5d superconformal field theories

(lecture 2/4)

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10th Asian Winter School, OIST
13 Jan 2016
References:

[some early works on 5d SCFTs]
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  Seiberg, 9608111
  Morrison, Seiberg, 9609070
  Intriligator, Morrison, Seiberg, 9702198

[instanton partition functions]
  Nekrasov, 0206161
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[Witten index of 1d gauge theories]
  Chiung Hwang, Joonho Kim, SK, Jaemo Park, 1406.6793
  Cordova, Shao, 1406.7853
  Hori, Heeyeon Kim, Piljin Yi, 1407.2567

[5d superconformal index and Z[S^5]]
  Hee-Cheol Kim, Sung-Soo Kim, Kimyeong Lee, 1206.6781
  Jafferis, Pufu, 1207.4359
5d SCFTs & SYM

• Consider 5d SCFT w/ Yang-Mills relevant deformation.

• SYM w/ simple gauge group G.
  - vector supermultiplet: \( A_\mu, \lambda^A, \phi \)
  - hypermultiplet in R representation of G:
    \[
    \varphi_A = (\varphi, \tilde{\varphi})^\dagger, \quad \Psi = (\psi_\alpha, \tilde{\psi}_\alpha) : 1 \text{ Dirac spinor}
    \]

• Coulomb branch: VEV for vector multiplet scalar
  - Easier to study physics. In particular, can constraint 5d SCFTs.
  \[
  S_{\text{Coulomb}} \sim \partial_i \partial_j F_i^j + \partial_i \phi^j \partial^j \phi^i + \partial_i \partial_j F_i^j \wedge F^k
  \]
  5d Chern-Simons level: should be constant & quantized
  - Physics is still interesting, especially in its soliton sector.
  - Can study some CFT observables.
  \[
  Z_{S^4 \times S^1}[x = e^{-\epsilon^+}, y = e^{-\epsilon^-}, m_i, q] = \int [d\alpha] Z_{\text{Nek}}^{R^4 \times S^1}(q, \epsilon_{1,2}, m_i, \alpha) Z_{\text{Nek}}^{R^4 \times S^1}(q^{-1}, \epsilon_{1,2}, m_i, \alpha)
  \]

5d CFT in Coulomb phase

- Cubic terms either come from
  - classical CS term for $G = SU(N)$
    $$\frac{k}{24\pi^2} \int \text{tr} \left( A \wedge F \wedge F + \frac{i}{2} A^3 \wedge F - \frac{1}{10} A^5 \right)$$
  - 1-loop effect in Coulomb phase [Witten] 96
    $$-\frac{\text{sign}(m)}{48\pi^2} \int A \wedge F \wedge F$$

$$\mathcal{F}_{1\text{-loop}} = \frac{1}{12} \left( \sum_{\alpha \in \text{root}(G)} |\alpha \cdot \phi|^3 - \sum_{i \in \text{hyper}} \sum_{\mu \in R_i} |\mu \cdot \phi + m_i|^3 \right)$$

- Too many matters imply Landau pole: incomplete QFT, need UV cutoff

- “coupling > 0” restricts # of matters:
  $$SU(2) \text{ with } N_f \text{ fund. matters}: g_{\text{eff}}^{-2} \sim (8 - N_f)|\phi|$$

Caveat: New massless particles at the “pole” may keep the coupling positive. [Bergman, Rodriguez-Gomez]
Main example

- Engineered on N D4’s probing Nf D8 + O8 [Seiberg] 1996

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- 5d SCFT for Nf ≤ 7 (Nf = 8: 6d SCFT)

- After YM deformation: Sp(N) w/ Nf fundamental + 1 antisymmetric hypers
  - scalars in Sp(N) vector: motion of D4’s along x^9
  - scalars in Sp(N) antisymmm. hyper: motion of D4’s along 8-branes, in 5678
  - fundamental hypers: quarks & superpartners from D4-D8 strings

- N=1: antisymm. hyper decouples. Sp(1) ~ SU(2) w/ Nf ≤ 7
Novel aspects of 5d SCFTs

