(Mock) Modular Forms
in String Theory and Moonshine

Miranda C. N. Cheng

Korteweg-de-Vries Institute of Mathematics and Institute of Physics,
University of Amsterdam, Amsterdam, the Netherlands

Abstract

Lecture notes for the Asian Winter School at OIST, Jan 2016.

Contents

1 Lecture 1: 2d CFT and Modular Objects

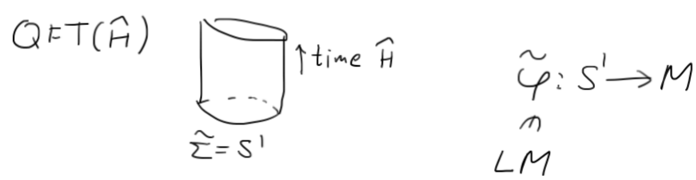
We assume basic knowledge of 2d CFTs.

1.1 Partition Function of a 2d CFT

For the convenience of discussion we focus on theories with a Lagrangian description and in particular have a description as sigma models. This includes, for instance, non-linear sigma models on Calabi–Yau manifolds and WZW models. The general lessons we draw are however applicable to generic 2d CFTs. An important restriction though is that the CFT has a discrete spectrum.

What are we quantising?

In the canonical quantisation of the Hamiltonian formalism:



Hence, the Hilbert space V is obtained by quantising LM = the free loop space of maps $S^1 \to M$.

Recall that in the usual radial quantisation of 2d CFTs we consider a plane with 2 special points: the point of origin and that of infinity. This plane is conformally equivalent to a cylinder via the exponential map: $z = e^{-2\pi i w}$, where z is the coordinate on the plane and $w = \sigma_1 + i\sigma_2$ is that on the cylinder. From this we can read off the Hamiltonian and the momentum on the cylinder via the transformation of the energy-momentum tensor under conformal transformations:

$$(\partial_z z')^2 T'(z') = T(z) - \frac{c}{12}\frac{2\partial_z^3 z'\,\partial_z z' - 3(\partial_z^2 z')^2}{2(\partial_z z')^2}\,. \tag{1.1}$$

They are given by

$$H = \int_0^1 d\sigma_1 T_{\sigma_2\sigma_2} = L_0 + \tilde{L}_0 - \frac{c+\tilde{c}}{24} := H_L + H_R\,, \tag{1.2}$$

$$P = L_0 - \tilde{L}_0 - \frac{c-\tilde{c}}{24} := H_L - H_R\,. \tag{1.3}$$

Partition Function

A partition function of the theory, is defined by

$$Z(\tau,\bar{\tau}) = \mathrm{Tr}_V e^{2\pi i \tau_1 \hat{P} - 2\pi\tau_2 \hat{H}} = \mathrm{Tr}_V q^{\hat{H}_L} \bar{q}^{\hat{H}_R}\ , \quad q = e^{2\pi i \tau}\ .$$

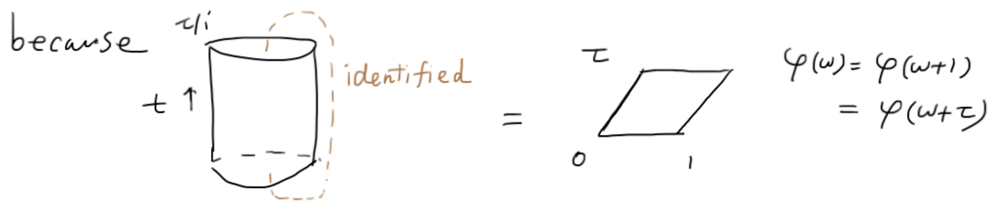
One way to think about it is to consider the $\hat{H}$ operator as measuring the energy of the state, and $\tau_2 = \beta = 1/T$ provides the Boltzmann factor as in statistical mechanics, where T denotes the temperature. The other way to think about it is the following: the Hamiltonian dictates the time translation of the theory, and the operator $e^{2\pi i \tau \hat{H}}$ evolves the system for a duration $-i\tau$ of time, and we want to know the trace of such an operator acting on the Hilbert space V. From this point of view, this quantity also has a very natural interpretation in terms of the Lagrangian formalism. In the latter point of view, taking the trace means we are also imposing a periodic boundary condition in the "time" direction. As a result, we end up performing a path integral with a doubly periodic boundary condition

$$\varphi(w) = \varphi(w+1) = \varphi(w+\tau)\ .$$

The first identification comes from the fact that $\tilde{\Sigma} = S^1$, and the second comes from taking the trace. In picture, this is:

$$Z(\tau,\bar\tau) \overset{(\hat H)}{=} \mathrm{Tr}_V\, q^{\hat H_L}\, \bar q^{\hat H_R}$$

$$\overset{(\mathcal{L})}{=} \int \mathcal{D}[\varphi : \lozenge_{0,1,\tau} \to M]\, e^{-\mathcal{L}(\varphi)}$$



But the above expression we have are now explicitly invariant under the torus mapping class group $SL_2(\mathbb{Z})$! To see this, note that the boundary condition $\varphi(z) = \varphi(z+1) = \varphi(z+\tau)$ is equivalent to the boundary condition $\varphi(z) = \varphi(z + a\tau + b) = \varphi(z + c\tau + d)$ for any integers a, b, c, d satisfying $ad - bc = 1$. Put differently, recall that any torus is flat and can be identified with $\mathbb{C}/(\alpha\mathbb{Z} + \tau\alpha\mathbb{Z})$ for some $\alpha, \tau \in \mathbb{C}$. Since we want to study a CFT and hence only care about the complex structure of the torus, we can rescale the torus such that $\alpha = 1$, $\tau \in \mathbb{H}$ where $\mathbb{H} = \{x + iy | y > 0\}$ is the upper-half plane. Note the lattice isomorphism $\mathbb{Z} + \tau\mathbb{Z} \cong (a\tau + b)\mathbb{Z} + (c\tau + d)\mathbb{Z}$ for all integers a, b, c, d satisfying $ad - bc = 1$. As a result, the complex structure moduli space of a torus is $\mathbb{H}/SL_2(\mathbb{Z})$, where $SL_2(\mathbb{Z})$ acts on $\mathbb{H}$ via

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \tau = \frac{a\tau + b}{c\tau + d} \tag{1.4}$$

A convenient presentation is

$$SL_2(\mathbb{Z}) = \left\{ S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},\ T = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix};\ S^2 = (ST)^3, S^4 = \mathbf{1} \right\}. \tag{1.5}$$

From this we can see that a fundamental domain is the keyhole region bounded by $\mathrm{Re}\tau = 1/2$, $\mathrm{Re}\tau = -1/2$ and $|\tau| = 1$. The cusps, corresponding to the degeneration limit of the tori, are $i\infty \cup \mathbb{Q}$.

Hence we conclude that the partition function of a 2d CFT should have the modular invariance

$$Z(\tau, \bar{\tau}) = Z\left(\frac{a\tau + b}{c\tau + d}, \frac{a\bar{\tau} + b}{c\bar{\tau} + d}\right) \quad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \, .$$

1.2 Modular Forms

We have seen that the partition function of a 2d CFT is invariant under the natural action of $SL_2(\mathbb{Z})$. This suggests an important role of the mathematical objects called modular forms in the study of various properties of 2d CFT.

Now we will define modular forms and give a few examples which will become useful later.

We start with defining weight 0 modular forms on the modular group $SL_2(\mathbb{Z})$, which are simply holomorphic functions on $\mathbb{H}$ that are invariant under the action of $SL_2(\mathbb{Z})$:

$$f(\tau) = f\left(\frac{a\tau + b}{c\tau + d}\right) \quad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}). \tag{1.6}$$

But this will turn out to be too restrictive: constants are the only such functions. We introduce modular forms on the modular group $SL_2(\mathbb{Z})$ of general weight k, which are holomorphic functions on $\mathbb{H}$ that transform under the action of $SL_2(\mathbb{Z})$ as:

$$f(\tau) = (c\tau + d)^{-k} f\left(\frac{a\tau + b}{c\tau + d}\right) \quad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}). \tag{1.7}$$

We will consider integral and half-integral weight k.[1]

With this definition we start to get some non-trivial examples. For instance the following Eisenstein series:

$$E_4(\tau) = 1 + 240 \sum_{n=1}^{\infty} \frac{n^3 q^n}{1 - q^n} = 1 + 240\, q + 2160\, q^2 + \ldots \tag{1.8}$$

$$E_6(\tau) = 1 - 504 \sum_{n=1}^{\infty} \frac{n^5 q^n}{1 - q^n} = 1 - 504\, q - 16632\, q^2 + \ldots \tag{1.9}$$

[1]Clearly, special care needs to be taken when k is half-integral. Strictly speaking, one should work with the metaplectic cover of $SL_2(\mathbb{Z})$ but we will avoid discussing the subtleties here.

are examples of modular forms of weight 4 and weight 6, respectively. But the definition is still somewhat too restrictive as in some sense there are only these two examples: The ring of modular forms on $SL_2(\mathbb{Z})$ is generated freely by E_4 and E_6. This is to say, any modular form of integral weight k can be written (uniquely) as a sum of monomials $E_4^\alpha E_6^\beta$ with $k = 4\alpha + 6\beta$.

We can further generalise the above definition in the following directions (and combinations thereof):

1. *Multipliers*: allowing for a phase $\psi : SL_2(\mathbb{Z}) \to \mathbb{C}^*$ in the transformation rule (1.7).

2. *Poles*: allowing the function to have exponential growth near the cusps. Such functions are said to be *weakly holomorphic modular forms*.

3. *Subgroups*: considering a subgroup $\Gamma \subset SL_2(\mathbb{Z})$ of the modular group for which we impose the transformation property.

4. *Vector-Valued*: instead of $f : \mathbb{H} \to \mathbb{C}$ we consider a vector-valued function $f : \mathbb{H} \to \mathbb{C}^n$ with n components.

Of course, the above generalisations can be combined. For instance one can consider a vector-valued modular form with multipliers for a subgroup Γ of $SL_2(\mathbb{Z})$. Obviously, now the multiplier ψ is no longer a phase but a matrix. Also, the above concepts are not entirely independent. For instance, a component of a vector-valued modular form for $SL_2(\mathbb{Z})$ can often be considered as a modular form for a subgroup of $SL_2(\mathbb{Z})$.

With these more general definitions we start to get a zoo of interesting examples.

