alpha)$$

[H.-C. Kim, S.-S. Kim, K. Lee] 2012

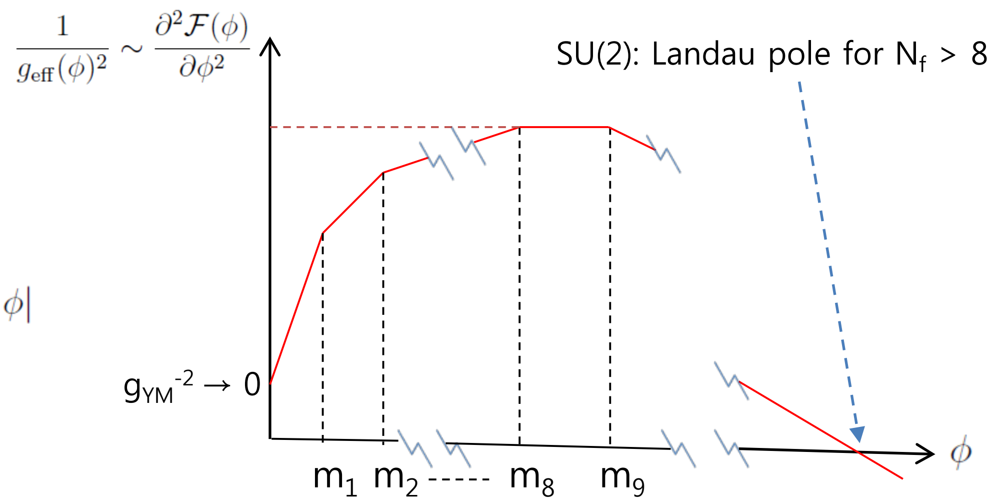
5d CFT in Coulomb phase

- Cubic terms either come from
- classical CS term for G = SU(N) $\frac{k}{24\pi^2}\int \mathrm{tr}\left(A\wedge F\wedge F+\frac{i}{2}A^3\wedge F-\frac{1}{10}A^5\right)$
- 1-loop effect in Coulomb phase [Witten] 96 $-\frac{\mathrm{sign}(m)}{48\pi^2}\int A\wedge F\wedge F$



$$\mathcal{F}_{\text{1-loop}}=\frac{1}{12}\left(\sum_{\alpha\in\mathrm{root}(G)}|\alpha\cdot\phi|^3-\sum_{i\in\mathrm{hyper}}\sum_{\mu\in R_i}|\mu\cdot\phi+m_i|^3\right)$$

- Too many matters imply Landau pole: incomplete QFT, need UV cutoff



- "coupling > 0" restricts # of matters:

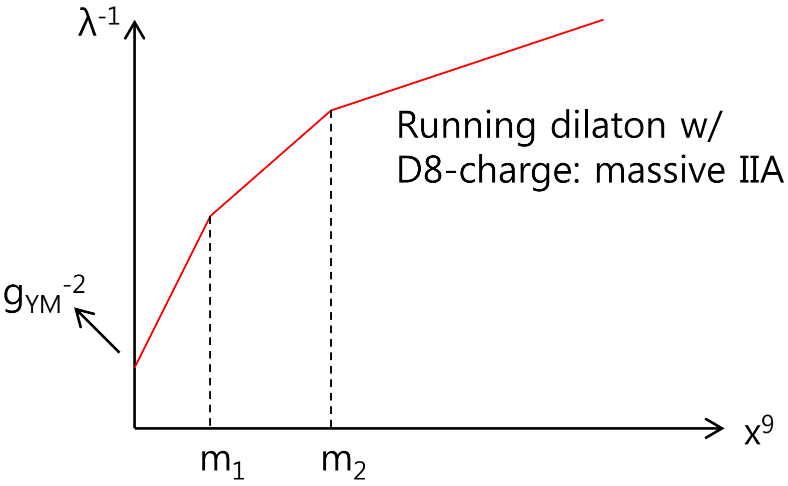
$$SU(2)\text{ with }N_f\text{ fund. matters}: g_{\text{eff}}^{-2}\sim(8-N_f)|\phi|$$

Caveat: New massless particles at the "pole" may keep the coupling positive. [Bergman, Rodriguez-Gomez]

Main example

- Engineered on N D4's probing Nf D8 + O8 [Seiberg] 1996

	0	1	2	3	4	5	6	7	8	9
D4	•	•	•	•	•					v
D8/O8	•	•	•	•	•	•	•	•	•	m_i



- 5d SCFT for Nf ≤ 7 (Nf = 8: 6d SCFT)