- UV fixed points exhibit properties which are invisible in SYM effective theory:
  - UV symmetry enhancements by instantons (instanton operators)
  - Expect $SO(2N_f) \times U(1) \to E_{N_f+1}$ enhanced global symmetry in UV [Seiberg].
  - Directly related to non-perturbative string dualities (9d HE vs. type I')

\[
E_8 \times E_8 \text{ heterotic on } S^1 \sim \text{ type I on } S^1 \sim \text{ IIA on } I = S^1/\mathbb{Z}_2 \equiv \text{ type I}'
\]

\[
SO(2N_f) \subset E_8
\]

$SO(2N_f) \times U(1)_w \to E_{N_f+1}$: perturbative

$SO(2N_f) \times U(1)_{D0} \to E_{N_f+1}$: nonperturbative
Some particles are "solitonic": Yang-Mills instantons

\[ F_{\mu\nu} = *_4 F_{\mu\nu} \quad k \equiv \frac{1}{8\pi^2} \int \text{tr} (F \wedge F') \in \mathbb{Z} \]

Often crucial for understanding CFT physics. (both in Coulomb & symmetric phases)
- k D0’s on D4-D8-O8.

5d BPS states:

\[ \{ Q^A_M, Q^B_N \} = P_\mu (\Gamma^\mu C)_{MN} \epsilon^{AB} + i \frac{4\pi^2 k}{g^2_{YM}} C_{MN} \epsilon^{AB} + i \text{tr}(qv) \epsilon^{AB} C_{MN} \]

\[ M = \frac{4\pi^2 k}{g^2_{YM}} + \text{tr}(qv) \quad \text{preserves } Q^A_\alpha \]

[Note: Exists other solitons in SYM, such as monopole strings.]
Instanton solitons & UV complete descriptions

- The solution of the self-dual eqn. comes with parameters: $4c_2k$ "moduli"
  \[
  \begin{align*}
  D_m \delta A_n - D_n \delta A_m &= \epsilon_{mnk} D_k \delta A_l \\
  D_n \delta A_n &= 0
  \end{align*}
  \]
  \[
  \begin{align*}
  \bar{\tau}^{\dot{\alpha}}_{\dot{\beta}} \bar{D}^{\dot{\beta}}{\dot{\alpha}} \delta A_{\alpha\dot{\alpha}} &= 0 \\
  \bar{D}^{\dot{\alpha}}{\dot{\alpha}} \delta A_{\alpha\dot{\alpha}} &= 0
  \end{align*}
  \]
  \[
  \bar{D}^{\dot{\alpha}}{\dot{\alpha}} A_{\alpha\dot{\beta}} = 0
  \]
  \[
  \bar{D}^{\dot{\alpha}}{\dot{\alpha}} \Psi_\alpha = 0 \text{ with } \Psi_\alpha \in \mathbb{R} : 2kD_R \text{ solutions}
  \]
  \[
  \text{tr}_R(T^a T^b) = D_R \delta^{ab}
  \]

- moduli space approximation to study the low $E$ dynamics on solitons

- $(0,4)$ SUSY non-linear sigma model w/ instanton moduli space as target space

  \[
  S_{QM} = \int dt \left[ g_{MN}(X) \dot{X}^M \dot{X}^N + \ldots \right]
  \]

- The sigma model is incomplete: YM description reliable when $\lambda \gg g_{YM}^2$

  E.g. single SU(N) instantons

  \[
  ds^2 = g_{MN}(X) dX^M dX^N = ds^2(\mathbb{R}^4) + d\lambda^2 + \lambda^2 \left[ ds^2(S^3/\mathbb{Z}_2) + ds^2(M_{4N-8}) \right]
  \]

  SU(2) orientation

  \[
  \frac{SU(N)}{SU(2) \times U(N-2)}
  \]

- Reflects UV incompleteness of 5d SYM.