- *Jacobi theta function*

  Consider a 1-dimensional lattice with bilinear form $\langle x, x\rangle = x^2$. The associated theta function is

$$\theta_3(\tau) = \sum_{n\in\mathbb{Z}} q^{n^2/2}\,. \tag{1.10}$$

  This simple function turns out to admit an expression in terms of infinite products

$$\theta_3(\tau) = \sum_{n\in\mathbb{Z}} q^{n^2/2} = \prod_{n=1}^{\infty}(1-q^n)(1+q^{n-1/2})^2\,, \tag{1.11}$$

and has nice modular properties. To describe the modular properties, it is most natural to introduce another two theta functions

$$\theta_2(\tau) = \sum_{n+\frac{1}{2}\in\mathbb{Z}} q^{n^2/2} = 2q^{1/8}\prod_{n=1}^{\infty}(1-q^n)(1+q^n)^2\,, \tag{1.12}$$

$$\theta_4(\tau) = \sum_{n\in\mathbb{Z}}(-1)^n q^{n^2/2} = \prod_{n=1}^{\infty}(1-q^n)(1-q^{n-1/2})^2. \tag{1.13}$$

It turns out that they are the three components of a vector-valued modular form for $SL_2(\mathbb{Z})$

$$\Theta(\tau) = \begin{pmatrix}\theta_2(\tau)\\ \theta_3(\tau)\\ \theta_4(\tau)\end{pmatrix} = \sqrt{\frac{i}{\tau}}S\,\Theta(-1/\tau) = T\,\Theta(\tau+1)\,, \tag{1.14}$$

where

$$S = \begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}, \quad T = \begin{pmatrix}e(-1/8)&0&0\\0&1&0\\0&0&1\end{pmatrix}. \tag{1.15}$$

Later we will see that they can in turn be most naturally considered as the specialisation at $z=0$ of the two-variable Jacobi theta functions

$$\begin{aligned}
\theta_1(\tau,z) &= -i\sum_{n+\frac{1}{2}\in\mathbb{Z}}(-1)^{n-\frac{1}{2}}y^n q^{n^2/2}\\
&= -iq^{1/8}(y^{1/2}-y^{-1/2})\prod_{n=1}^{\infty}(1-q^n)(1-yq^n)(1-y^{-1}q^n)\,,\\
\theta_2(\tau,z) &= \sum_{n+\frac{1}{2}\in\mathbb{Z}} y^n q^{n^2/2}\\
&= (y^{1/2}+y^{-1/2})q^{1/8}\prod_{n=1}^{\infty}(1-q^n)(1+yq^n)(1+y^{-1}q^n)\,,\\
\theta_3(\tau,z) &= \sum_{n\in\mathbb{Z}} y^n q^{n^2/2} = \prod_{n=1}^{\infty}(1-q^n)(1+yq^{n-1/2})(1+y^{-1}q^{n-1/2})\,,\\
\theta_4(\tau,z) &= \sum_{n\in\mathbb{Z}}(-1)^n y^n q^{n^2/2} = \prod_{n=1}^{\infty}(1-q^n)(1-yq^{n-1/2})(1-y^{-1}q^{n-1/2})\,,
\end{aligned}$$

which transform in the following way. Define

$$\Theta(\tau,z) = \begin{pmatrix} \theta_1(\tau,z) \\ \theta_2(\tau,z) \\ \theta_3(\tau,z) \\ \theta_4(\tau,z) \end{pmatrix} \tag{1.16}$$

we have

$$\Theta(\tau,z)\sqrt{\frac{i}{\tau}}e(-z^2/(2\tau))S\,\Theta(-1/\tau,z/\tau) = T\,\Theta(\tau+1,z)\,, \tag{1.17}$$

where

$$S = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} e(-1/8) & 0 & 0 & 0 \\ 0 & e(-1/8) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{1.18}$$

- *Dedekind eta function*

$$\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n) \tag{1.19}$$

is a weight 1/2 modular form with multipliers, satisfying

$$\eta(\tau) = \sqrt{\frac{i}{\tau}}\eta(-1/\tau), \quad \eta(\tau) = e(-1/24)\,\eta(\tau+1), \tag{1.20}$$

where we write $e(x) := e^{2\pi i x}$. It is related to the theta functions by

$$\eta(\tau)^3 = \frac{1}{2}\theta_2(\tau)\theta_3(\tau)\theta_4(\tau). \tag{1.21}$$

1.3 Example: The Free Boson

To illustrate the previous discussion we will now do a simple example in which $M = \mathbb{R}$ and the Lagrangian is free (of interaction terms). In this part I will not be careful with the factors of π and i. But apart from this which can be fixed by carefully going through the steps, all formulas should be correct.

We are looking at the maps $\tilde{\varphi} : S^1 \to \mathbb{R}$. The free loop space has the following coordinates

$$\tilde{\varphi}(s) = \tilde{\varphi}(s+1) = \sum_{n\in\mathbb{Z}} e(ns)\varphi_n \ .$$

The free Lagrangian is given by

$$\mathcal{L} = \int_\Sigma d\varphi \wedge \star d\varphi = \sum_{n\in\mathbb{Z}} \left(\dot{\varphi}_n \dot{\varphi}_{-n} - n^2 \varphi_n \varphi_{-n}\right) \ .$$

Canonical quantisation renders

$$[\varphi_n, \pi_m] = i\delta_{n,m} \ ,$$

where we read out from the Lagrangian (or the Hamiltonian) that $\pi_n \sim \dot{\varphi}_{-n}$. Using instead the alternative basis

$$\varphi_n = \frac{1}{n}\left(a_n - \tilde{a}_{-n}\right) \ , \quad \pi_n = a_n + \tilde{a}_{-n} \ ,$$

we get the commutator relation

$$[a_n, a_m] = n\delta_{n+m,0} = [\tilde{a}_n, \tilde{a}_m], \ \ [a_n, \tilde{a}_m] = 0 \ ,$$

and the Hamiltonian reads

$$H = \pi_0^2 + \sum_{n\neq 0}(a_{-n}a_n + \tilde{a}_{-n}\tilde{a}_n) \ . \tag{1.22}$$

Using the Hamiltonian to time-evolve the system and go to the Heisenberg picture, we finally obtain

$$\partial_z \varphi(z,\bar{z}) = \pi_0 z^{-1} + \sum_{n\neq 0} a_n z^{-n-1} \ , \ \partial_{\bar{z}} \varphi(z,\bar{z}) = \pi_0 \bar{z}^{-1} + \sum_{n\neq 0} \tilde{a}_n \bar{z}^{-n-1} \ .$$

From this we can compute and discuss some basic quantities of this theory.

Conformal Symmetry

From the Lagrangian we can read out the energy-momentum tensor $T(z) \sim \partial\varphi\partial\varphi$. We can check that it satisfies the Virasoro algebra with $c = 1 = \tilde{c}$.

Essentially, this can be understood by using the zeta function regularisation $1+2+3+\cdots=-\frac{1}{12}$ when dealing with the ordering ambiguity of quantisation.

Ground State

Focusing on the "oscillators" $a_n, \tilde{a}_n, n \neq 0$, we see that the ground states must satisfy

$$a_n|0\rangle = 0 = \tilde{a}_n|0\rangle \quad , \quad \text{for } n>0\,.$$

In fact, note that there is no restriction on the eigenvalue of the "zero modes" π_0 and hence there is a continuum of ground states

$$\pi_0|0;p\rangle = p|0;p\rangle \tag{1.23}$$

$$a_n|0;p\rangle = 0 = \tilde{a}_n|0;p\rangle \quad , \quad \text{for } n>0\,. \tag{1.24}$$

Primary Fields

Some examples are given by $\partial^n\varphi, \bar{\partial}^m\varphi, e^{ik\varphi}, \ldots$.

Partition Function

We want to compute the quantity $Z(\tau,\bar{\tau}) = \mathrm{Tr}_V q^{\hat{H}_L}\bar{q}^{\hat{H}_R}$. Notice that the Hamiltonian (1.22) splits into two parts: the "zero modes" π_0 and the "oscillators" $a_n, \tilde{a}_n, n \neq 0$. It is clear that the latter part is simply given by the tensor product of two copies of the Heisenberg algebra and its contribution is simply $|\prod_{n=1}^{\infty}\frac{1}{1-q^n}|^2$. The first part, on the other hand, doesn't factor into the left- and right-moving part and π_0 has a continuous spectrum which leads us to doing the following integral

$$\int \frac{dk}{2\pi} e^{-\pi k^2 \mathrm{Im}\tau} \sim \frac{1}{\sqrt{\mathrm{Im}\tau}}\,.$$

Combining ingredients including the zero-mode energy $-c/24 = -1/24$, we get

$$Z(\tau,\bar{\tau}) = \frac{1}{\sqrt{\mathrm{Im}\tau}}\left|q^{-1/24}\prod_{n=1}^{\infty}\frac{1}{1-q^n}\right|^2 = \frac{1}{\sqrt{\mathrm{Im}\tau}}\left|\frac{1}{\eta(\tau)}\right|^2\,.$$

Now we can check that this is indeed modular *invariant*. This is to be contrasted with the partition function for the Heisenberg algebra, namely when we forget about the zero-modes and take only the chiral half. In that case the partition function $1/\eta(\tau)$ is a modular form with non-zero weight.

1.4 Example: Ising Model

A minimal model have a finite number of primary fields. This means the Hilbert space has the structure

$$\mathcal{H} = \bigoplus_{h,\tilde{h}} M_{h,\tilde{h}}\, V_h \otimes V_{\tilde{h}} \ , \tag{1.25}$$

where $M_{h,\tilde{h}}$ is the multiplicity and $V_h(V_{\tilde{h}})$ are the left(right)-moving Virasoro irreducible modules, and subsequently the partition function has the structure

$$Z(\tau,\bar{\tau}) = \sum_{h,\tilde{h}} M_{h,\tilde{h}}\, \chi_h(\tau) \otimes \chi_{\tilde{h}}(\bar{\tau}) \ , \tag{1.26}$$

where χ_h is the character of the irreducible Virasoro module. Since this is a finite sum, the only way $Z(\tau,\bar{\tau})$ is invariant under $SL_2(\mathbb{Z})$ is that the characters themselves have modular properties. This "explains" the modular properties of the Virasoro characters and similarly the characters for other Kac–Moody algebras: they often transform as weight 0 modular forms for some subgroup of $SL_2(\mathbb{Z})$.

The simplest example is probably the Ising model, for which we have

$$\mathcal{H} = V_0 \otimes V_0 \oplus V_{1/2} \otimes V_{1/2} \oplus V_{1/16} \otimes V_{1/16} \ , \tag{1.27}$$

with the corresponding characters are given by

$$\chi_0 = \frac{1}{2}\left(\sqrt{\frac{\theta_3}{\eta}} + \sqrt{\frac{\theta_4}{\eta}}\right) , \tag{1.28}$$

$$\chi_{1/2} = \frac{1}{2}\left(\sqrt{\frac{\theta_3}{\eta}} - \sqrt{\frac{\theta_4}{\eta}}\right) , \tag{1.29}$$

$$\chi_{\frac{1}{16}} = \sqrt{\frac{\theta_2}{2\eta}} \, . \tag{1.30}$$

One can check explicitly that

$$Z_{\text{Ising}}(\tau,\bar{\tau}) = |\chi_0|^2 + |\chi_{1/2}|^2 + |\chi_{1/16}|^2 = \frac{1}{2}\left(\left|\frac{\theta_2}{\eta}\right| + \left|\frac{\theta_3}{\eta}\right| + \left|\frac{\theta_4}{\eta}\right|\right) \tag{1.31}$$

is indeed invariant under $SL_2(\mathbb{Z})$.

1.5 $\mathcal{N} = 2$ SCA, Elliptic Genus, and Jacobi Forms

In the context of string theory we often study 2d CFTs with supersymmetries. The presence of supersymmetry means that there is now an extra $\mathbb{Z}_2$ grading on the Hilbert space: $V = V_0 \oplus V_1$. For instance, in the context of type II superstrings compactified on Calabi-Yau manifolds, the relevant "internal" CFT is a non-linear sigma model with $\mathcal{N} = 2$ supersymmetry. When its target space is a Calabi-Yau manifold, the theory has the $\mathcal{N} = 2$ extension of Virasoro symmetry, given by the so-called $\mathcal{N} = 2$ superconformal algebra (SCA).

The "$\mathcal{N} = 2$" refers to the fact that we include 2 fermionic currents in the algebra on top of the bosonic energy-momentum tensor $T(z)$. Furthermore, there's now an extra automorphism, which we call the R-symmetry, that rotates different fermionic currents onto each other.

We denote the two fermionic currents by $G_+(z)$ and $G_-(z)$ and the $U(1)$ R-symmetry current rotating the two by $J(z)$. The algebra reads

$$\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\,\delta_{m+n,0} \\
[J_m, J_n] &= \frac{c}{3}m\,\delta_{m+n,0} \\
[L_n, J_m] &= -m\,J_{m+n} \\
[L_n, G_r^\pm] &= (\frac{n}{2} - r)\,G_{r+n}^\pm \\
[J_n, G_r^\pm] &= \pm G_{r+n}^\pm \\
\{G_r^+, G_s^-\} &= 2L_{r+s} + (r-s)J_{r+s} + \frac{c}{3}\,(r^2 - \frac{1}{4})\,\delta_{r+s,0}\ ,
\end{aligned} \tag{1.32}$$

and all other (anti-)commutators are zero. As before we have two possible periodic conditions for the fermions

$$\begin{cases} 2r = 0 \bmod 2 & \text{for R sector} \\ \\ 2r = 1 \bmod 2 & \text{for NS sector}\ . \end{cases} \tag{1.33}$$

Two comments about this algebra are in order here. First, we have now two generators of the Cartan subalgebra: $[L_0, J_0] = 0$. As a result, the representations will now be graded by two "quantum numbers" that are the eigenvalues of the L_0 and J_0 of the highest weight vector. The second new feature is that there is a non-trivial inner automorphism of the algebra, which means that the algebra remains the same under the following redefinition

$$\begin{aligned} L_n &\to L_n + \eta J_n + m\,\eta^2\,\delta_{n,0} \\ J_n &\to J_n + 2m\,\eta\,\delta_{n,0} \\ G_r^\pm &\to G_{r\pm\eta}^\pm \end{aligned} \tag{1.34}$$

with $\eta \in \mathbb{Z}$. In the above we have written $m := c/6$. If instead we choose $\eta \in \mathbb{Z}+1/2$ we exchange the Ramond and the Neveu-Schwarz algebra. Note that the only operator (up to scaling and the addition of central terms) invariant under such a transformation is $4mL_0 - J_0^2$.