- After YM deformation: Sp(N) w/ N_f fundamental + 1 antisymmetric hypers
- scalars in Sp(N) vector: motion of D4's along x^9
- scalars in Sp(N) antisymm. hyper: motion of D4's along 8-branes, in 5678
- fundamental hypers: quarks & superpartners from D4-D8 strings

- N=1: antisymm. hyper decouples. Sp(1) ~ SU(2) w/ Nf ≤ 7

Novel aspects of 5d SCFTs

- UV fixed points exhibit properties which are invisible in SYM effective theory:

- UV symmetry enhancements by instantons (instanton operators)

- Expect $SO(2N_f) \times U(1) \rightarrow E_{Nf+1}$ enhanced global symmetry in UV [Seiberg].

- Directly related to non-perturbative string dualities (9d HE vs. type I')

$$E_8 \times E_8 \text{ heterotic on } S^1 \;\sim\; \text{type I on } S^1 \;\sim\; \text{IIA on } I = S^1/\mathbb{Z}_2 \;\equiv\; \text{type I}'$$

$SO(2N_f) \subset E_8$ $\qquad\qquad$ $SO(2N_f) \subset E_8$

$SO(2N_f) \times U(1)_w \rightarrow E_{N_f+1}$: perturbative $\qquad\qquad$ $SO(2N_f) \times U(1)_{D0} \rightarrow E_{N_f+1}$: nonperturbative

Solitons of 5d SYM

- Some particles are "**solitonic**": Yang-Mills instantons

$$F_{\mu\nu} = \star_4 F_{\mu\nu} \qquad\qquad k \equiv \frac{1}{8\pi^2}\int \mathrm{tr}\,(F\wedge F) \in \mathbb{Z}$$

- Often crucial for understanding CFT physics. (both in Coulomb & symmetric phases)

- k D0's on D4-D8-O8.

- 5d BPS states:

$$\{Q_M^A, Q_N^B\} = P_\mu(\Gamma^\mu C)_{MN}\epsilon^{AB} + i\frac{4\pi^2 k}{g_{YM}^2}C_{MN}\epsilon^{AB} + i\mathrm{tr}(qv)\epsilon^{AB}C_{MN}$$

$$M = \frac{4\pi^2 k}{g_{YM}^2} + \mathrm{tr}(qv) \qquad \text{preserves } \bar{Q}_{\dot\alpha}^A$$

[Note: Exists other solitons in SYM, such as monopole strings.]

Instanton solitons & UV complete descriptions

- The solution of the self-dual eqn. comes with parameters: $4c_2k$ "moduli"

$$\begin{bmatrix} \mathcal{D}_m\delta A_n - \mathcal{D}_n\delta A_m = \epsilon_{mnkl}\mathcal{D}_k\delta A_l \\ \mathcal{D}_n\delta A_n = 0 \end{bmatrix} \to \begin{bmatrix} \vec{\tau}^{\dot\alpha}{}_{\dot\beta}\bar{\mathcal{D}}^{\dot\beta\alpha}\delta A_{\alpha\dot\alpha} = 0 \\ \bar{\mathcal{D}}^{\dot\alpha\alpha}\delta A_{\alpha\dot\alpha} = 0 \end{bmatrix} \to \bar{\mathcal{D}}^{\dot\alpha\alpha}\delta A_{\alpha\dot\beta} = 0$$

$$\bar{\mathcal{D}}^{\dot\alpha\alpha}\Psi_\alpha = 0 \text{ with } \Psi_\alpha \in \mathbf{R} : 2kD_{\mathbf{R}} \text{ solutions}$$

$$\mathrm{tr}_{\mathbf{R}}(T^aT^b) = D_{\mathbf{R}}\delta^{ab}$$

- moduli space approximation to study the low E dynamics on solitons
- (0,4) SUSY non-linear sigma model w/ instanton moduli space as target space

$$S_{QM} = \int dt\left[g_{MN}(X)\dot{X}^M\dot{X}^N + \cdots\right]$$

coordinates of $4c_2k$ dimensional instanton moduli space

- The sigma model is incomplete: YM description reliable when $\lambda \gg g_{YM}^2$

E.g. single SU(N) instantons

$$ds^2 = g_{MN}(X)dX^MdX^N = ds^2(\mathbb{R}^4) + d\lambda^2 + \lambda^2\left[ds^2(S^3/\mathbb{Z}_2) + ds^2(\mathcal{M}_{4N-8})\right]$$

center-of-mass; **instanton "size"**; $SU(2)$ orientation; $\frac{SU(N)}{SU(2)\times U(N-2)}$

- Reflects UV incompleteness of 5d SYM.
- Sometimes, we know Lagrangian UV completions of 1d N=(0,4) gauge theories.