- Sometimes, we know Lagrangian UV completions of 1d $N=(0,4)$ gauge theories.
1d gauge theories on 5d solitons

• This is the well-known ADHM descriptions (but subtleties later)

• Construction of instantons: E.g. for SU(N) k-instantons,

\[ A_\mu = iv^\dagger \partial v \quad (v_{(N+2k)\times N}, \quad v^\dagger v = 1_{N\times N}) \]

\[ U^\dagger v = 0, \quad U_{(N+2k)\times 2k} = \left( \begin{array}{c} \bar{q}_{N\times 2k} \\ (a_{\alpha \dot{\beta}})_{k\times k} - x_{\alpha \dot{\beta}} \otimes 1_{k\times k} \end{array} \right) \quad D^I \equiv q_{\dot{\alpha}} (\tau^I)^{\dot{\alpha}}_{\dot{\beta}} \bar{q}^\dot{\beta} + (\tau^I)^{\dot{\alpha}}_{\dot{\beta}} [a_{\alpha \dot{\lambda}}, a^{\alpha \dot{\beta}}] = 0 \]

• Motivated by (0,4) gauge theories for light open strings on “Dp-D(p+4)-branes”

\[ \mathcal{L} = \frac{1}{g_{1d/2d}^2} \text{tr} \left[ -\frac{1}{2} (D_\mu a_m)^2 - |D_\mu q_{\dot{\alpha}}|^2 - \frac{1}{2} (D_I)^2 - \frac{1}{4} (F_{\mu \nu})^2 + \text{fermions} \right] \]

• However, it is often useful to regard it as an abstract UV completion (lecture 4)

• Can UV complete a subsector of 5d SCFT (after relevant deformation)

• Can compute Coulomb phase observables: e.g. instanton partition function…

• Can compute some CFT observables: e.g. superconformal index
N=(0,4) quantum mechanics for instantons

- Other names: N=4B mechanics (unlike N=(2,2) or (0,2), largely unexplored, both in 1d/2d)

- (0,4) multiplets (on-shell): supercharges: $Q_{A\dot{\alpha}}$

  vector: $A_\mu$, $\phi$, $\lambda_{A\dot{\alpha}}$

  hyper: $\varphi_{\dot{\alpha}}$, $\psi_A$

  twisted hyper: $\varphi_A$, $\psi_{\dot{\alpha}}$

  Fermi: $\Psi$

- E.g. k instantons of $G = ABCD \sim SU(N)$, Sp(N), SO(N) pure SYM:

  5d gauge group: $G_N = U(N), SO(N), Sp(N)$

  QM gauge group: $\hat{G}_k = U(k), Sp(k), O(k)$

  QM vector multiplet: $\varphi$, $A_\mu$, $\lambda^A_{\dot{\alpha}}$ ($\hat{G}_k$ adjoint)

  $\hat{G}_k$ adjoint/antisymmetric/symmetric hyper: $a_{\alpha\beta}$, $\Psi^A_{\dot{\alpha}}$

  $G_N \times \hat{G}_k$ bi-fundamental hyper: $q_{\dot{\alpha}}$, $\psi^A$

  $$L_{QM} = \frac{1}{g_{QM}} \text{tr} \left[ \frac{1}{2}(D_\mu \varphi)^2 + \frac{1}{2}(D_\mu a_m)^2 + D_\mu q_{\dot{\alpha}} D_\mu \bar{q}^{\dot{\alpha}} + \frac{1}{2} [\varphi, a_m]^2 - (\varphi \bar{q}^{\dot{\alpha}} - q^{\dot{\alpha}} \varphi)(q_{\dot{\alpha}} \varphi - v q_{\dot{\alpha}}) - D_{\beta}^{\dot{\alpha}} D^{\dot{\alpha}} + \cdots \right]$$

- SO/Sp fields are derivable by putting extra O4- or O4+ plane.