Again we will focus on the Ramond algebra and define the Ramond ground states of $\mathcal{N}=2$ SCFT:

Ramond Ground States and the Witten Index

As before, we require the ground states to be annihilated by all the positive modes:

$$L_n|\phi\rangle = J_m|\phi\rangle = G_r^\pm|\phi\rangle = 0 \quad \text{for all} \quad m,n,r>0\ .$$

Moreover, they have to annihilated by the zero modes of the fermionic currents

$$G_0^\pm|\phi\rangle = 0\ .$$

Again this condition fixes their L_0-eigenvalue to be

$$\frac{1}{2}\{G_0^+,G_0^-\}|\phi\rangle = \Big(L_0 - \frac{c}{24}\Big)|\phi\rangle = 0\ .$$

Let's ignore the right-moving part of the spectrum for a moment and consider a chiral Hilbert space V. We define its Witten index as

$$\mathrm{WI}(\tau,V) = \mathrm{Tr}_V\big((-1)^{J_0}q^{L_0-\frac{c}{24}}\big)\ .$$

If a state $|\psi\rangle$ is not annihilated by G_0^+, then the states $|\psi\rangle$ and $G_0^+|\psi\rangle$ together contribute 0 to $\mathrm{WI}(\tau,V)$ since $[L_0,G_0^+]=0$ while $[J_0,G_0^+]=G_0^+$. The same argument holds for G_0^- and we conclude that only Ramond ground states can contribute to the Witten index. As a result, the Witten index $\mathrm{WI} : \mathcal{N}=2\ \mathrm{SCFT} \to \mathbb{Z}$ is independent of τ and counts (with signs) the number of Ramond ground states in V.

Notice moreover that the Witten index for $\mathcal{N}=2$ SCFT acquires an interpretation as computing the graded dimension of the cohomology of the Q_0^+ operator, satisfying $(Q_0^+)^2=0$. For $\{G_0^+,(G_0^+)^\dagger\}=\{G_0^+,G_0^-\}=L_0-\frac{c}{24}$, the Ramond ground states have the interpretation as the harmonic representative in the cohomology. This fact underlies the rigidity property of the Witten index and the elliptic genus which we will define now.

The same analysis can be trivially extended when one has a non-chiral with both left- and right-moving degrees of freedom: the Witten index

$$\mathrm{WI}(\tau,\bar\tau,V)=\mathrm{Tr}_V\big((-1)^{\tilde J_0+J_0}\bar q^{\tilde L_0-\frac{\tilde c}{24}}q^{L_0-\frac{c}{24}}\big)$$

counts states that are Ramond ground states for both the left- and the right-moving copy of $\mathcal{N}=2$ SCA.

The $\mathcal{N}=2$ Elliptic Genus

It is fine to be able to compute the graded dimension of a cohomology, but we can go further and compute more interesting properties of this vector space. For instance, we have learned that the representations of $\mathcal{N}=2$ SCA are labeled by two quantum numbers corresponding to the Cartan generators L_0 and J_0. It will hence be natural to consider the following quantity which computes the dimension of $\tilde Q_0^+$ cohomology graded by the left-moving quantum numbers L_0, J_0.

Definition: Elliptic Genus (CFT)

The elliptic genus of a $\mathcal{N}=(2,2)$ SCFT is the quantity

$$\mathcal{Z}(\tau,z)=\mathrm{Tr}_{\mathcal{H}_{\mathrm{RR}}}\Big((-1)^{J_0+\bar J_0}y^{J_0}q^{L_0-c/24}\bar q^{\bar L_0-c/24}\Big)\ ,\quad y=e^{2\pi iz}\ , \tag{1.35}$$

where the "RR" in $\mathcal{H}_{\mathrm{RR}}$ denotes the fact that we are considering the Ramond sector of the $\mathcal{N}=2$ SCA both for the left- and right-moving copy of the algebra.

Note that the elliptic genus provides a compromise between the partition function and the Witten index in the following sense. The former contains a lot more information of the latter which only knows about the states that are Ramond ground states with respect to both the left- and the right-moving copy of $\mathcal{N}=2$ SCA. The elliptic genus on the other hands contains information of states that are Ramond ground states with respect to only the the right-moving copy of $\mathcal{N} = 2$ SCA but still has the rigidity property of the Witten index which makes it possible to compute for many SCFTs.

Modular Properties

As we argued before, a path integral interpretation of the elliptic genus suggests it has nice transformation property under the torus mapping class group. Moreover, the inner automorphism of the algebra (the spectral flow symmetry) implies that the graded dimension of a L_0-, J_0- eigenspace should only depends on its eigenvalue under the eigenvalue of the combined operator $4mL_0 - J_0^2$ and the charge of J_0 mod $2m$ where $m = c/6$. Hence, the Fourier expansion of the elliptic genus should take the form

$$\mathcal{Z}(\tau, z) = \sum_{n,\ell} q^n y^\ell \, c_{\ell\,(2m)}(4mn - \ell^2) \; .$$

From these facts one can deduce that the elliptic genus of an $\mathcal{N} = (2,2)$ SCFT is a *weak Jacobi form*.

Definition: Jacobi Form

If the function $\phi(\tau, z) : \mathbb{H} \times \mathbb{C} \to \mathbb{C}$ transformas in the following way under the Jacobi group $SL(2,\mathbb{Z}) \ltimes \mathbb{Z}^2$:

$$\begin{aligned} \mathcal{Z}\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right) &= (c\tau+d)^k \, e^{2\pi i t \frac{cz^2}{c\tau+d}} \mathcal{Z}(\tau, z) \; , \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) \\ \mathcal{Z}(\tau, z + \lambda\tau + \mu) &= e^{-2\pi i t(\lambda^2\tau + 2\lambda z)} \mathcal{Z}(\tau, z) \; , \quad \lambda, \mu \in \mathbb{Z} \; , \end{aligned} \tag{1.36}$$

and have furthermore the expansion

$$\mathcal{Z}(\tau, z) = \sum_{n \geq 0, \ell + t \in \mathbb{Z}} c_{\ell\,(2t)}(4tn - \ell^2) q^n y^\ell \; . \tag{1.37}$$

with $c_{\ell\,(2t)}(4tn - \ell^2) = 0$ for all $4tn - \ell^2 < 0$ ($n < 0$) with some integral or half-integral k and t, then the function is called a holomorphic (weak) Jacobi form of weight k and index t.

From this definition, we have the following **fact:** The elliptic genus of an $\mathcal{N} = (2,2)$ SCFT with central charge $c = 6t$ is a weight zero, index t weak Jacobi form.

Some Examples: $K3$ and T^4

There are two topologically distinct Calabi-Yau two-folds: $K3$ and T^4. We expect their elliptic genus to be weight zero weak Jacobi forms with index 1. Coincidentally, the dimension of the space of such a form is one: $\mathbb{C} \otimes \varphi_{0,1}(\tau, z)$ where

$$\varphi_{0,1}(\tau, z) = 4 \sum_{i=2,3,4} \big(\frac{\theta_i(\tau, z)}{\theta_i(\tau, 0)}\big)^2 . \tag{1.38}$$

Hence we only need one topological invariant of the Calabi-Yau two-folds to fix the whole elliptic genus. From

$$\mathcal{Z}(\tau, z = 0; T^4) = \chi(T^4) = 0 \quad , \quad \mathcal{Z}(\tau, z = 0; K3) = \chi(K3) = 24$$

and

$$\varphi_{0,1}(\tau, z = 0) = 12$$

we obtain

$$\mathcal{Z}(\tau, z; T^4) = 0 \quad , \quad \mathcal{Z}(\tau, z; K3) = 2\varphi_{0,1}(\tau, z) .$$

This clearly demonstrates the power of modularity in gaining extremely non-trivial information about the spectrum of $\mathcal{N} = (2, 2)$ SCFT.

For later use we will introduce another basic Jacobi form, this time of weight -2 and index 1. It is given by

$$\varphi_{-2,1} = -\frac{\theta_1(\tau, z)^2}{\eta^6(\tau)}. \tag{1.39}$$

Together with $\varphi_{0,1}$ it generates the ring of weak Jacobi forms of even weight.

1.6 Symmetries and Twined Functions

Consider a special situation when the target manifold M has a non-trivial automorphism group G that is a finite simple group. There are two interesting things we can do in such a situation: one is the so-called "twisting" (or "twining") where more refined information about the spectrum can be obtained. The other is the so-called "orbifolding", which is a procedure that allows one to construct a new conformal field theory, with now the orbifold M/G as target space, from the old one with target space M. Of course, in general the orbifold is not a smooth manifold. However, as we will see, it does not hinder us to construct the corresponding conformal field theory. Hence we say that this type of singularity can be "dealt with" by string theory.

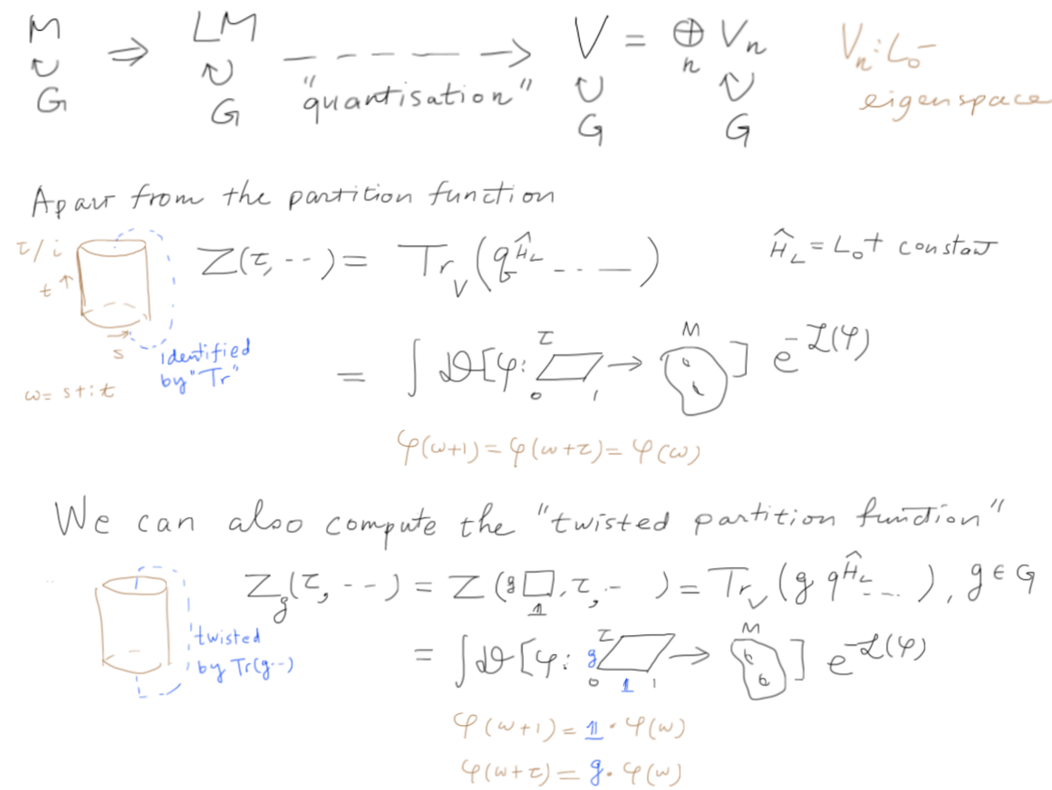
For simplicity we will limit ourselves to Abelian groups. We will comment briefly on the non-Abelian orbifold in the end of this lecture.