1d gauge theories on 5d solitons

- This is the well-known ADHM descriptions (but subtleties later)
- Construction of instantons: E.g. for SU(N) k-instantons,

$$A_\mu = i v^\dagger \partial v \qquad (v_{(N+2k)\times N},\ v^\dagger v = \mathbf{1}_{N\times N})$$

$$U^\dagger v = 0\ ,\quad U_{(N+2k)\times 2k} = \begin{pmatrix} \bar{q}_{N\times 2k} \\ (a_{\alpha\dot\beta})_{k\times k} - x_{\alpha\dot\beta}\otimes \mathbf{1}_{k\times k} \end{pmatrix} \qquad D^I \equiv q_{\dot\alpha}(\tau^I)^{\dot\alpha}{}_{\dot\beta}\bar{q}^{\dot\beta} + (\tau^I)^{\dot\alpha}{}_{\dot\beta}[a_{\alpha\dot\alpha}, a^{\alpha\dot\beta}] = 0$$

- Motivated by (0,4) gauge theories for light open strings on "Dp-D(p+4)-branes"

$$\mathcal{L} = \frac{1}{g^2_{\rm 1d/2d}}{\rm tr}\left[-\frac{1}{2}(D_\mu a_m)^2 - |D_\mu q_{\dot\alpha}|^2 - \frac{1}{2}(D^I)^2 - \frac{1}{4}(F_{\mu\nu})^2 + {\rm fermions}\right]$$

- However, it is often useful to regard it as an abstract UV completion (lecture 4)

- Can UV complete a subsector of 5d SCFT (after relevant deformation)
- Can compute Coulomb phase observables: e.g. instanton partition function…
- Can compute some CFT observables: e.g. superconformal index

N=(0,4) quantum mechanics for instantons

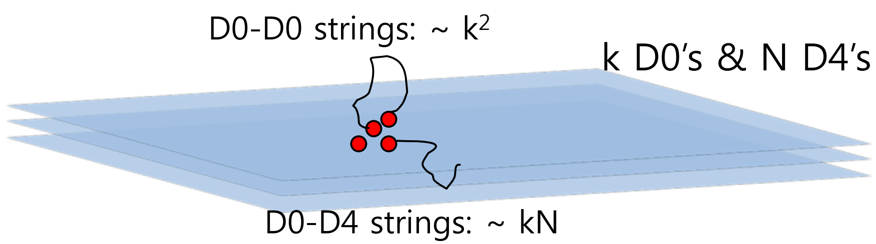
- Other names: N=4B mechanics (unlike N=(2,2) or (0,2), largely unexplored, both in 1d/2d)
- (0,4) multiplets (on-shell): supercharges : $Q_{A\dot\alpha}$

vector : $A_t,\ \phi,\ \lambda_{A\dot\alpha}$ hyper : $\varphi_{\dot\alpha},\ \psi_A$ twisted hyper : $\varphi_A,\ \psi_{\dot\alpha}$ Fermi : Ψ

- E.g. k instantons of G = ABCD ~ SU(N), Sp(N), SO(N) pure SYM:

5d gauge group: $G_N = U(N), SO(N), Sp(N)$

QM gauge group: $\hat{G}_k = U(k), Sp(k), O(k)$



QM vector multiplet: $\varphi, A_t, \bar\lambda^A_{\dot\alpha}$ ($\hat{G}_k$ adjoint)

$\hat{G}_k$ adjoint/antisymmetric/symmetric hyper: $a_{\alpha\dot\beta}, \Psi^A_\alpha$

$G_N \times \hat{G}_k$ bi-fundamental hyper: $q_{\dot\alpha}, \psi^A$

$$L_{\rm QM} = \frac{1}{g_{QM}}\mathrm{tr}\left[\frac{1}{2}(D_t\varphi)^2 + \frac{1}{2}(D_t a_m)^2 + D_t q_{\dot\alpha} D_t \bar{q}^{\dot\alpha} + \frac{1}{2}[\varphi, a_m]^2 - (\varphi\bar{q}^{\dot\alpha} - \bar{q}^{\dot\alpha}v)(q_{\dot\alpha}\varphi - v q_{\dot\alpha}) - D^{\dot\alpha}{}_{\dot\beta} D^{\dot\beta}{}_{\dot\alpha} + \cdots\right]$$

- SO/Sp fields are derivable by putting extra O4- or O4+ plane.
- Again, although helpful to think w/ branes, often better to think abstractly.