- Again, although helpful to think w/ branes, often better to think abstractly.
• 5d hypermultiplets: extra 0-modes in instanton background

\[ \mathcal{D}^{\dot{\alpha} \alpha} \Psi_\alpha = 0 \quad \text{with} \quad \Psi_\alpha \in \mathbb{R} : 2kD_R \text{ solutions} \]
\[ \mathcal{D}^\mu \mathcal{D}_\mu \phi = 0 : \text{unique solution for given VEV } v = \phi(|x| \to \infty) \]

Only extra fermion 0-modes caused by matters in NLSM

• UV completions w/ extra 0-modes: sometimes very nontrivial…

- \( N_f \) fundamental hypers in \( G \): \( N_f \) fundamental Fermi’s in \( \tilde{G} \)

\[ c_{\text{fund}} = \frac{1}{2} \text{ for } SU(N), \ Sp(N) ; \quad c_{\text{fund}} = 1 \text{ for } SO(N) \]

- Generally, needs extra bosons in UV: in IR, either massive or decouple

- 1d: should really rely on branes. Otherwise, hard to guess. (e.g. [Gaiotto, H.-C. Kim] 2015)

- 2d: easier to guess. more constraints, e.g. anomaly (lecture 4)

• Our interest: \( Sp(N) \) antisymmetric hyper.
ADHM quantum mechanics

• In summary, the UV completion of non-linear sigma model is known for
  - instantons with classical gauge groups
  - hypers in certain representations (prescription [Shadchin] related to brane constructions)
  - Possibly in any product representations of fundamentals

• The UV completion is NOT known for…
  - instantons in exceptional gauge theories
  - hypermultiplet matters in SO(N) spinor representations

(But I will say something more than this on Jan.15)
Observables

• SUSY observable of 5d SCFT: perturbed by relevant deformations, in Coulomb phase
    
    \[ Z_{\text{Nek}} = \text{Tr} \left[ (-1)^F q^k e^{-\beta \{Q, Q^\dagger\}} e^{-\epsilon_+ (2J_R^3 + J_1 + J_2)} e^{-\epsilon_- (J_1 - J_2)} e^{-\text{tr}(\Pi)} \right] \]
    \[ Q = -Q_2^2 = Q^{2i}, \quad Q_1^\dagger = Q_2^\dagger \]
    \[ \epsilon_\pm \equiv \frac{\epsilon_1 \pm \epsilon_2}{2} \]
    \[ J_R^3: \text{Cartan of } SU(2)_R, \quad J_1, J_2: \text{SO(4) rotation} \]
    \[ \Pi: \text{electric charge, } v: \text{its chemical potential} \]
  - Tr over 5d QFT’s Hilbert space
  - UV: light particle spectra (small VEVs & relevant deformations, reflecting CFT’s physics)
  - Can compute in IR: Witten index is generally expected to be invariant
  - At low E, 5d “bulk” & soliton physics at given k decouple
    \[ Z_{\text{Nek}} = Z_{\text{pert}} Z_{\text{inst}} (q) \]
    \[ Z_{\text{inst}} = \sum_{k=0}^{\infty} Z_k q^k \]
  - 1d Witten index for \( Z_k \): [Hwang, J. Kim, SK, Park] [Cordova, Shao] [Kim, Hori, Yi] (2014)
The Witten index of gauge theories

- $Z_k$ can basically be computed from our (0,4) gauge theories.

- View them as 1d N=2 theories (S1 reduction of 2d N=(0,2) theories)

- Multiplet decompositions: from (0,4) to (0,2)
  - vector multiplet $\rightarrow$ vector + Fermi multiplet
  - hypermultiplet $\rightarrow$ chiral + chiral

- The index $Z_k$ is given by a SUSY path integral on a circle.
  - Gaussian integral over non-0-modes, and then exact integral over 0-modes.
    (closely follows similar analysis in 2d elliptic genus [Benini, Eager, Hori, Tachikawa] 2013)

- The 0-modes have to be treated with great care: bosonic + fermionic
Result

• 0-modes of path integral: \( r \) (\( = \)rank) complex variables \( \phi = \varphi + iA_r \)

• Gaussian path integral: holomorphic measure \cite{Nekrasov} 2002 \cite{Nekrasov, Shadchin}

\[
\text{chiral: } \left[ 2 \sinh \left( \frac{\rho \cdot \phi + J\epsilon_+ + Fz}{2} \right) \right]^{-1} \quad \text{vector: } 2 \sinh \left( \frac{\alpha \cdot \phi}{2} \right) \quad \text{Fermi: } 2 \sinh \left( \frac{\rho \cdot \phi + J\epsilon_+ + Fz}{2} \right)
\]

• \( r \) dimensional “contour integral” = summation over a set of residues.