Twining

To understand the procedure of twisting (or twining), let us note that the free loop space LM also inherit the automorphism group G. Upon quantisation, the quantum Hilbert space V hence also has a G-symmetry. Moreover, from the above description we expect the G action on V to commute with the grading of L_0. Now, when we have a G-module V_n, apart from its dimension we can also compute its character $\mathrm{Tr}_{V_n} g$, for $g \in G$. Moreover, knowing the character for all conjugacy classes $[g]$ of the group allows us to pin down the action of G on it. This procedure of "twisting by g" leads to the computation of the so-called "twisted (or twining) partition function" of the theory.

Apart from this Hamiltonian description of the twisted partition function, it is obvious that it also allows a natural interpretation in terms of path integral. Namely, instead of performing a functional integral over maps from a torus into the target space with doubly periodic boundary condition, we are integrating over maps with boundary conditions that are modified by g along the Euclidean "time" direction.

Here we illustrate what we said in terms of pictures:



1.7 Orbifolding

Twisted Sectors

As mentioned before, we can construct a new CFT by quantising the free loop space $L(M/G)$ of the orbifold instead. A very useful decomposition of the loop space can be found by considering the larger space

$$IM = \{\breve{\varphi} : [0,1] \to M\} \, .$$

The loop space $L(M/G)$ of the orbifold can be identified with a subspace of IM in the following way:

$$L(M/G) = \{\tilde{\varphi} \in IM | \tilde{\varphi}(1) = h \cdot \tilde{\varphi}(0) \text{ for some } h \in G\} \, .$$

Using this identification, we see that the loop space has a natural decomposition

$$L(M/G) = \oplus_h L^h(M/G)$$

defined by $\tilde{\varphi} \in L^h(M/G)$ iff $\varphi(1) = h \cdot \tilde{\varphi}(0)$. Related to this loop sub-space is the following quantity $Z(\mathbb{1}\underset{h}{\square};\tau)$, which is an important building block of the orbifold theory:

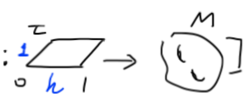
$$IM = \{\tilde{\varphi} : [0,1] \to M\}$$

$$\cup$$

$$L(M/G) = \bigoplus_{h\in G} L^h(M/G)$$

$$L^h(M/G) = \{\tilde{\varphi} \in IM \mid \tilde{\varphi}(1) = h \cdot \tilde{\varphi}(0)\}$$

The corresponding path integral reads

$$Z(\mathbb{1}\underset{h}{\square};\tau,\ldots) = \int \mathcal{D}[\varphi :$$  $$] \; e^{-\mathcal{L}(\varphi)}$$

$$\varphi(w+1) = h \cdot \varphi(w)$$
$$\varphi(w+\tau) = \mathbb{1} \cdot \varphi(w)$$

Upon quantisation we should obtain the decomposition of the physical Hilbert space

$$V^G = \oplus_{h\in G} V^{G,h} \ .$$

We call $V^{G,h}$ the quantum Hilbert space of the *h-twisted sector* of the orbifold theory. We might expect $V^{G,h}$ to be simply the quantisation of the space $L^h(M/G)$ seen as a subspace of IM. There is a very crucial subtlety we would be overlooking if we take this viewpoint, however. Namely, the quantum Hilbert space of the orbifold theory on M/G has to be, by definition, G-invariant. This invariance can be achieved by, in a path integral language, averaging over the boundary condition along the Euclidean time circle. From this prescription, and summing over all twisted sectors, we finally obtain an expression for the orbifold theory in terms of path integrals over maps into the *original* manifold M:

$$Z^{G,h}(\tau,\dots) = \mathrm{Tr}_{V_{G,h}}\left(q^{\hat{H}_L}\dots\right) = \frac{1}{|G|}\sum_{g\in G} Z\left(g\underset{h}{\square};\tau,\dots\right)$$

$$Z\left(g\underset{h}{\square};\tau,\dots\right) = \int \mathcal{D}\varphi\left[\varphi : \underset{0\quad h\quad 1}{{}^{\tau}_{g}\square} \to M\right] e^{-\mathcal{I}(\varphi)}$$

$$\begin{aligned} Z^{G}(\tau,\dots) &= \mathrm{Tr}_{V_G}\left(q^{\hat{H}_L}\dots\right) \\ &= \sum_{h\in G} Z^{G,h}(\tau,\dots) \\ &= \frac{1}{|G|}\sum_{g,h} Z\left(g\underset{h}{\square};\tau,\dots\right) \\ &= \int \mathcal{D}\check{\varphi}\left[\check{\varphi} : \underset{0\qquad 1}{{}^{\tau}\square} \to M/G\right] e^{-\mathcal{I}(\check{\varphi})} \end{aligned}$$

For non-Abelian finite simple group G, the projection onto G-invariant state to obtain $V^{G,h}$ is achieved by summing over the boundary condition twisted by elements of the centralising subgroup $C(h)$, and as a result we get the modified formula

$$Z^{G}(\tau,\dots) = \sum_{h\in G} Z^{G,h}(\tau,\dots) = \frac{1}{|G|}\sum_{h\in G}\sum_{g\in C(h)} Z(g\underset{h}{\square};\tau)\ .$$

<u>*Modular Transformation*</u>

One natural question to ask is how these twisted partition functions and twisted sector partition functions transform under the torus mapping class group. Recall that we argued for the invariance under $SL(2,\mathbb{Z})$ by evoking the invariance of the path integral under such a transformation of the torus, a natural way to answer this question is to again examine how the (now twisted) path integral transforms.

Using the so-called box notation as above to denote the boundary condition, we see that upon a redefinition of the A- and B-cycles of the elliptic curve

$$A \to c\,B + d\,A\,, B \to a\,B + b\,A\,, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})\,,$$

the path integral with different boundary conditions transform into each other as

$$Z(g \underset{h}{\square}; \tau) = \varepsilon(\gamma, g, h)\, Z\left(g^a h^b \underset{g^c h^d}{\square} ; \frac{a\tau + b}{c\tau + d} \right) \quad , \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})\,,$$

where $\varepsilon(\gamma, g, h)$ is a phase. As a result, typically these $Z(g \underset{h}{\square}; \tau)$ are only invariant (up to a phase) under a subgroup $\Gamma_{g,h}$ of the full modular group. On the other hand, the total partition function $Z^G(\tau, \dots)$ is invariant under $SL(2,\mathbb{Z})$.

See [1, 2] for a systematic discussion on orbifolds CFT. Note that, although for the purpose of illustration we use the geometric language in which the finite simple group G is a geometric symmetry, the orbifold construction discussed here can be straightforwardly applied as long as G is a symmetry of the Hilbert space V, a condition that is more general that the geometric statement.

An Example: $\mathbb{Z}_2$ Orbifold

First we consider a close cousin of the free boson example of the first lecture: the compactified boson. Namely, we quantise the loop space $LS^1 : \{\tilde{\varphi} : S^1 \to \mathbb{R}/\mathbb{Z} \simeq S^1\}$ of a circle of radius R. The oscillator part is identical as before and given by two copies of the Heisenberg algebras. The only difference now is that the spectrum of the so-called "zero-modes" are no longer continuous. Instead we have

$$\partial\varphi(z) = \frac{1}{2}\left(\frac{n}{R} + mR\right) + \sum_{k\neq 0} a_k z^{-1-k}\,.$$

The integer n comes from the requirement that the vertex operator $e^{ip\varphi}$ has to be invariant under $\varphi \to \varphi + 2\pi R$ and hence the eigenvalue of p is quantised to be $p = \frac{n}{R}$. In general, when M is the torus $M = \mathbb{R}^n/\Lambda$ with some n-dimensional lattice Λ, the momentum p has to be in the dual lattice $p \in \Lambda^*$. The second integer m comes from the fact that we can now consider the map $\breve{\varphi} : S^1 \to \mathbb{R}$ with $\breve{\varphi}(s+1) = \breve{\varphi}(s) + 2\pi Rm$.

With this modification in mind, repeat the calculation in 1.3 we obtain the partition function

$$Z(\tau) = \frac{1}{|\eta(\tau)|^2} \sum_{n,w\in\mathbb{Z}} q^{\frac{1}{2}(\frac{n}{R}+mR)^2} \bar{q}^{\frac{1}{2}(\frac{n}{R}-mR)^2} \ .$$

From this theory we would like to construct an orbifold theory by quantising $\tilde{\varphi} : S^1 \to S^1/\mathbb{Z}_2$ where the $\mathbb{Z}_2$ acts as $x \to -x$. According to the above prescription, we need to compute

$$Z^{\mathbb{Z}_2}(\tau) = \tfrac{1}{4}\big(Z(+\underset{+}{\square};\tau) + Z(-\underset{+}{\square};\tau) + Z(+\underset{-}{\square};\tau)Z(-\underset{-}{\square};\tau)\big) \ .$$

It is now not difficult to compute these quantities, first we have of course

$$Z(+\underset{+}{\square};\tau) = Z(\tau) \ . \tag{1.40}$$

It is also straightforward to compute the partition function twisted by the $\mathbb{Z}_2$ symmetry: First, the only zero-modes which survives the twisting is when $m = n = 0$. This collapses the theta function to 1. Second, we have $\prod_n \frac{1}{1+q^n}$ instead of $\prod_n \frac{1}{1-q^n}$ because the oscillators are all odd under the $\mathbb{Z}_2$-action. Putting it together we get

$$Z(-\underset{+}{\square};\tau) = \left| q^{-1/24} \prod_{n=1}^{\infty} \frac{1}{1+q^n} \right|^2 \ . \tag{1.41}$$

And then we have to quantise the twisted sector $\tilde{\varphi}(s+1) = -\tilde{\varphi}(s)$. The mode expansion now takes the form $\tilde{\varphi}(s) = \sum_{n\in\mathbb{Z}} \frac{\psi_n}{z^{1/2+n}}$. The ground state energy can be computed using zeta-function regularisation $1 + 2 + 3 + \cdots = -\frac{1}{12}$ as in the case of free bosons and yields 1/48. Moreover, quantisation of the zero-modes ψ_0 leads to representations of the (in this case 1-dimensional) Clifford algebra and gives an extra factor of two. Putting things together we obtain

$$Z(+\underset{-}{\square};\tau) = \left| 2q^{1/48} \prod_{n=1}^{\infty} \frac{1}{1-q^{n-1/2}} \right|^2 \tag{1.42}$$

$$Z(-\underset{+}{\square};\tau) = \left| 2q^{1/48} \prod_{n=1}^{\infty} \frac{1}{1+q^{n-1/2}} \right|^2 \ . \tag{1.43}$$

Combining all the results, we can check explicitly that $Z^{\mathbb{Z}_2}$ is invariant under the modular group.

2 Lecture 2: Moonshine and Physics

2.1 Monstrous Moonshine

[The content of this subsection is based on the introduction section of [50].]

The term *monstrous moonshine* was coined by Conway [3] in order to describe the unexpected and mysterious connections between the representation theory of the largest sporadic group—the *Fischer–Griess monster*, $\mathbb{M}$—and modular functions that stemmed from McKay's observation that $196883 + 1 = 196884$, where the summands on the left are degrees of irreducible representations of $\mathbb{M}$ and the number on the right is the coefficient of q in the Fourier expansion of the *elliptic modular invariant*

$$\begin{aligned} J(\tau) &= \sum_{m \geq -1} a(m) q^m \\ &= q^{-1} + 196884q + 21493760q^2 + 864299970q^3 + \cdots . \end{aligned} \tag{2.44}$$

Thompson expanded upon McKay's observation in [4] and conjectured the existence of an infinite-dimensional monster module

$$V = \bigoplus_{m \geq -1} V_m \ , \tag{2.45}$$

with $\dim V_m = a(m)$ for all m. He also proposed [5] to consider the series, now known as *McKay–Thompson series*, given by

$$T_g(\tau) = \sum_{m \geq -1} \operatorname{tr}_{V_m}(g)\, q^m \ , \tag{2.46}$$

for $g \in \mathbb{M}$, and detailed explorations [3] by Conway–Norton led to the astonishing *moonshine conjecture*:

> For each $g \in \mathbb{M}$ the function T_g is a principal modulus for some genus zero group Γ_g.