N=(0,4) quantum mechanics for instantons

- 5d hypermultiplets: extra 0-modes in instanton background

$$\bar{\mathcal{D}}^{\dot{\alpha}\alpha}\Psi_\alpha = 0 \text{ with } \Psi_\alpha \in \mathbf{R} : 2kD_{\mathbf{R}} \text{ solutions}$$

$$\mathcal{D}^\mu \mathcal{D}_\mu \phi = 0 : \text{ unique solution for given VEV } v = \phi(|x| \to \infty)$$

Only extra **fermion** 0-modes caused by matters in NLSM

- UV completions w/ extra 0-modes: sometimes very nontrivial…

- N_f fundamental hypers in G: Nf fundamental Fermi's in $\hat{G}$

$$c_{\text{fund}} = \frac{1}{2} \text{ for } SU(N),\ Sp(N)\ ;\ \ c_{\text{fund}} = 1 \text{ for } SO(N)$$

- Generally, needs extra bosons in UV: in IR, either massive or decouple
- 1d: should really rely on branes. Otherwise, hard to guess. (e.g. [Gaiotto, H.-C. Kim] 2015)
- 2d: easier to guess. more constraints, e.g. anomaly (lecture 4)

- Our interest: Sp(N) antisymmetric hyper.

$$\begin{aligned}
\text{D0-D0} \ &: \ O(k) \text{ antisymmetric} \quad (A_t, \varphi)\ ,\ \ (\bar{\lambda}^A_{\dot{\alpha}}, \underline{\lambda^a_\alpha}) \\
&\ \ O(k) \text{ symmetric} \quad (a_{\alpha\dot{\beta}}, \underline{\varphi_{aA}})\ ,\ \ (\lambda^A_\alpha, \underline{\bar{\lambda}^a_{\dot{\alpha}}}) \\
\text{D0-D4} \ &: \ Sp(N) \times O(k) \text{ bif.} \quad (q_{\dot{\alpha}})\ ,\ \ (\psi^A, \underline{\psi^a}) \\
\text{D0-D8} \ &: \ SO(2N_f) \times O(k) \text{ bif.} \quad (\underline{\Psi_l})
\end{aligned}$$

ADHM quantum mechanics

- In summary, the UV completion of non-linear sigma model is known for
- instantons with classical gauge groups
- hypers in certain representations (prescription [Shadchin] related to brane constructions)
- Possibly in any product representations of fundamentals

- The UV completion is NOT known for…
- instantons in exceptional gauge theories
- hypermultiplet matters in SO(N) spinor representations

(But I will say something more than this on Jan.15)

Observables

- SUSY observable of 5d SCFT: perturbed by relevant deformations, in Coulomb phase
- Nekrasov partition function [Nekrasov] (2002)

$$Z_{\rm Nek} = {\rm Tr}\left[(-1)^F q^k e^{-\beta\{Q,Q^\dagger\}} e^{-\epsilon_+\left(2J_R^3+J_1+J_2\right)} e^{-\epsilon_-(J_1-J_2)} e^{-{\rm tr}(v\Pi)} ({\rm flavor\ fugacities})\right]$$

$Q = -\bar{Q}^2_{\dot{2}} = \bar{Q}^{2\dot{1}}$, $Q^\dagger = \bar{Q}^1_{\dot{1}} = \bar{Q}_{2\dot{1}}$ $\quad$ J_R^3: Cartan of $SU(2)_R$, J_1, J_2: $SO(4)$ rotation

$\epsilon_\pm \equiv \frac{\epsilon_1 \pm \epsilon_2}{2}$ $\quad$ Π : electric charge , v: its chemical potential

- Tr over 5d QFT's Hilbert space
- UV: light particle spectra (small VEVs & relevant deformations, reflecting CFT's physics)
- Can compute in IR: Witten index is generally expected to be invariant
- At low E, 5d "bulk" & soliton physics at given k decouple

$$Z_{\rm Nek} = Z_{\rm pert} Z_{\rm inst}(q) \qquad Z_{\rm inst} = \sum_{k=0}^{\infty} Z_k q^k$$

- 1d Witten index for Z_k: [Hwang, J. Kim, SK, Park] [Cordova, Shao] [Kim, Hori, Yi] (2014)

The Witten index of gauge theories

- Z_k can basically be computed from our (0,4) gauge theories.