• The “contour prescriptions” were given by Nekrasov only for some examples
• But the full derivation remained unclear for more than 10 years.
• In particular, many interesting gauge theories from 5d/6d CFTs could be studied.
The “contour”

- The residues to be picked are determined by the Jeffrey-Kirwan residues:

\[
JK-\text{Res}(Q_*, \eta) \frac{d\phi_1 \wedge \cdots \wedge d\phi_r}{Q_{j_1}(\phi) \cdots Q_{j_r}(\phi)} = \begin{cases} 
|\det(Q_{j_1}, \cdots, Q_{j_r})|^{-1} & \text{if } \eta \in \text{Cone}(Q_{j_1}, \cdots, Q_{j_r}) \\
0 & \text{otherwise}
\end{cases}
\]

\[
Z = \frac{1}{|W|} \sum_{\phi_*} JK-\text{Res}(Q(\phi_*), \eta) \ Z_{\text{1-loop}}(\phi, \epsilon_+, z)
\]

- Here, \(\eta\) should be taken to be proportional to FI parameter of the 1d U(k) gauge theory.
- For other groups, it can be any vector without affecting the results.

Can be derived by closely following [Benini, Eager, Hori, Tachikawa] 2013 for elliptic genus.
[Hwang, J.Kim, SK, Park] [Cordova, Shao] [Hori, H. Kim, Yi] 2014
Remarks

- Often, this is NOT a 5d QFT observable. May contain extra light d.o.f. even in IR
- Decoupled states in string theory (UV completion) don’t belong to QFT Hilbert space

- This implies factorization: \( Z_{\text{extra}} Z_{\text{QFT}} \)
- Should know how to compute \( Z_{\text{extra}} \), by our knowledge of string theory background, etc.

- Notes:
  - Full quantum meaning of “instanton partition function” is obscure. (non-renormalizable QFT)
  - With \( Z_{\text{extra}} \) discarded, we abstractly get a CFT partition function, not referring to 5d SYM
  - There may be continuum, from which index acquires fractional coefficients. This goes to \( Z_{\text{extra}} \), not \( Z_{\text{QFT}} \).
  - In principle, there could be “wall crossings” in 1d Witten indices, but not in our \( Z_{\text{QFT}} \).
Sp(N) theory with Nf ≤ 7

- O(k) gauge theory. Should also sum over two sectors of holonomies on S1.

\[
U_- = e^{\phi_-} = \begin{cases} 
\text{diag}(e^{\sigma_2\phi_1}, \ldots, e^{\sigma_2\phi_{2n-1}}, \sigma_3) & \text{for even } k = 2n \\
\text{diag}(e^{\sigma_2\phi_1}, \ldots, e^{\sigma_2\phi_{2n}}, -1) & \text{for odd } k = 2n+1 
\end{cases}
\]

\[
U_+ = e^{\phi_+} = \begin{cases} 
\text{diag}(e^{\sigma_2\phi_1}, \ldots, e^{\sigma_2\phi_n}) & \text{for even } k = 2n \\
\text{diag}(e^{\sigma_2\phi_1}, \ldots, e^{\sigma_2\phi_n}, 1) & \text{for odd } k = 2n+1 
\end{cases}
\]

\[
Z^k = \frac{Z_+^k + Z_-^k}{2}
\]

- Before studying the 5d CFT, one should first understand the decoupled factors.
  - Decoupled factors themselves have interesting physics.
  - Provides supports of dualities suggested in 90’s.

