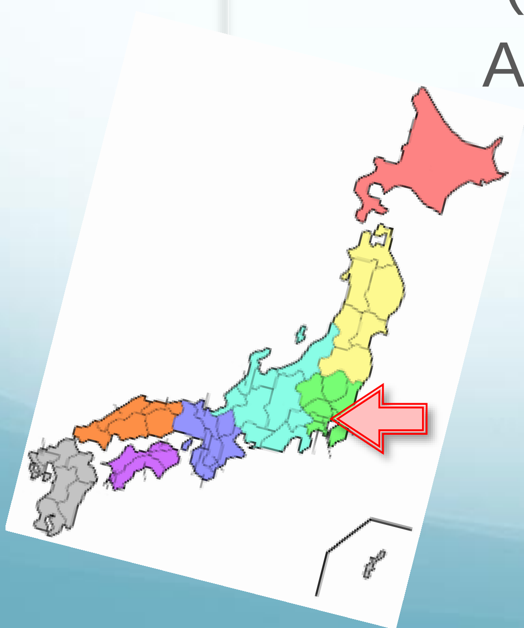
# Hydrodynamics on non-commutative space

Based on PTEP 2014, 103B03
(arXiv:1408.3885v2 [hep-th])

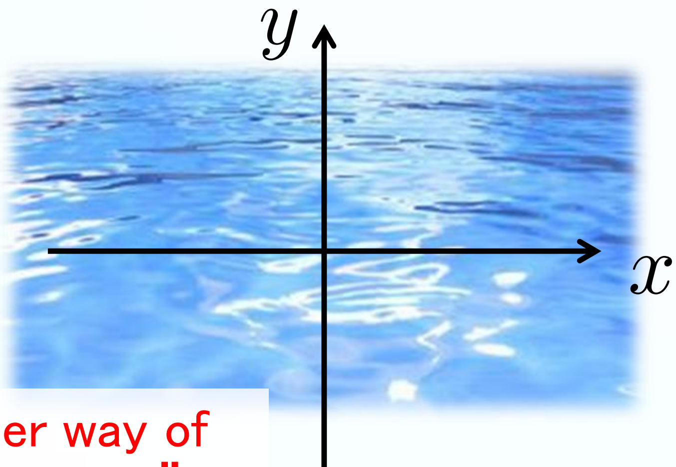
A. Sugamoto, K. Bamba, M.S.

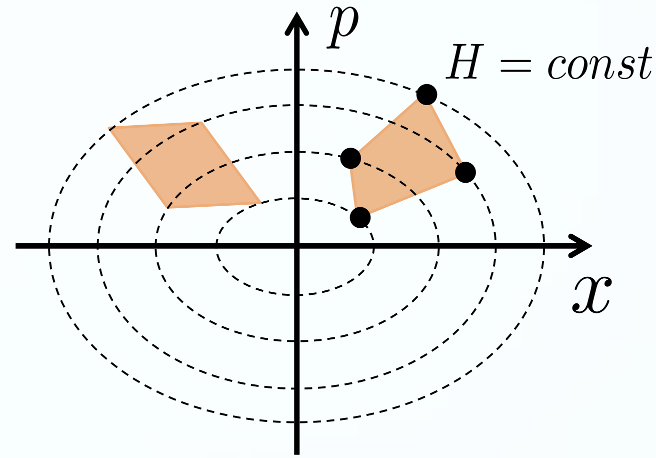


Mayumi Saitou (Ochanomizu Univ.)
麻由美 齋藤 (お茶の水女子大学)

2D hydrodynamics

2D phase space




The other way of "quantization"

"Quantization"

Quantization

New hydrodynamics

Y. Nambu ('11)

Quantum dynamics

New effects

"Uncertainty relation"