- View them as 1d N=2 theories (S1 reduction of 2d N=(0,2) theories)
- Multiplet decompositions: from (0,4) to (0,2)

$$Q = \bar{Q}^1_{\dot{1}} = \bar{Q}_{2\dot{1}} \ , \ Q^\dagger = -\bar{Q}^2_{\dot{2}} = \bar{Q}^{2\dot{1}}$$

- vector multiplet → vector + Fermi multiplet
- hypermultiplet → chiral + chiral

- The index Zk is given by a SUSY path integral on a circle.
- Gaussian integral over non-0-modes, and then exact integral over 0-modes.

(closely follows similar analysis in 2d elliptic genus [Benini, Eager, Hori, Tachikawa] 2013)

- The 0-modes have to be treated with great care: bosonic + fermionic

Result

- 0-modes of path integral: r (=rank) complex variables $\phi = \varphi + i A_\tau$
- Gaussian path integral: holomorphic measure [Nekrasov] 2002 [Nekrasov, Shadchin]

$$\text{chiral}: \ \left[2\sinh\left(\frac{\rho\cdot\phi + J\epsilon_+ + Fz}{2}\right)\right]^{-1} \qquad \text{vector}: \ 2\sinh\left(\frac{\alpha\cdot\phi}{2}\right) \qquad \text{Fermi}: \ 2\sinh\left(\frac{\rho\cdot\phi + J\epsilon_+ + Fz}{2}\right)$$

- r dimensional "contour integral" = summation over a set of residues.

- The "contour prescriptions" were given by Nekrasov only for some examples
- But the full derivation remained unclear for more than 10 years.
- In particular, many interesting gauge theories from 5d/6d CFTs could be studied.

The "contour"

- The residues to be picked are determined by the Jeffrey-Kirwan residues:

$$\text{JK-Res}(\mathbf{Q}_*, \eta)\frac{d\phi_1 \wedge \cdots \wedge d\phi_r}{Q_{j_1}(\phi)\cdots Q_{j_r}(\phi)} = \begin{cases} |\det(Q_{j_1}, \cdots, Q_{j_r})|^{-1} & \text{if } \eta \in \text{Cone}(Q_{j_1}, \cdots, Q_{j_r}) \\ 0 & \text{otherwise} \end{cases}$$

$$Z = \frac{1}{|W|}\sum_{\phi_*} \text{JK-Res}(\mathbf{Q}(\phi_*), \eta)\ Z_{\text{1-loop}}(\phi, \epsilon_+, z)$$

- Here, η should be taken to be proportional to FI parameter of the 1d U(k) gauge theory.
- For other groups, it can be any vector without affecting the results.

Can be derived by closely following [Benini, Eager, Hori, Tachikawa] 2013 for elliptic genus.

[Hwang, J.Kim, SK, Park] [Cordova, Shao] [Hori, H. Kim, Yi] 2014

Remarks

- Often, this is NOT a 5d QFT observable. May contain extra light d.o.f. even in IR
- Decoupled states in string theory (UV completion) don't belong to QFT Hilbert space

- This implies factorization: $Z_{extra}\ Z_{QFT}$
- Should know how to compute Z_{extra}, by our knowledge of string theory background, etc.

- Notes:

- Full quantum meaning of "instanton partition function" is obscure. (non-renormalizable QFT)
- With Z_{extra} discarded, we abstractly get a CFT partition function, not referring to 5d SYM
- There may be continuum, from which index acquires fractional coefficients. This goes to Z_{extra} , not Z_{QFT}.
- In principtple, there could be "wall crossings" in 1d Witten indices, but not in our Z_{QFT}.