(A discrete group $\Gamma < SL_2(\mathbb{R})$ is said to have *genus zero* if the Riemann surface $\Gamma \backslash \mathbb{H}$ is isomorphic to the Riemann sphere minus finitely many points, and a holomorphic function f on $\mathbb{H}$ is called a *principal modulus* for a genus zero group Γ if it generates the field of Γ-invariant functions on $\mathbb{H}$.)

Thompson's conjecture was verified numerically by Atkin, Fong and Smith (cf. [6, 7]). A more constructive verification was obtained by Frenkel, Lepowsky and Meurman [8, 9] with the explicit construction of a monster module $V = V^\natural$ with graded dimension given by the Fourier expansion (2.44) of the elliptic modular invariant. They used *vertex operators*—structures originating in the dual resonance theory of particle physics and finding contemporaneous application [10, 11] to affine Lie algebras—to recover the non-associative *Griess algebra* structure (developed in the first proof [12] of the existence of the monster) from a subspace of $V^\natural$. Borcherds found a way to attach vertex operators to every element of $V^\natural$ and determined the precise sense in which these operators could be given a commutative associative composition law, and thus arrived at the notion of *vertex algebra* [13], an axiomatisation of the operator product expansion of chiral conformal field theory. The closely related notion of *vertex operator algebra (VOA)* was subsequently introduced by Frenkel–Lepowsky–Meurman [15] and they established that the monster is precisely the group of automorphisms of a VOA structure on $V^\natural$; the Frenkel–Lepowsky–Meurman construction of $V^\natural$ would ultimately prove to furnish the first example of an *orbifold conformal field theory.*

Borcherds introduced the notion of *generalised Kac–Moody algebra* in [16] and by using the VOA structure on $V^\natural$ was able to construct a particular example—the monster Lie algebra—and use the corresponding equivariant denominator identities to arrive at a proof [17] of the Conway–Norton moonshine conjectures. Thus by 1992 monstrous moonshine had already become a phenomenon encompassing elements of finite group theory, modular forms, vertex algebras and generalised Kac–Moody algebras, as well as aspects of conformal field theory and string theory.

2.2 M_{24} Moonshine

[The content of this subsection is a modification of parts of [26].]

As mentioned above, the Jacobi form property greatly constrains the possibilities for the elliptic genus of a Calabi–Yau manifold M. For example, when M is a $K3$ surface the Euler number $\mathbf{EG}(\tau, z = 0; K3) = 24$ forces the elliptic genus to be

$$\mathcal{Z}(\tau, z) = \mathbf{EG}(\tau, z; K3) = 8 \sum_{i=2,3,4} \left(\frac{\theta_i(\tau, z)}{\theta_i(\tau, 0)} \right)^2 , \qquad (2.47)$$

as computed in [18]. In particular this function is independent of the choice of $K3$ surface and so there is no ambiguity in writing $K3$ in place of M in (2.47).

It is to be expected that some further special properties should hold for the elliptic genus when M is a $K3$ surface because two (complex) dimensional Calabi–Yau manifolds are not only Kähler but also hyper-Kähler. As a result, the $U(1)_R$ symmetry can be extended to $SU(2)_R$ and the sigma model has enhanced $\mathcal{N}=4$ superconformal symmetry for both the left and the right movers. This leads to a specific decomposition of the elliptic genus of an $\mathcal{N}=4$ SCFT that we will now explain.

Since the underlying CFT admits an action by the $\mathcal{N}=4$ superconformal algebra the Hilbert space decomposes into a direct sum of (unitary) irreducible representations of this algebra. Hence the elliptic genus (1.35) can be written as a sum of characters of representations of this algebra with some multiplicities (cf. (2.51)). A natural embedding of the $U(1)$ current algebra of the $\mathcal{N}=2$ superconformal algebra into the $SU(2)$ current algebra of the $\mathcal{N}=4$ superconformal algebra is obatined by choosing $J_0^3 \sim J_0$. As a result, the $\mathcal{N}=4$ highest weight representations are again labeled by two quantum numbers h,ℓ, corresponding to the operators L_0, J_0^3 respectively, and the character of an irreducible representation $V_{h,\ell}$ say is defined as

$$ch_{h,\ell}(\tau,z) = \mathrm{tr}_{V_{h,\ell}}\left((-1)^{J_0} y^{J_0} q^{L_0-c/24}\right).$$

For the central charge $c=6$, there are two supersymmetric (also called 'BPS' or 'massless') representations in the Ramond sector and they have the quantum numbers

$$h=\tfrac{1}{4}\,,\ \ell = 0, \tfrac{1}{2}\,.$$

Their characters are given by [19, 20, 21]

$$\begin{aligned} ch_{\frac{1}{4},0}(\tau,z) &= \frac{\theta_1^2(\tau,z)}{\eta^3(\tau)}\mu(\tau,z)\,, \\ ch_{\frac{1}{4},\frac{1}{2}}(\tau,z) &= q^{-\frac{1}{8}}\frac{\theta_1(\tau,z)}{\eta^3(\tau)} - 2\frac{\theta_1(\tau,z)^2}{\eta^3(\tau)}\mu(\tau,z)\,, \end{aligned} \tag{2.48}$$

where $\mu(\tau,z)$ denotes the so-called Appell-Lerch sum

$$\mu(\tau,z) = \frac{-iy^{1/2}}{\theta_1(\tau,z)} \sum_{\ell=-\infty}^{\infty} \frac{(-1)^\ell y^\ell q^{\ell(\ell+1)/2}}{1-yq^\ell}\,.$$

Notice that the supersymmetric representations have non-vanishing Witten index

$$ch_{\frac{1}{4},0}(\tau, z=0) = 1\,, \quad ch_{\frac{1}{4},0}(\tau, z=0) = -2\,.$$

On the other hand the massive (or 'non-BPS' or 'non-supersymmetric') representations with

$$h = \tfrac{1}{4} + n\,, \quad \ell = \tfrac{1}{2}\,, \qquad n = 1, 2, \ldots$$

have the character given by

$$ch_{\frac{1}{4}+n,\frac{1}{2}}(\tau, z) = q^{-\frac{1}{8}+n} \frac{\theta_1^2(\tau, z)}{\eta^3(\tau)}\,,$$

which has, by definition, vanishing Witten index $ch_{\frac{1}{4}+n,\frac{1}{2}}(\tau, z=0) = 0$.

Rewriting the $K3$ elliptic genus in terms of these characters, we arrive at the following specific expression for the weak Jacobi form $\mathcal{Z}(\tau, z)$ [18, 24, 22]

$$\mathcal{Z}(\tau, z) = \frac{\theta_1^2(\tau, z)}{\eta^3(\tau)} \left(a\,\mu(\tau, z) + q^{-1/8}\big(b + \sum_{n=1}^{\infty} t_n q^n\big) \right), \tag{2.49}$$

where $a, b \in \mathbb{Z}$ and $t_n \in \mathbb{Z}$ for all positive integers n, and the t_n count the number of non-supersymmetric representations with $h = n + 1/4$ which contribute to the elliptic genus. As the notation suggests, from (2.47) we can compute the above integers to be $a = 24, b = -2$, and the first few t_n's are indeed as we have seen in the last section

$$2 \times 45,\ 2 \times 231,\ 2 \times 770,\ 2 \times 2277,\ 2 \times 5796, \ldots$$

Later, we will see that

$$H(\tau) = 2q^{-\frac{1}{8}} \left(-1 + 45q + 231q^2 + \ldots\right)$$

defined in terms of

$$\mathcal{Z}(\tau, z) = \frac{\theta_1^2(\tau, z)}{\eta^3(\tau)} \left(24\,\mu(\tau, z) + H(\tau)\right) \tag{2.50}$$

enjoys a modified modular property and is an example of the so-called mock modular forms. From the above we see that it can be interpreted as the generating function of multiplicities of massive irreducible representations of the $\mathcal{N}=4$ superconformal algebra in the $K3$ elliptic genus.

In (2.49) we have seen the appearance of the infinite-dimensional M_{24}-module K in the elliptic genus $\mathcal{Z}(\tau,z)$. One might wonder whether M_{24} acts on the other part of the decomposition as well. A simple observation is that $a=24$ is the dimension of the defining permutation representation R of M_{24}, and hence a naive guess will be that M_{24} acts on the massless $\mathcal{N}=4$ multiplets as the direct sum of R and a two dimensional (odd) trivial representation and this suggests that we write $24=\mathrm{tr}_R\,\mathbb{1}$. Together with this assumption, the conjecture (2.3) of the previous section implies that M_{24} acts on the states of the $K3$ sigma model contributing to the elliptic genus and moreover commutes with the superconformal algebra. Therefore, it is natural to consider also the twisted (or equivariant) elliptic genus which is expected to have the decomposition

$$\begin{aligned}\mathcal{Z}_g(\tau,z) &= \frac{\theta_1^2(\tau,z)}{\eta^3(\tau)}\big(\chi(g)\,\mu(\tau,z)+H_g(\tau)\big)\\ &= \sum_{n\geq 0,\ell\in\mathbb{Z}} c_g(4n-\ell^2)q^n y^\ell\ , \quad \text{with } \ \chi(g)=\mathrm{tr}_R\, g.\end{aligned} \tag{2.51}$$

Again, a non-trivial connection between weak Jacobi forms and M_{24} arises if all such $\mathcal{Z}_g(\tau,z)$ display interesting modular properties. Moreover, consistency with the CFT interpretation requires this to be true. More specifically it requires that $\mathcal{Z}_g(\tau,z)$ transform nicely under the action of $\Gamma_0(n_g)$. Indeed, from the (mock) modularity of $H_g(\tau)$ (cf. (2.2)) it is now easy to show

Proposition 2.1. *For all $g\in M_{24}$, the function*

$$\mathcal{Z}_g(\tau,z)=\frac{\chi(g)}{12}\,\varphi_{0,1}(\tau,z)+\tilde{T}_g(\tau)\,\varphi_{-2,1}(\tau,z) \tag{2.52}$$

is a weak Jacobi form of weight 0 and index 1 for the group $\Gamma_0(N_g)$. Moreover, we have

$$\mathcal{Z}_g(\tau,z)=\rho_{n_g|h_g}(\gamma)\,e\left(-\frac{cz^2}{c\tau+d}\right)\mathcal{Z}_g(\gamma(\tau,z))\,,$$

for $\gamma\in\Gamma_0(n_g)$.

In the above $\varphi_{0,1}(\tau,z)$ and $\varphi_{-2,1}(\tau,z)$ are weight 0 and weight 2, index 1 weak Jacobi forms whose explicit expressions can be found in (1.38) and (1.39). The weight two modular forms $\tilde{T}_g(\tau)$ are listed in Table 1 of [26].

2.3 Mock Modular Forms

In the previous section we have seen the relation between M_{24} and the following q-series

$$H(\tau) = 2q^{-\frac{1}{8}}\left(-1 + 45q + 231q^2 + \dots\right) = q^{-\frac{1}{8}}\left(-2 + \sum_{n=1}^{\infty} t_n q^n\right). \qquad (2.53)$$

Then the observation made in [25] is that the first few t_n's read

$$2 \times 45,\ 2 \times 231,\ 2 \times 770,\ 2 \times 2277,\ 2 \times 5796,\ \dots$$

and the integers 45, 231, 770, 2277 and 5796 are dimensions of irreducible representations of M_{24}.