- Two possible reasons for decoupled factors:
  - continuum from $\phi$: massive IIA dilaton lifts it.
  - D0 unbound to D4, but bound to D8-O8: 8+1d particle
  - This leads to nontrivial $Z_{\text{extra}}$. This is the Witten index of D0-D8-O8 system.
The bulk enhanced symmetry & duality

- The index of D0-D8-O8:
  \[ Z_{N_f=0} = \text{PE} \left[ \frac{t^2}{(1 - tu)(1 - t/u)(1 - tv)(1 - t/v)} \right] \]
  \[ Z_{1 \leq N_f \leq 5} = \text{PE} \left[ \frac{t^2}{(1 - tu)(1 - t/u)(1 - tv)(1 - t/v)} q^{\chi(y_i)^{SO(2N_f)}} \right] \]
  \[ Z_{N_f=6} = \text{PE} \left[ \frac{t^2}{(1 - tu)(1 - t/u)(1 - tv)(1 - t/v)} \left( q^{\chi(y_i)^{SO(12)}} + q^2 \right) \right] \]
  \[ Z_{N_f=7} = \text{PE} \left[ \frac{t^2}{(1 - tu)(1 - t/u)(1 - tv)(1 - t/v)} \left( q^{\chi(y_i)^{SO(14)}} + q^2 \chi(y_i)^{SO(14)} \right) \right] \]

  \[ PE[f(x)] = \exp \left[ \sum_{n=1}^{\infty} \frac{1}{n} f(x^n) \right] \]

- These combine with the index of SO(2Nf) 9d SYM living on D8+O8,
- makes an \( E_{N_f+1} \) vector multiplet:
  \[ E_4 = SU(5) : \quad 24 \rightarrow 1_0 + 15_0 + 4_1 + 4_{-1} \]
  \[ E_5 = SO(10) : \quad 45 \rightarrow 1_0 + 28_0 + (8_s)_1 + (8_s)_{-1} \]
  \[ E_6 : \quad 78 \rightarrow 1_0 + 45_0 + 16_1 + 16_{-1} \]
  \[ E_7 : \quad 133 \rightarrow 1_0 + 66_0 + 32_1 + 32_{-1} + 1_2 + 1_{-2} \]
  \[ E_8 : \quad 248 \rightarrow 1_0 + 91_0 + 64_1 + 64_{-1} + 14_2 + 14_{-2} \]

- Nonperturbative gauge symmetry enhancement by D0-branes
- So this way, we get the CFT partition function by dividing out this factor.
The superconformal index

\[
I(t, u, m_i, q) = \text{Tr} \left[ (-1)^F e^{-\beta(Q,S)} t^{2(J_r + J_R)} u^{2J_I} e^{-F \cdot m_k^2} \right] = \int [da] Z_{\text{pert}}(ia, t, u, m_i) Z_{\text{inst}}(ia, t, u, m_i, q) Z_{\text{inst}}(-ia, t, u, -m_i, q^{-1})
\]

- **Sp(1) theory, SO(14) x U(1) \rightarrow E_8:** [H.-C.Kim, S.-S.Kim, K.Lee] [Hwang, J.Kim, SK, Park]
\[
I = 1 + \chi_{248}^E t^2 + \chi_2(u) \left[ 1 + \chi_{248}^E \right] t^3 + \left[ 1 + \chi_{27000}^E + \chi_3(u) (1 + \chi_{248}^E) \right] t^4 + \left[ \chi_2(u) \left( 1 + \chi_{248}^E + \chi_{27000}^E + \chi_{30380}^E \right) + \chi_4(u) \left( 1 + \chi_{248}^E \right) \right] t^5 + \left[ 2\chi_{248}^E + \chi_{30380}^E + \chi_{1763125}^E + \chi_3(u) \left( 2 + 2\chi_{133}^E + \chi_{3875}^E + 2\chi_{27000}^E + \chi_{30380}^E \right) + \chi_5(u) \left( 1 + \chi_{248}^E \right) \right] t^6 + O(t^7).
\]