Sp(N) theory with Nf ≤ 7

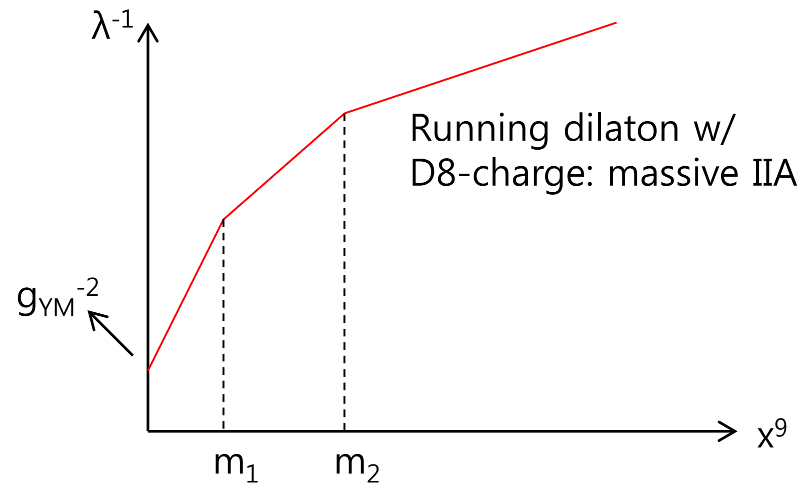
- O(k) gauge theory. Should also sum over two sectors of holonomies on S1.

$$U_- = e^{\phi_-} = \begin{cases} \mathrm{diag}(e^{\sigma_2\phi_1}, \cdots, e^{\sigma_2\phi_{n-1}}, \sigma_3) & \text{for even } k = 2n \\ \mathrm{diag}(e^{\sigma_2\phi_1}, \cdots, e^{\sigma_2\phi_n}, -1) & \text{for odd } k = 2n+1 \end{cases}$$

$$U_+ = e^{\phi_+} = \begin{cases} \mathrm{diag}(e^{\sigma_2\phi_1}, \cdots, e^{\sigma_2\phi_n}) & \text{for even } k = 2n \\ \mathrm{diag}(e^{\sigma_2\phi_1}, \cdots, e^{\sigma_2\phi_n}, 1) & \text{for odd } k = 2n+1 \end{cases}$$

$$Z^k = \frac{Z_+^k + Z_-^k}{2}$$

- Before studying the 5d CFT, one should first understand the decoupled factors.
- Decoupled factors themselves have interesting physics.
- Provides supports of dualities suggested in 90's.



- Two possible reasons for decoupled factors:
- continuum from φ: massive IIA dilaton lifts it.
- D0 unbound to D4, but bound to D8-O8: 8+1d particle
- This leads to nontrivial Z_{extra}. This is the Witten index of D0-D8-O8 system.

The bulk enhanced symmetry & duality

- The index of D0-D8-O8:

$$Z_{N_f=0} = \text{PE}\left[-\frac{t^2 q}{(1-tu)(1-t/u)(1-tv)(1-t/v)}\right]$$

$$Z_{1\leq N_f\leq 5} = \text{PE}\left[-\frac{t^2}{(1-tu)(1-t/u)(1-tv)(1-t/v)} q\chi(y_i)^{SO(2N_f)}_{\mathbf{2^{N_f-1}}}\right]$$

$$Z_{N_f=6} = \text{PE}\left[-\frac{t^2}{(1-tu)(1-t/u)(1-tv)(1-t/v)}\left(q\chi(y_i)^{SO(12)}_{\mathbf{32}} + q^2\right)\right]$$

$$Z_{N_f=7} = \text{PE}\left[-\frac{t^2}{(1-tu)(1-t/u)(1-tv)(1-t/v)}\left(q\chi(y_i)^{SO(14)}_{\mathbf{64}} + q^2\chi(y_i)^{SO(14)}_{\mathbf{14}}\right)\right]$$

$$PE\left[f(x)\right] = \exp\left[\sum_{n=1}^{\infty}\frac{1}{n}f(x^n)\right]$$

- These combine with the index of SO(2Nf) 9d SYM living on D8+O8,
- makes an E_{Nf+1} vector multiplet:

$$E_4 = SU(5) \;:\; \mathbf{24} \to \mathbf{1}_0 + \mathbf{15}_0 + \mathbf{4}_1 + \overline{\mathbf{4}}_{-1}$$

$$E_5 = SO(10) \;:\; \mathbf{45} \to \mathbf{1}_0 + \mathbf{28}_0 + (\mathbf{8_s})_1 + (\mathbf{8_s})_{-1}$$

$$E_6 \;:\; \mathbf{78} \to \mathbf{1}_0 + \mathbf{45}_0 + \mathbf{16}_1 + \overline{\mathbf{16}}_{-1}$$

$$E_7 \;:\; \mathbf{133} \to \mathbf{1}_0 + \mathbf{66}_0 + \mathbf{32}_1 + \mathbf{32}_{-1} + \mathbf{1}_2 + \mathbf{1}_{-2}$$

$$E_8 \;:\; \mathbf{248} \to \mathbf{1}_0 + \mathbf{91}_0 + \mathbf{64}_1 + \overline{\mathbf{64}}_{-1} + \mathbf{14}_2 + \mathbf{14}_{-2}$$

- Nonperturbative gauge symmetry enhancement by D0-branes

- So this way, we get the CFT partition function by dividing out this factor.