This function $H(\tau)$, defined in (2.50), enjoys a special relationship with the group $SL_2(\mathbb{Z})$; namely, it is a *weakly holomorphic mock modular form of weight* $1/2$ on $SL_2(\mathbb{Z})$ with *shadow* $24\,\eta(\tau)^3$ [24, 31]. This means that if we define the *completion* $\hat{H}(\tau)$ of the holomorphic function $H(\tau)$ by setting

$$\hat{H}(\tau) = H(\tau) + 24\,(4i)^{-1/2} \int_{-\bar{\tau}}^{\infty} (z+\tau)^{-1/2}\, \overline{\eta(-\bar{z})^3}\, \mathrm{d}z\,, \qquad (2.54)$$

then $\hat{H}(\tau)$ transforms as a modular form of weight $1/2$ on $SL_2(\mathbb{Z})$ with multiplier system conjugate to that of $\eta(\tau)^3$. In other words, we have

$$\epsilon(\gamma)^{-3}\hat{H}\left(\frac{a\tau+b}{c\tau+d}\right)(c\tau+d)^{-1/2} = \hat{H}(\tau)\,, \quad \text{for} \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})\,,$$

where $\epsilon : SL_2(\mathbb{Z}) \to \mathbb{C}^*$ is the multiplier system for $\eta(\tau)$. See (1.20) for an explicit description of ϵ.

More generally, a mock modular form (MMF) embodies an interesting generalisation of the concept of modular forms: it transforms as a modular form only after the addition of a non-holomorphic function given by the *shadow* function of the MMF. In more details, a holomorphic function $h(\tau)$ on $\mathbb{H}$ is called a *(weakly holomorphic) mock modular form of weight* w for a discrete group Γ (e.g. a congruence subgroup of $SL_2(\mathbb{R})$) if it has at most exponential growth as $\tau \to \alpha$ for any $\alpha \in \mathbb{Q}$, and if there exists a holomorphic modular form $f(\tau)$ of weight $2-w$ on Γ such that $\hat{h}(\tau)$, given by

$$\hat{h}(\tau) = h(\tau) + (4i)^{w-1} \int_{-\bar{\tau}}^{\infty} (z+\tau)^{-w}\, \overline{f(-\bar{z})}\, \mathrm{d}z, \qquad (2.55)$$

is a (non-holomorphic) modular form of weight w for Γ for some multiplier system ψ say. In this case the function f is called the *shadow* of the mock modular form h and ψ is called the multiplier system of h. Evidently ψ is the conjugate of the multiplier system of f. The completion $\hat{h}(\tau)$ satisfies interesting differential equations. For instance, completions of mock modular forms were identified as Maass forms in [32]. As was observed in [31] we have the identity

$$2^{1-w}\pi\Im(\tau)^w\frac{\partial\hat{h}(\tau)}{\partial\bar{\tau}} = -2\pi i\overline{f(\tau)}$$

when f is the shadow of h.

From a physical point of view, as demonstrated in a series of recent works, the "mockness" of mock modular forms is often related to the non-compactness of relevant spaces in the theory. See, for instance, [88, 89, 90, 91, 79]. Let us take 2d CFTs with a non-compact target space as an example. Recall that in the previous lecture we carefully assume that the CFT we discuss has a discrete spectrum. The non-compactness of the target space often lead to a continuous part of the spectrum. In this case the arguments in the previous lecture might fail and we might obtain a non-holomorphic elliptic genus as a result. In this case it would be the completion of a mock modular object, while the mock modular object itself has the interpretation as the contribution to the elliptic genus from the discrete part of the spectrum. See the for instance [89, 83, 82, 85, 84, 86] for details for some specific examples.

Another source of mock modular forms in (mathematical) physics is the characters of supersymmetric infinite algebras, such as the $\mathcal{N}=2$ and $\mathcal{N}=4$ superconformal algebras mentioned in section 2.2. Some more examples can be found in for instance [87] and references therein.

M_{24} Moonshine Revisited

In this context of $K3$ elliptic genus, the origin of the mock modularity of $H(\tau)$ can be understood in the following way. From the fact that the elliptic genus (2.49) transforms nicely under $SL_2(\mathbb{Z})$, so must the combination $24\mu(\tau,z)+H(\tau)$. Now, the Appell-Lerch sum itself does not transform nicely, rather its non-holomorphic completion

$$\hat{\mu}(\tau;z) = \mu(\tau;z) - \frac{1}{2}\int_{-\bar{\tau}}^{\infty} \mathrm{d}z\,\frac{\eta^3(-z)}{\sqrt{i(z+\tau)}}$$

transforms like a weight 1/2 theta function. This has been demonstrated in [23] as part of a systematic treatment of mock θ-functions. Therefore, the mock modularity of the q-series $H(\tau)$ is directly related to the mock modularity of the massless (BPS) $\mathcal{N}=4$ characters (2.48).

Given the observation regarding the first few Fourier coefficients of $H(\tau)$ indicated above one would like to conjecture that the entire set of values t_n for $n \in \mathbb{Z}^+$ encode the graded dimension of a naturally defined $\mathbb{Z}$-graded M_{24} module $K = \bigoplus_{n=1}^{\infty} K_n$ with $\dim K_n = t_n$. Of course, this conjecture by itself is an empty statement, since all positive integers can be expressed as dimensions of representations of any group, since we may always consider trivial representations. However, the fact that the first few t_n can be written so nicely in terms of irreducible representations suggests that the K_n should generally be non-trivial, and given any particular guess for a M_{24}-module structure on the K_n we can test its merit by considering the twists of this mock modular form H obtained by replacing the identity element in $\dim K_n = \operatorname{tr}_{K_n} \mathbb{1}$ with an element g of the group M_{24}. We would then call the resulting q-series

$$q^{-\frac{1}{8}} \left(-2 + \sum_{n=1}^{\infty} \operatorname{tr}_{K_n} g q^n \right) \tag{2.56}$$

the *McKay–Thompson series* attached to g. A non-trivial connection between mock modular forms and M_{24} arises if all such McKay–Thompson series of the M_{24}-module K display interesting (mock) modular properties. In fact, since a function with good modular properties is generally determined by the first few of its Fourier coefficients, it is easier in practice to guess the McKay–Thompson series than it is to guess the representations K_n. Not long after the original observation was announced in [25] candidates for the McKay–Thompson series had been proposed for all conjugacy classes $[g] \subset M_{24}$ in [33, 34, 35, 36], and with functions $\tilde{T}_g(\tau)$ defined as in Table 1 of [26] the following result was established.

Proposition 2.2. *Let $H : \mathbb{H} \to \mathbb{C}$ be given by (2.50). Then for all $g \in M_{24}$, the function*

$$H_g(\tau) = \frac{\chi(g)}{24} H(\tau) - \frac{\tilde{T}_g(\tau)}{\eta(\tau)^3} \,, \tag{2.57}$$

is a (mock) modular form for $\Gamma_0(N_g)$ of weight 1/2 with shadow $\chi(g)\eta(\tau)^3$. Moreover, we have

$$\hat{H}_g(\tau) = \psi(\gamma) \operatorname{jac}(\gamma, \tau)^{1/4} \hat{H}_g(\gamma\tau),$$

for $\gamma \in \Gamma_0(n_g)$ where

$$\hat{H}_g(\tau) = H_g(\tau) + \chi(g)\,(4i)^{-1/2} \int_{-\bar{\tau}}^{\infty} (z+\tau)^{-1/2} \overline{\eta(-\bar{z})^3} \mathrm{d}z.$$

and the multiplier system is given by $\psi(\gamma) = \epsilon(\gamma)^{-3} \rho_{n_g|h_g}(\gamma)$.

Our discussion above leads to the following conjecture.

Conjecture 2.3. *The weight 1/2 (mock) modular forms H_g defined in (2.57) satisfy*

$$H_g(\tau) = q^{-\frac{1}{8}}\big(-2 + \sum_{n=1}^{\infty} q^n\,(\mathrm{tr}_{K_n}\, g)\big) \tag{2.58}$$

for a certain $\mathbb{Z}$-graded, infinite-dimensional M_{24} module $K = \bigoplus_{n=1}^{\infty} K_n$.

Moreover, the representations K_n are even in the sense that they can all be written in the form $K_n = k_n \oplus k_n^*$ for some M_{24}-modules k_n where k_n^* denotes the module dual to k_n.

The first few Fourier coefficients of the q-series $H_g(\tau)$ and the corresponding M_{24}-representations are given in [26].

A proof of the first part of the above conjecture, namely the existence of an M_{24}-module $K = \bigoplus_{n=1}^{\infty} K_n$ such that (2.58) holds, has been attained in [37].

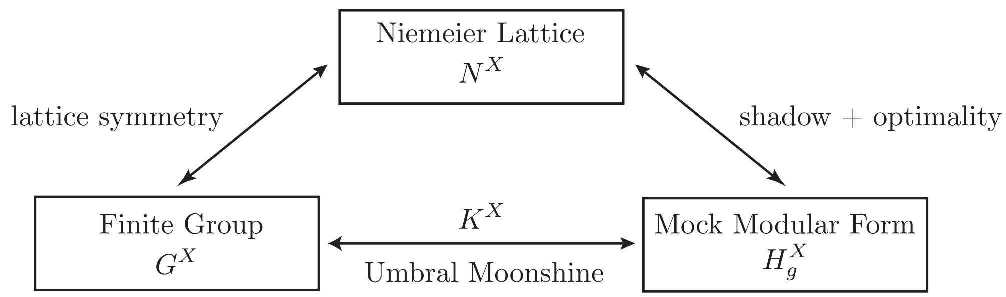
The conjecture 2.3 on the relationship between $H_g(\tau)$ and M_{24} now implies that the Fourier coefficients $c_g(4n-\ell^2)$ of the twisted $K3$ elliptic genus encode the supercharacter of a $\mathbb{Z}$-graded super (or virtual) representation $\hat{K} = \bigoplus \hat{K}_{4n-\ell^2}$ for M_{24}.

2.4 Umbral Moonshine

Later it was realised that the connection between certain mock modular forms and finite groups is just tip of the iceberg. Or less metaphorically just one case out of a series of such relations, called "umbral moonshine" [50, 41]. There are in total 23 instances of umbral moonshine, which admit a uniform construction.

The starting point of this uniform construction are the 23 special lattices N^X (even, self-dual lattices in 24 dimensions with non-trivial root systems X) classified by Niemeier. The 23 *Niemeier lattices* are uniquely labelled by the root systems X, which are precisely one of the 23 unions of simply-laced

Figure 1: The construction of umbral moonshine.



(ADE) root systems $X = \cup_i Y_i$ satisfying the conditions 1) All components have the same Coxeter number $\mathrm{Cox}(Y_i) = \mathrm{Cox}(Y_j)$; 2) the total rank equals the rank of the lattice $\sum_i \mathrm{rk}(Y_i) = 24$. Some examples out of the 23 include $24A_1$, $2A_{12}$, $8E_8$ and $D_{16}E_8$.

One interesting feature of this UM construction is the fact that it includes the aforementioned M_{24} observation as a special case: for the simplest Niemeier lattice corresponding to A_1 in the ADE classification, and in particular $X = 24A_1$, the corresponding finite group G^X is the Mathieu group M_{24} and the mock modular form H^X is nothing but the q-series in (2.50). As a result, the discovery of UM not only greatly extends the case observed in 2010 but also provides the general framework in which this paradigm should be studied. In particular, it offers a crucial hint regarding the origin of the finite group symmetry.

In more details, for each X the symmetry of the lattice readily leads to a finite group G^X via $G^X := \mathrm{Aut}(N^X)/\mathrm{Weyl}(X)$. On the other hand, we use the root system X to specify certain mock modular forms displaying relation to the finite group G^X. We will now briefly explain how this is done. From a mathematical point of view, the functions T_g for $g \in \mathbb{M}$ and H_g for $g \in M_{24}$ have very special properties [38, 39]. Basically, the properties guarantees that the modular property together with the pole structure of the (mock) modular form are sufficient to determine the whole q-series. To construct the functions relevant for umbral moonshine we assume that the analogous mathematical property which we shall refer to as the "optimality" holds, but this time the functions will be vector-valued mock modular forms of weight 1/2. The umbral moonshine construction then uses the root system X of the Niemeier lattice to specify the mock modular property of the MMF in a way that is very reminiscent of the ADE classification of modular invariants discussed in [42]. The shadow in particular determines a unique "optimal" mock module form H^X, where the optimality condition is defined in terms of a certain analyticity structure.