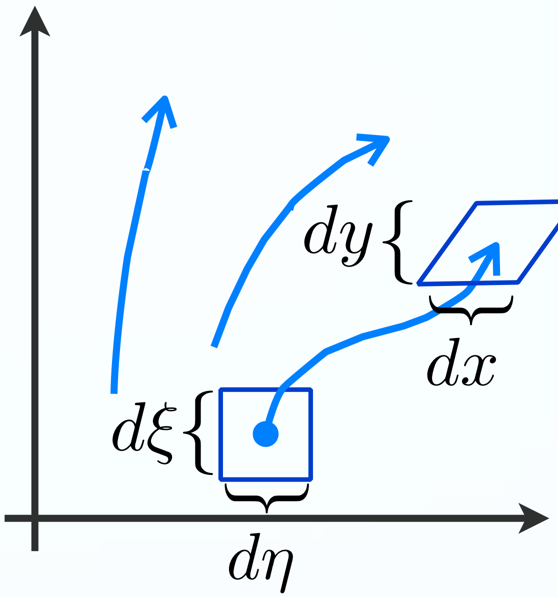
$$\Delta x \cdot \Delta y \geq const.$$

Uncertainty relation

$$\Delta q \cdot \Delta p \geq \frac{\hbar}{2}$$

Now we conjecture that the flow is

- 2-dimensional ideal fluid
- Incompressibilitive
- Non Relativistic




initial coordinate: $(\xi, \eta)$
material coordinate: $x(\xi, \eta), y(\xi, \eta)$

[Jacobian]

$$\frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \eta} - \frac{\partial y}{\partial \xi}\frac{\partial x}{\partial \eta} = 1$$

$$\{x, y\}_{\xi,\eta} = 1$$

… [Poisson bracket]

[equation of continuity]

$$\nabla \cdot \overrightarrow{v} = 0 \leftrightarrow \frac{d}{dt}\{x, y\} = 0$$

[Jacobian]

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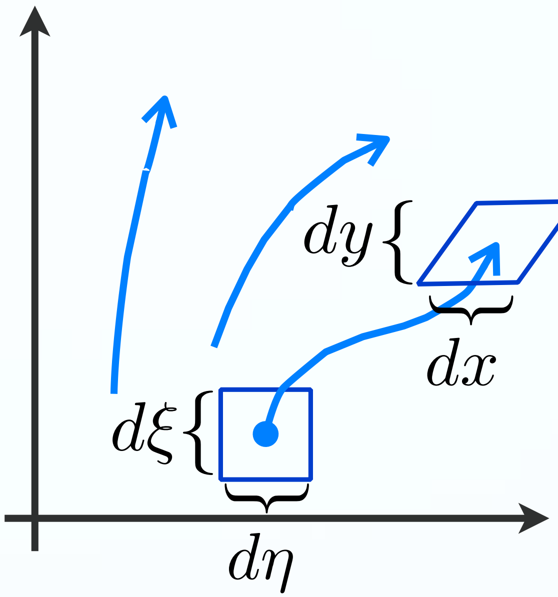


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[equation of continuity]

$$\nabla \cdot \vec{v} = 0 \leftrightarrow \frac{d}{dt}\{x, y\} = 0$$

[stream function]

$$\varphi = \varphi(x, y; t)$$

Y. Nambu ('11)

$$\dot{x} \equiv \{x, \varphi\},\ \ \dot{y} \equiv \{y, \varphi\} \qquad i.e.\quad \frac{\partial \varphi}{\partial y} = \dot{x},\ -\frac{\partial \varphi}{\partial x} = \dot{y}$$

[Jacobian]

$$\frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \eta} - \frac{\partial y}{\partial \xi}\frac{\partial x}{\partial \eta} = 1$$

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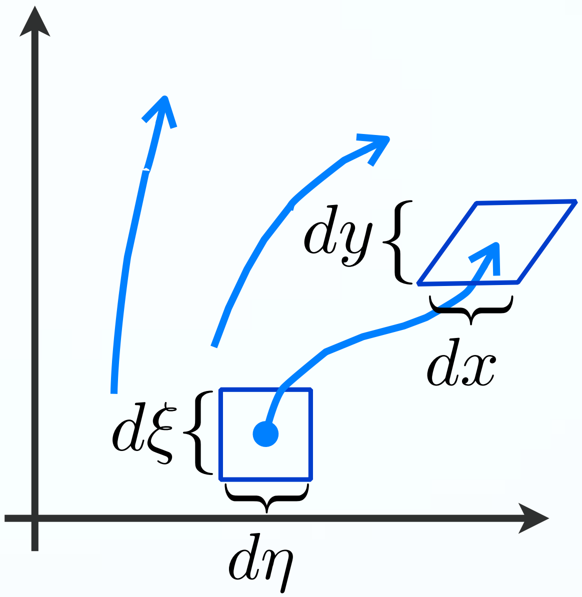
… [Poisson bracket]