The superconformal index

- CFT observables: [H.-C.Kim, S.-S.Kim, K.Lee] 2012, [Pestun] 2012

$$\{Q,S\} = E - 2J_r - 3J_R \geq 0$$

$$I(t,u,m_i,q) = \mathrm{Tr}\left[(-1)^F e^{-\beta\{Q,S\}} t^{2(J_r+J_R)} u^{2J_l} e^{-F\cdot m} q^k\right]$$
$$= \int [da] Z_{\text{pert}}(ia,t,u,m_i) Z_{\text{inst}}(ia,t,u,m_i,q) Z_{\text{inst}}(-ia,t,u,-m_i,q^{-1})$$

- Sp(1) theory, SO(14) x U(1) → E_8 : [H.-C.Kim, S.-S.Kim, K.Lee] [Hwang, J.Kim, SK, Park]

$$\begin{aligned} I = & 1 + \chi^{E_8}_{248} t^2 + \chi_2(u)\left[1+\chi^{E_8}_{248}\right] t^3 + \left[1 + \chi^{E_8}_{27000} + \chi_3(u)\left(1+\chi^{E_8}_{248}\right)\right] t^4 \\ & + \left[\chi_2(u)\left(1+\chi^{E_8}_{248}+\chi^{E_8}_{27000}+\chi^{E_8}_{30380}\right) + \chi_4(u)\left(1+\chi^{E_8}_{248}\right)\right] t^5 \\ & + \left[2\chi^{E_8}_{248} + \chi^{E_8}_{30380} + \chi^{E_8}_{1763125} + \chi_3(u)\left(2 + 2\chi^{E_8}_{133} + \chi^{E_8}_{3875} + 2\chi^{E_8}_{27000} + \chi^{E_8}_{30380}\right)\right. \\ & \left. + \chi_5(u)\left(1+\chi^{E_8}_{248}\right)\right] t^6 + \mathcal{O}(t^7) \ , \end{aligned}$$

$$\begin{aligned} \mathbf{248} &= \mathbf{1}_0 + \mathbf{14}_2 + \mathbf{14}_{-2} + \mathbf{64}_{-1} + \overline{\mathbf{64}}_1 + \mathbf{91}_0, \\ \mathbf{3875} &= \mathbf{1}_4 + \mathbf{1}_0 + \mathbf{1}_{-4} + \mathbf{14}_2 + \mathbf{14}_{-2} + \mathbf{64}_3 + \mathbf{64}_{-1} + \overline{\mathbf{64}}_1 + \overline{\mathbf{64}}_{-3} + \mathbf{91} \\ &\quad + \mathbf{104}_0 + \mathbf{364}_2 + \mathbf{364}_{-2} + \mathbf{832}_{-1} + \overline{\mathbf{832}}_1 + \mathbf{1001}_0, \\ \mathbf{27000} &= 2\times\mathbf{1}_0 + \mathbf{14}_2 + \mathbf{14}_{-2} + 2\times\mathbf{64}_{-1} + 2\times\overline{\mathbf{64}}_1 + 2\times\mathbf{91}_0 \\ &\quad + \mathbf{104}_4 + \mathbf{104}_0 + \mathbf{104}_{-4} + \mathbf{364}_2 + \mathbf{364}_{-2} \\ &\quad + \mathbf{832}_3 + \mathbf{832}_{-1} + \overline{\mathbf{832}}_1 + \overline{\mathbf{832}}_{-3} + \mathbf{896}_2 + \mathbf{896}_{-2} + \mathbf{1001}_0 \\ &\quad + \mathbf{1716}_{-2} + \overline{\mathbf{1716}}_2 + \mathbf{3003}_0 + \mathbf{3080}_0 + \mathbf{4928}_{-1} + \overline{\mathbf{4928}}_1, \\ \mathbf{30380} &= \mathbf{1}_0 + 2\times\mathbf{14}_2 + 2\times\mathbf{14}_{-2} + \mathbf{64}_3 + 2\times\mathbf{64}_{-1} + 2\times\overline{\mathbf{64}}_1 + \overline{\mathbf{64}}_{-3} \\ &\quad + \mathbf{91}_4 + 3\times\mathbf{91}_0 + \mathbf{91}_{-4} + \mathbf{104}_0 + \mathbf{364}_2 + \mathbf{364}_{-2} \\ &\quad + \mathbf{832}_3 + 2\times\mathbf{832}_{-1} + 2\times\overline{\mathbf{832}}_1 + \overline{\mathbf{832}}_{-3} + \mathbf{896}_2 + \mathbf{896}_{-2} \\ &\quad + \mathbf{1001}_0 + \mathbf{2002}_2 + \mathbf{2002}_{-2} + \mathbf{3003}_0 + \mathbf{4004}_0 + \mathbf{4928}_{-1} + \overline{\mathbf{49}} \end{aligned}$$