For instance, for the case $X = 12A_2$ the vector-valued mock modular form has two independent components, H_1^X and H_2^X. Their q-expansion reads

$$H_1^X(\tau) = 2q^{-1/12}(-1 + 16\,q + 55\,q^2 + 144\,q^3 + \dots) \tag{2.59}$$

$$H_2^X(\tau) = 2q^{8/12}(10 + 44\,q + 110\,q^2 + \dots). \tag{2.60}$$

At the same time, the symmetries of the corresponding Niemeier lattice gives $G^X \cong 2.M_{12}$. The relation between the finite group G^X and the vector-valued mock modular form H^X can be seen by the simple fact that the group $2.M_{12}$ has irreducible representations of dimensions $16, 55, 144$ as well as $10, 44, 110$, analogous to the case of monstrous and M_{24} moonshine that we discussed before.

This construction can be readily extended to construct a MMF H_g^X for each conjugacy class $[g]$ of G^X. After constructing the MMFs H_g^X as well as the finite group G^X, the *umbral moonshine conjecture* then states that the coefficients of H_g^X encode the dimensions (and g-characters more generally) of a natural defined graded G^X-module K^X, called the *umbral module*. In more explicit terms, the conjecture states that for each power of q there is a corresponding representation of the group G^X such that its g-character coincides with the corresponding Fourier coefficient in the q-expansion of the MMF H_g^X. The existence of this umbral module has been proven mathematically in the meanwhile [37, 43], although what it really is still mysterious at the moment.

2.5 Moonshine and String Theory

Monstrous Moonshine

The mystery of monstrous moonshine, as we mentioned above, is largely understood in terms of the chiral CFT $V^\natural$ constructed in [8, 9]. The generalised Kac-Moody algebra Borcherds attached to $V^\natural$ can be thought of as arising from a full string theory and not just the chiral CFT. The most natural setup to explicitly realise such a construction is believed to be the heterotic string theory, with its non-supersymmetric side with $c = 26$ compactified on $V^\natural$ and a two torus. The details is currently being worked out in [44].

M_{24} and Umbral Moonshine

Five years after the observation relating the elliptic genus of $K3$ and the sporadic group M_{24} [25], the mystery of M_{24} moonshine remains. In the meantime, great progress has been made in the understanding of both the nature of this type of moonshine and the symmetries of $K3$ sigma models and $K3$ string theory in general. See [49, 53, 51, 73, 57, 74, 56, 58, 55, 75, 54, 62, 64, 65, 66, 60, 63, 68, 50, 41, 61, 52, 72, 78, 59].

In the former category, it was realized that M_{24} moonshine is but one out of 23 cases of the umbral moonshine as we described above. In the latter category, we have learned a lot about the symmetries of $K3$ sigma models in the past years. First, a CFT analogue of Mukai's classification theorem of hyper-Kähler-preserving (or symplectic) automorphisms of $K3$ surfaces [27] has been established for $K3$ sigma models. Extending the lattice arguments in [28], it was shown in [30] that all symmetries of non-singular $K3$ CFTs preserving $\mathcal{N} = (4, 4)$ superconformal symmetry are necessarily subgroups of the Conway group (Co_0, often known as the automorphism group of the Leech lattice) that moreover preserve at least a four-dimensional subspace in the irreducible 24-dimensional representation of the group. (We will call such subgroups the "4-plane preserving subgroups.") This classification was later rephrased in terms of automorphisms of derived categories on $K3$ in [46], and was moreover proposed to govern the symmetries of the appropriately defined moduli space relevant for $K3$ curve counting [47]. The relation between umbral moonshine and $K3$ CFTs is proposed in [79] and further explored in [80]. The relation between Conway moonshine (see also 2.6) and $K3$ CFTs is proposed in [78].

One important upshot from the above is that $\mathcal{N} = 4$-supersymmetry-preserving symmetries of $K3$ CFT is closely related to the symmetries of M_{24} and umbral moonshine but it *cannot* be the whole story. One way to see this is that the umbral groups G^X is generically larger than any of its 4-plane preserving subgroups: it often contains group elements that does not preserve a 4-plane in the natural 24-dimensional representation. Hence, the reason why M_{24} and other umbral groups G^X appear to be related through umbral moonshine to the elliptic genus of $K3$ is still puzzling, and there is at the moment no consensus regarding what the relevant physical context of umbral moonshine is. A few possibilities that have been suggested include

- Non-perturbative states in type II theory compactified on $K3 \times T^2$: The idea is roughly to extend the symmetry consideration from the realm of CFT to the full BPS states arising from string theory compactified on $K3 \times T^2$. See [33] and [75]. This development has led to nice new insights into string dualities [71].

- Combining symmetries realised at different points in the moduli space of $K3$ CFT:
  The idea is that although the full CFT never has large enough symmetries, an object carrying only the information of BPS states of the theory might carry symmetries larger than that realised in specific points in the moduli space and indeed admit actions of the relevant umbral groups. See [29, 49, 55, 80] for some results of explorations in this direction.

- Heterotic string theory in the background of $K3$ surfaces:
  Its plausibility lies in the fact that its moduli space contains the moduli space of $\mathcal{N} = (4,4)$ $K3$ CFT as a sub locus. This route has been somewhat explored in [57, 81] but a lot remains to be done.

- Five-brane dynamics:
  This idea is natural in the following (and possibly more) ways. First it also extends the consideration and takes certain non-perturbative elements of string theory into account. Second the NS five-branes natural admit ADE classification and the same ADE plays an important role in the construction of umbral moonshine through the root systems X. See [58, 64] for some results in this direction.

- ???
  As none of the above approach has led to a definite answer so far, a logically possible idea is that the connections we observed between $K3$ elliptic genera and umbral moonshine is just a coincidence and the physical context (if any) of umbral moonshine lies completely somewhere else.

As the reader can see, answering the above question is an active research area at the moment and is arguably the holy grail in the study of umbral moonshine at the moment.

2.6 Other Moonshine

Apart from monstrous and umbral moonshine there are a few more known cases of similar connections between modular objects and finite groups. In particular, recently a new Thompson moonshine has been proposed [45]. It would be very interesting to understand its significance in physics. Closer in spirit to monstrous moonshine is the Conway moonshine, developed in [15, 76, 77, 78]. An excellent review on (pre-umbral) moonshine including lots of background material is [14]. Other (post-umbral) moonshine reviews include [26, 65, 67].

References

[1] L. J. Dixon, J. A. Harvey, C. Vafa and E. Witten, "Strings on Orbifolds," Nucl. Phys. B **261** (1985) 678. doi:10.1016/0550-3213(85)90593-0

[2] L. J. Dixon, J. A. Harvey, C. Vafa and E. Witten, "Strings on Orbifolds. 2.," Nucl. Phys. B **274** (1986) 285. doi:10.1016/0550-3213(86)90287-7

[3] J. H. Conway and S. P. Norton, "Monstrous Moonshine," *Bull. London Math. Soc.* **11** (1979) 308 339.

[4] J. G. Thompson, "Some numerology between the Fischer-Griess Monster and the elliptic modular function," *Bull. London Math. Soc.* **11** (1979) no. 3, 352–353.

[5] J. G. Thompson, "Finite groups and modular functions," *Bull. London Math. Soc.* **11** (1979) no. 3, 347–351.

[6] P. Fong, "Characters arising in the Monster-modular connection," in *The Santa Cruz Conference on Finite Groups (Univ. California, Santa Cruz, Calif., 1979)*, vol. 37 of *Proc. Sympos. Pure Math.*, pp. 557–559. Amer. Math. Soc., Providence, R.I., 1980.

[7] S. D. Smith, "On the head characters of the Monster simple group," in *Finite groups—coming of age (Montreal, Que., 1982)*, vol. 45 of *Contemp. Math.*, pp. 303–313. Amer. Math. Soc., Providence, RI, 1985. http://dx.doi.org/10.1090/conm/045/822245.

[8] I. B. Frenkel, J. Lepowsky, and A. Meurman, "A natural representation of the Fischer-Griess Monster with the modular function J as character," *Proc. Nat. Acad. Sci. U.S.A.* **81** (1984) no. 10, Phys. Sci., 3256–3260.

[9] I. B. Frenkel, J. Lepowsky, and A. Meurman, "A moonshine module for the Monster," in *Vertex operators in mathematics and physics (Berkeley, Calif., 1983)*, vol. 3 of *Math. Sci. Res. Inst. Publ.*, pp. 231–273. Springer, New York, 1985.

[10] I. B. Frenkel and V. G. Kac, "Basic representations of affine Lie algebras and dual resonance models," *Invent. Math.* **62** (1980/81) no. 1, 23–66.

[11] G. Segal, "Unitary representations of some infinite-dimensional groups," *Comm. Math. Phys.* **80** (1981) no. 3, 301–342. `http://projecteuclid.org/getRecord?id=euclid.cmp/1103919978`.

[12] R. L. Griess, Jr., "The friendly giant," *Invent. Math.* **69** (1982) no. 1, 1–102.

[13] R. Borcherds, "Vertex algebras, Kac-Moody algebras, and the Monster," *Proceedings of the National Academy of Sciences, U.S.A.* **83** (1986) no. 10, 3068–3071.

[14] T. Gannon, *Moonshine Beyond Monster*, Cambridge University Press, Cambridge, UK, 2006.

[15] I. Frenkel, J. Lepowsky, and A. Meurman, *Vertex operator algebras and the Monster*, vol. 134 of *Pure and Applied Mathematics*. Academic Press Inc., Boston, MA, 1988.

[16] R. Borcherds, "Generalized Kac-Moody algebras," *J. Algebra* **115** (1988) no. 2, 501–512. `http://dx.doi.org/10.1016/0021-8693(88)90275-X`.

[17] R. E. Borcherds, "Monstrous moonshine and monstrous Lie superalgebras," *Invent. Math.* **109, No.2** (1992) 405–444.

[18] T. Eguchi, H. Ooguri, A. Taormina, and S.-K. Yang, "Superconformal Algebras and String Compactification on Manifolds with SU(N) Holonomy," *Nucl. Phys.* **B315** (1989) 193.

[19] T. Eguchi and A. Taormina, "Unitary Representations of $N = 4$ Superconformal Algebra," *Phys. Lett.* **B196** (1987) 75.

[20] T. Eguchi and A. Taormina, “Character Formulas for the $N = 4$ Superconformal Algebra,” *Phys. Lett.* **B200** (1988) 315.

[21] T. Eguchi and A. Taormina, “On the Unitary Representations of $N = 2$ and $N = 4$ Superconformal Algebras,” *Phys. Lett.* **B210** (1988) 125.

[22] T. Eguchi and K. Hikami, “Superconformal Algebras and Mock Theta Functions 2. Rademacher Expansion for $K3$ Surface,” *Communications in Number Theory and Physics* **3,** (Apr., 2009) 531–554, `0904.0911`.

[23] S. Zwegers, *Mock Theta Functions*. PhD thesis, Utrecht University, 2002.

[24] T. Eguchi and K. Hikami, “Superconformal algebras and mock theta functions,” *J.Phys.A* **42:304010,2009** (Dec., 2008) , `0812.1151`.

[25] T. Eguchi, H. Ooguri, and Y. Tachikawa, “Notes on the $K3$ Surface and the Mathieu group M_{24},” `1004.0956`.

[26] M. C. N. Cheng and J. F. R. Duncan, “The Largest Mathieu Group and (Mock) Automorphic Forms,” arXiv:1201.4140 [math.RT].

[27] S. Mukai, “Finite groups of automorphisms of $K3$ surfaces and the Mathieu group,” *Invent. Math.* **94, No.1** (1988) 183–221.

[28] S. Kondo, Shigeyuki (Mukai, “Niemeier lattices, Mathieu groups, and finite groups of symplectic automorphisms of $K3$ surfaces. With an appendix by Shigeru Mukai.,” *Duke Math. J* **92, No.3** (1998) 593–603.