\[
\begin{align*}
248 &= 1_0 + 14_2 + 14_{-2} + 64_{-1} + \overline{64}_1 + 91_0, \\
3875 &= 1_4 + 1_0 + 1_{-4} + 1_2 + 14_{-2} + 64_3 + 64_{-1} + \overline{64}_1 + \overline{64}_{-3} + 91_0 + 104_0 + 364_2 + 364_{-2} + 832_{-1} + \overline{832}_1 + 1001_0, \\
27000 &= 2 \times 1_0 + 14_2 + 14_{-2} + 2 \times 64_{-1} + 2 \times \overline{64}_1 + 2 \times 91_0 + 104_0 + 104_4 + 104_{-4} + 364_2 + 364_{-2} + 832_3 + 832_{-1} + \overline{832}_1 + \overline{832}_{-3} + 896_2 + 896_{-2} + 1001_0 + 1716_{-2} + 3003_0 + 3080_0 + 4928_{-1} + \overline{4928}_1, \\
30380 &= 1_0 + 2 \times 14_2 + 2 \times 14_{-2} + 64_3 + 2 \times 64_{-1} + 2 \times \overline{64}_1 + \overline{64}_{-3} + 91_4 + 3 \times 91_0 + 91_{-4} + 104_0 + 364_2 + 364_{-2} + 832_3 + 2 \times 832_{-1} + 2 \times \overline{832}_1 + \overline{832}_{-3} + 896_2 + 896_{-2} + 1001_0 + 2002_2 + 2002_{-2} + 3003_0 + 4004_0 + 4928_{-1} + \overline{4928}_1 + 49 \]
\end{align*}
\]

\[
1763125 = 2 \times 1_0 + 2 \times 14_2 + 2 \times 14_{-2} + 3 \times 64_{-1} + 3 \times \overline{64}_1 + 3 \times 91_0 + 104_0 + 104_4 + 104_{-4} + 364_2 + 364_{-2} + 546_5 + 546_{-5} + 546_{-1} + 2 \times 832_3 + 2 \times 832_{-1} + 2 \times \overline{832}_1 + 832_3 + 2 \times 832_{-3} + 2 \times 896_2 + 2 \times 896_{-2} + 2 \times 1001_0 + 2 \times 1716_{-2} + 2 \times \overline{1716}_2 + 2002_2 + 2002_{-2} + 3 \times 3003_0 + 2 \times 3080_0 + 4004_0 + 2 \times 4004_0 + 4004_{-4} + 3 \times 4928_{-1} + 3 \times \overline{4928}_1 + 5625_4 + 5625_{-4} + 5625_{-5} + \overline{5824}_3 + \overline{5824}_5 + \overline{5824}_{-3} + \overline{5824}_{-5} + 5824_5 + 5824_1 + 5824_{-3} + 11648_2 + 11648_{-2} + 17472_3 + 17472_{-1} + \overline{17472}_1 + 17472_{-3} + 18200_2 + 18200_{-2} + 21021_0 + 21021_{-4} + 21021_4 + 21021_0 + 24024_{-2} + 24024_{-2} + 27456_3 + 27456_{-3} + 36608_3 + 36608_{-2} + 40768_{-1} + \overline{40768}_1 + 45760_3 + 45760_{-1} + \overline{45760}_1 + 45760_{-3} + 58344_0 + 58344_{-2} + 64064_{-1} + \overline{64064}_1 + 115830_{-2} + \overline{115830}_1 + 146432_1 + 146432_{-1} + 200200_0.
\]
Concluding remarks

• 5d solitons’ Witten indices (or $Z_{\text{Nekrasov}}$) are clearly understood only recently.
  - understanding the precise contour integral
  - proper interpretation of decoupled factors (the issue is spread everywhere, e.g. AGT)

• This can be used as building blocks of other CFT observables

• Some nontrivial aspects of 5d SCFTs understood using various BPS observables

• Classification of 5d SCFTs? Should be much more challenging than 6d SCFTs