$$\begin{aligned} \mathbf{1763125} &= 2\times\mathbf{1}_0 + 2\times\mathbf{14}_2 + 2\times\mathbf{14}_{-2} + 3\times\mathbf{64}_{-1} + 3\times\overline{\mathbf{64}}_1 + 3\times\mathbf{91}_0 \\ &\quad + \mathbf{104}_4 + \mathbf{104}_0 + \mathbf{104}_{-4} + \mathbf{364}_2 + \mathbf{364}_{-2} + \mathbf{546}_6 + \mathbf{546}_2 + \mathbf{546}_{-2} + \mathbf{546}_{-6} \\ &\quad + 2\times\mathbf{832}_3 + 2\times\mathbf{832}_{-1} + 2\times\overline{\mathbf{832}}_1 + 2\times\overline{\mathbf{832}}_{-3} + 2\times\mathbf{896}_2 + 2\times\mathbf{896}_{-2} \\ &\quad + 2\times\mathbf{1001}_0 + 2\times\mathbf{1716}_{-2} + 2\times\overline{\mathbf{1716}}_2 + \mathbf{2002}_2 + \mathbf{2002}_{-2} \\ &\quad + 3\times\mathbf{3003}_0 + 2\times\mathbf{3080}_0 + \mathbf{4004}_4 + 2\times\mathbf{4004}_0 + \mathbf{4004}_{-4} \\ &\quad + 3\times\mathbf{4928}_{-1} + 3\times\overline{\mathbf{4928}}_1 + \mathbf{5625}_4 + \mathbf{5625}_0 + \mathbf{5625}_{-4} \\ &\quad + \mathbf{5824}_3 + \mathbf{5824}_{-1} + \mathbf{5824}_{-5} + \overline{\mathbf{5824}}_5 + \overline{\mathbf{5824}}_1 + \overline{\mathbf{5824}}_{-3} \\ &\quad + \mathbf{11648}_2 + \mathbf{11648}_{-2} + \mathbf{17472}_3 + \mathbf{17472}_{-1} + \overline{\mathbf{17472}}_1 + \overline{\mathbf{17472}}_{-3} \\ &\quad + \mathbf{18200}_2 + \mathbf{18200}_{-2} + \mathbf{21021}_0 + \mathbf{21021}_{-4} + \overline{\mathbf{21021}}_4 + \overline{\mathbf{21021}}_0 \\ &\quad + \mathbf{24024'}_2 + \mathbf{24024'}_{-2} + \mathbf{27456}_3 + \overline{\mathbf{27456}}_{-3} + \mathbf{36608}_2 + \mathbf{36608}_{-2} \\ &\quad + \mathbf{40768}_{-1} + \overline{\mathbf{40768}}_1 + \mathbf{45760}_3 + \mathbf{45760}_{-1} + \overline{\mathbf{45760}}_1 + \overline{\mathbf{45760}}_{-3} \\ &\quad + \mathbf{58344}_0 + \mathbf{58968}_0 + \mathbf{64064'}_{-1} + \overline{\mathbf{64064'}}_1 + \mathbf{115830}_{-2} + \overline{\mathbf{115830}}_2 \\ &\quad + \mathbf{146432}_{-1} + \overline{\mathbf{146432}}_1 + \mathbf{200200}_0. \end{aligned}$$

Concluding remarks

- 5d solitons' Witten indices (or $Z_{Nekrasov}$) are clearly understood only recently.
- understanding the precise contour integral
- proper interpretation of decoupled factors (the issue is spread everywhere, e.g. AGT)

- This can be used as building blocks of other CFT observables

- Some nontrivial aspects of 5d SCFTs understood using various BPS observables

- Classification of 5d SCFTs? Should be much more challenging than 6d SCFTs