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[stream function]

$$\varphi = \varphi(x, y; t)$$

Y. Nambu ('11)

$$\dot{x} \equiv \{x, \varphi\}, \ \ \dot{y} \equiv \{y, \varphi\}$$

$$\rho \frac{D\overrightarrow{v}}{Dt} + \nabla p - \eta \Delta \overrightarrow{v} = 0$$

Rewrite with Poisson brackets

$$\rho \left( \{x_i, \dot{\varphi}\} + \{\{x_i, \varphi\}, \varphi\} \right) + \epsilon^{ij} \{p, x_j\} - \eta \Delta \{x_i, \varphi\} = 0$$

We replaced Poisson brackets by Moyal brackets.

$$\{A, B\}_P \xrightarrow{replace} \frac{1}{i\theta}[A(x), B(x)]_M$$

J. E. Moyal ('49)

## Definition for Moyal bracket

$$A(x) * B(x) = \exp\left(\frac{i}{2!}\theta_{ab}\frac{\partial^2}{\partial y^a \partial z^b}\right) A(y)B(z)\Big|_{y,z\to x}$$

constant parameter: $\theta_{ab}$

$$\theta_{ab} = \epsilon\ \theta$$

$$[A(x), B(x)]_M = \sum_{A,B} \epsilon_{AB}\ A(x) * B(x)$$

# In two-dimensional hydrodynamics,

$$\rho \frac{D\vec{v}}{Dt} + \nabla p - \eta \Delta \vec{v} = 0$$

: Navie-Stokes equation

Rewrite with Poisson brackets

$$\rho \left( \{x_i, \dot{\varphi}\} + \{\{x_i, \varphi\}, \varphi\} \right) + \epsilon^{ij}\{p, x_j\} - \eta \Delta \{x_i, \varphi\} = 0$$

Replacement by Moyal brackets

$$\rho \left( [x_i, \dot{\varphi}]_M + \left[ \frac{1}{\theta}[x_i, \varphi]_M ,\ \varphi \right]_M \right) + \epsilon^{ij}[p, x_j]_M - \eta \Delta [x_i, \varphi]_M = 0$$

Expansion by constant parameter

$$\rho \frac{D\vec{v}}{Dt} + \nabla p - \eta \Delta \vec{v} = \vec{K}$$

$$K_i = \rho \frac{\theta^2}{24} \left[ \left( \partial_{y^1} \partial_{z^2} - \partial_{y^2} \partial_{z^1} \right)^3 v_i(y^1, y^2)\ \varphi(z^1, z^2) \right]_{y^a, z^b \to x} + O\left(\theta^4\right)$$

$$[A(x), B(x)]_M = A(x) * B(x) - B(x) * A(x)$$

$$A(x) * B(x) \equiv \exp\left(\frac{i}{2!}\theta\ \epsilon_{ab}\frac{\partial^2}{\partial y^a \partial z^b}\right) A(y)B(z)\Big|_{y,z\to x}$$

$$A(x) * B(x) = 1 + \frac{i\theta}{2!}\{A(x), B(x)\} + \sum_2^n \frac{1}{n!}\frac{(i\theta)^n}{2^n}\left(\partial_{y_1}\partial_{z_2} - \partial_{y_2}\partial_{z_1}\right)^n A(y)B(x)\Big|_{y,z\to x}$$

Replacement by Moyal brackets

$$\rho\left([x_i, \dot{\varphi}]_M + \left[\frac{1}{\theta}[x_i, \varphi]_M,\ \varphi\right]_M\right) + \epsilon^{ij}[p, x_j]_M - \eta\Delta[x_i, \varphi]_M = 0$$

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For 3-dimensional case, we could get new terms as natural expansions of 2-dimensional case.

# Summary

- We discussed a new hydrodynamics on non-commutative space, through the replacement of the Poison brackets by Moyal brackets.

- The appeared terms would involve effects as minimum size of fluid, from their non-commutativity. We expect that they describe dynamics of granular materials whose size is related to the constant parameter $\theta$.

- We will check our results by computer simulation.
- We would like to construct relationships between the hydrodynamics and a world volume theory.