[29] A. Taormina and K. Wendland, “The symmetries of the tetrahedral Kummer surface in the Mathieu group M_{24},” `1008.0954`.

[30] M. R. Gaberdiel, S. Hohenegger, and R. Volpato, “Symmetries of $K3$ sigma models,” `1106.4315`.

[31] A. Dabholkar, S. Murthy and D. Zagier, “Quantum Black Holes, Wall Crossing, and Mock Modular Forms,” arXiv:1208.4074 [hep-th].

[32] K. Bringmann and K. Ono, “The $f(q)$ mock theta function conjecture and partition ranks,” *Invent. Math.* **165** (2006) no. 2, 243–266. `http://dx.doi.org/10.1007/s00222-005-0493-5`.

[33] M. C. N. Cheng, "$K3$ Surfaces, $N = 4$ Dyons, and the Mathieu Group M_{24}," 1005.5415.

[34] M. R. Gaberdiel, S. Hohenegger, and R. Volpato, "Mathieu twining characters for $K3$," 1006.0221.

[35] M. R. Gaberdiel, S. Hohenegger, and R. Volpato, "Mathieu Moonshine in the elliptic genus of $K3$," 1008.3778.

[36] T. Eguchi and K. Hikami, "Note on Twisted Elliptic Genus of $K3$ Surface," 1008.4924.

[37] T. Gannon, "Much ado about Mathieu," arXiv:1211.5531 [math.RT].

[38] J. F. R. Duncan and I. B. Frenkel, "Rademacher sums, moonshine and gravity," *Commun. Number Theory Phys.* **5** (2011) no. 4, 1–128.

[39] M. C. N. Cheng and J. F. R. Duncan, "On rademacher sums, the largest mathieu group, and the holographic modularity of moonshine," 1110.3859.

[40] M. C. N. Cheng, J. F. R. Duncan, and J. A. Harvey, "Umbral Moonshine," arXiv:1204.2779 [math.RT].

[41] M. C. N. Cheng, J. F. R. Duncan, and J. A. Harvey, "Umbral Moonshine and the Niemeier Lattices," arXiv:1307.5793 [math.RT].

[42] Andrea Cappelli, C. Itzykson, and J.B. Zuber. The ADE Classification of Minimal and A1(1) Conformal Invariant Theories. *Commun.Math.Phys.*, 113:1, 1987.

[43] J. F. Duncan, M. J. Griffin, and K. Ono, "Proof of the Umbral Moonshine Conjecture," arXiv:1503.01472 [math.RT].

[44] N. M. Paquette, D. Persson and R. Volpato,"Work in Progress,"

[45] J. A. Harvey and B. C. Rayhaun, "Traces of Singular Moduli and Moonshine for the Thompson Group," arXiv:1504.08179 [math.RT].

[46] D. Huybrechts, "On derived categories of $K3$ surfaces and Mathieu groups", arXiv:1309.6528 [math.AG].

[47] M. C. N. Cheng, J. F. R. Duncan, S. M. Harrison and S. Kachru, “Equivariant K3 Invariants”, arXiv:1508.02047 [hep-th].

[48] J. F. R. Duncan, M. J. Griffin and K. Ono, “Moonshine,” Research in the Mathematical Sciences (2015) 2:11 [arXiv:1411.6571 [math.RT]].

[49] A. Taormina and K. Wendland, “The overarching finite symmetry group of Kummer surfaces in the Mathieu group M_{24}”, JHEP **1308**, 125 (2013), [arXiv:1107.3834 [hep-th]].

[50] M. C. Cheng, J. F. Duncan, and J. A. Harvey, “Umbral Moonshine”, Commun. Num. Theor Phys. **08** (2014) 101–242, [arXiv:1204.2779 [math.RT]].

[51] M. R. Gaberdiel and R. Volpato, “Mathieu Moonshine and Orbifold K3s”, Contrib. Math. Comput. Sci. **8**, 109 (2014), [arXiv:1206.5143 [hep-th]].

[52] T. Creutzig, G. Hoehn, and T. Miezaki, “The McKay-Thompson series of Mathieu Moonshine modulo two”, arXiv:1211.3703 [math.NT].

[53] M. R. Gaberdiel, D. Persson, H. Ronellenfitsch and R. Volpato, “Generalized Mathieu Moonshine”, Commun. Num. Theor Phys. **07**, 145 (2013), [arXiv:1211.7074 [hep-th]].

[54] M. R. Gaberdiel, D. Persson and R. Volpato, “Generalised Moonshine and Holomorphic Orbifolds”, arXiv:1302.5425 [hep-th].

[55] A. Taormina and K. Wendland, “Symmetry-surfing the moduli space of Kummer K3s”, arXiv:1303.2931 [hep-th].

[56] A. Taormina and K. Wendland, “A twist in the M24 moonshine story”, arXiv:1303.3221 [hep-th].

[57] M. C. N. Cheng, X. Dong, J. Duncan, J. Harvey, S. Kachru and T. Wrase, “Mathieu Moonshine and N=2 String Compactifications”, JHEP **1309**, 030 (2013), [arXiv:1306.4981 [hep-th]].

[58] J. A. Harvey and S. Murthy, “Moonshine in Fivebrane Spacetimes”, JHEP **1401**, 146 (2014), [arXiv:1307.7717 [hep-th]].

[59] S. Harrison, S. Kachru and N. M. Paquette, "Twining Genera of (0,4) Supersymmetric Sigma Models on K3," JHEP **1404**, 048 (2014) [arXiv:1309.0510 [hep-th]].

[60] T. Wrase, "Mathieu moonshine in four dimensional $\mathcal{N} = 1$ theories", JHEP **1404**, 069 (2014), [arXiv:1402.2973 [hep-th]].

[61] K. Ono, L. Rolen, and S. Trebat-Leder, "Classical and umbral moonshine: connections and p-adic properties", J. Ramanujan Math. Soc. **30** (2015) no. 2, 135–159, [arXiv:1403.3712 [math.NT]].

[62] M. C. N. Cheng, X. Dong, J. F. R. Duncan, S. Harrison, S. Kachru and T. Wrase, "Mock Modular Mathieu Moonshine Modules", Research in the Mathematical Sciences (2015) 2:13, [arXiv:1406.5502 [hep-th]].

[63] N. M. Paquette and T. Wrase, "Comments on M_{24} representations and CY_3 geometries", JHEP **1411**, 155 (2014), [arXiv:1409.1540 [hep-th]].

[64] J. A. Harvey, S. Murthy and C. Nazaroglu, "ADE Double Scaled Little String Theories, Mock Modular Forms and Umbral Moonshine", JHEP **1505**, 126 (2015), [arXiv:1410.6174 [hep-th]].

[65] J. F. R. Duncan, M. J. Griffin and K. Ono, "Moonshine", arXiv:1411.6571 [math.RT].

[66] N. Benjamin, S. M. Harrison, S. Kachru, N. M. Paquette and D. Whalen, "On the elliptic genera of manifolds of Spin(7) holonomy", arXiv:1412.2804 [hep-th].

[67] R. Volpato, "Mathieu Moonshine and symmetries of K3 sigma models," Fortsch. Phys. **60** (2012) 1112 doi:10.1002/prop.201200017 [arXiv:1201.6172 [hep-th]].

[68] M. C. N. Cheng, S. M. Harrison, S. Kachru and D. Whalen, "Exceptional Algebra and Sporadic Groups at c=12", arXiv:1503.07219 [hep-th].

[69] J. F. R. Duncan and S. Mack-Crane, "Derived Equivalences of $K3$ Surfaces and Twined Elliptic Genera", arXiv:1506.06198 [math.RT].

[70] D. Persson and R. Volpato, "Second Quantized Mathieu Moonshine," Commun. Num. Theor. Phys. **08** (2014) 403 doi:10.4310/CNTP.2014.v8.n3.a2 [arXiv:1312.0622 [hep-th]].

[71] D. Persson and R. Volpato, "Fricke S-duality in CHL models," JHEP **1512** (2015) 156 doi:10.1007/JHEP12(2015)156 [arXiv:1504.07260 [hep-th]].

[72] T. Creutzig and G. Höhn, "Mathieu moonshine and the geometry of K3 surfaces", Commun. Num. Theor Phys. **8** (2014) no. 2, 295–328, [arXiv:1309.2671 [math.QA]].

[73] M. R. Gaberdiel, S. Hohenegger and R. Volpato, "Mathieu Moonshine in the elliptic genus of K3", JHEP **1010**, 062 (2010), [arXiv:1008.3778 [hep-th]].

[74] M. R. Gaberdiel, A. Taormina, R. Volpato and K. Wendland, "A K3 sigma model with $\mathbb{Z}_2^8 : \mathbb{M}_{20}$ symmetry," JHEP **1402**, 022 (2014), [arXiv:1309.4127 [hep-th]].

[75] D. Persson and R. Volpato, "Second Quantized Mathieu Moonshine", Commun. Num. Theor. Phys. **08**, 403 (2014), [arXiv:1312.0622 [hep-th]].

[76] J. F. Duncan, "Super-moonshine for Conway's largest sporadic group", Duke Math. J., 139(2):255-315, (2007), [arXiv:math/0502267].

[77] J. F. R. Duncan and S. Mack-Crane, "The Moonshine Module for Conway's Group", SIGMA **3**, e10 (2015), [arXiv:1409.3829 [math.RT]].

[78] J. F. R. Duncan and S. Mack-Crane, "Derived Equivalences of $K3$ Surfaces and Twined Elliptic Genera", arXiv:1506.06198 [math.RT].

[79] M. C. N. Cheng and S. Harrison, "Umbral Moonshine and $K3$ Surfaces", Commun. Math. Phys. **339**, no. 1, 221 (2015), [arXiv:1406.0619 [hep-th]].

[80] M. C. N. Cheng, F. Ferrari, S. M. Harrison and N. M. Paquette, "Landau-Ginzburg Orbifolds and Symmetries of K3 CFTs," arXiv:1512.04942 [hep-th].

[81] S. Harrison, S. Kachru, and N. M. Paquette, "Twining Genera of (0,4) Supersymmetric Sigma Models on K3," *JHEP* **04** (2014) 048, `arXiv:1309.0510 [hep-th]`.

[82] T. Eguchi and Y. Sugawara, "SL(2,R)/U(1) supercoset and elliptic genera of noncompact Calabi-Yau manifolds," *JHEP* **0405** (2004) 014, `arXiv:hep-th/0403193 [hep-th]`.

[83] T. Eguchi and Y. Sugawara, "Non-holomorphic Modular Forms and SL(2,R)/U(1) Superconformal Field Theory," *JHEP* **1103** (2011) 107, arXiv:1012.5721 [hep-th].

[84] S. K. Ashok and J. Troost, "A Twisted Non-compact Elliptic Genus," *JHEP* **1103** (2011) 067, arXiv:1101.1059 [hep-th].

[85] S. K. Ashok, N. Doroud, and J. Troost, "Localization and real Jacobi forms," arXiv:1311.1110 [hep-th].

[86] S. Murthy, "A holomorphic anomaly in the elliptic genus," arXiv:1311.0918 [hep-th].

[87] V. G. Kac and M. Wakimoto, "Representations of affine superalgebras and mock theta functions II," arXiv:1402.0727 [math.RT].

[88] C. Vafa and E. Witten, "A strong coupling test of S-duality," *Nuclear Phys. B* **431** (1994) no. 1-2, 3–77. http://dx.doi.org/10.1016/0550-3213(94)90097-3.

[89] J. Troost, "The non-compact elliptic genus: mock or modular," *JHEP* **1006** (2010) 104, arXiv:1004.3649 [hep-th].

[90] A. Dabholkar, S. Murthy, and D. Zagier, "Quantum Black Holes, Wall Crossing, and Mock Modular Forms," arXiv:1208.4074 [hep-th].

[91] S. Alexandrov, J. Manschot, and B. Pioline, "D3-instantons, Mock Theta Series and Twistors," *JHEP* **1304** (2013) 002, arXiv:1207.1109 [hep-th].