Scattering Amplitudes

LECTURE 4

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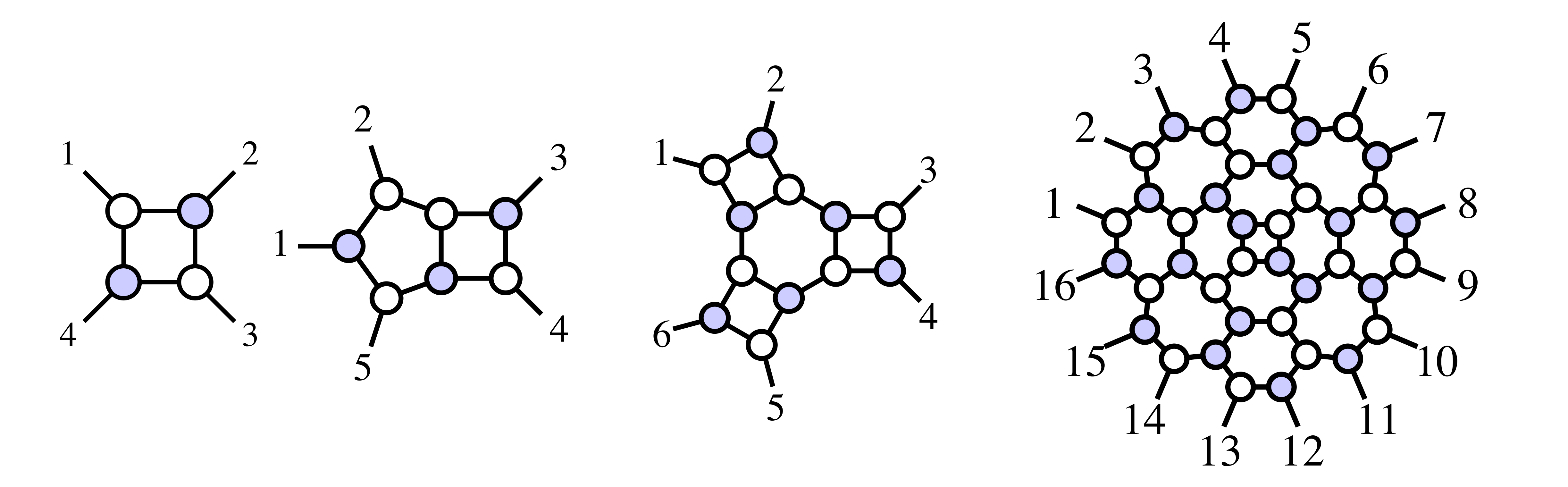
Review of Lecture 3

On-shell diagrams

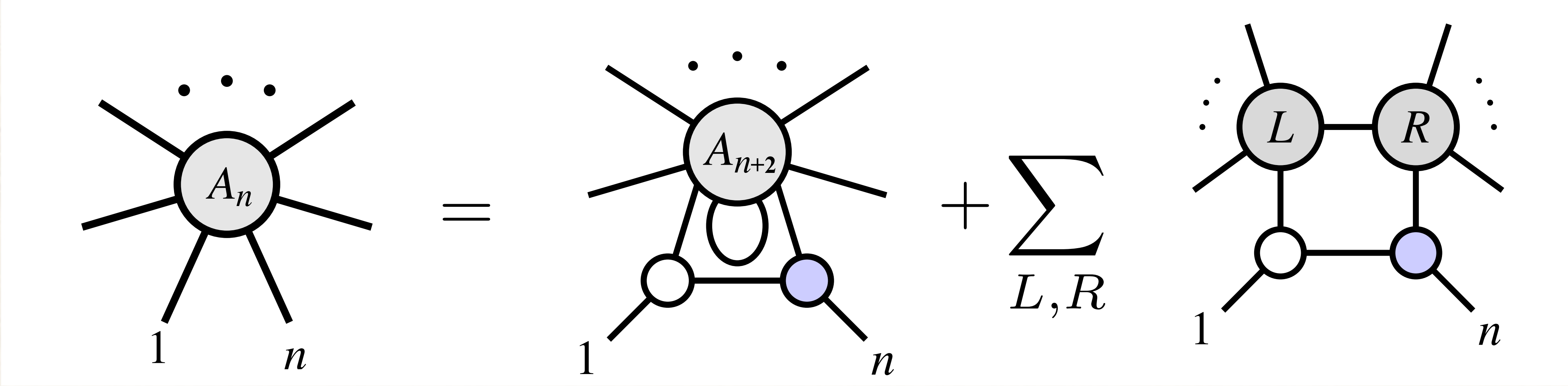
- Draw arbitrary graph with three point vertices



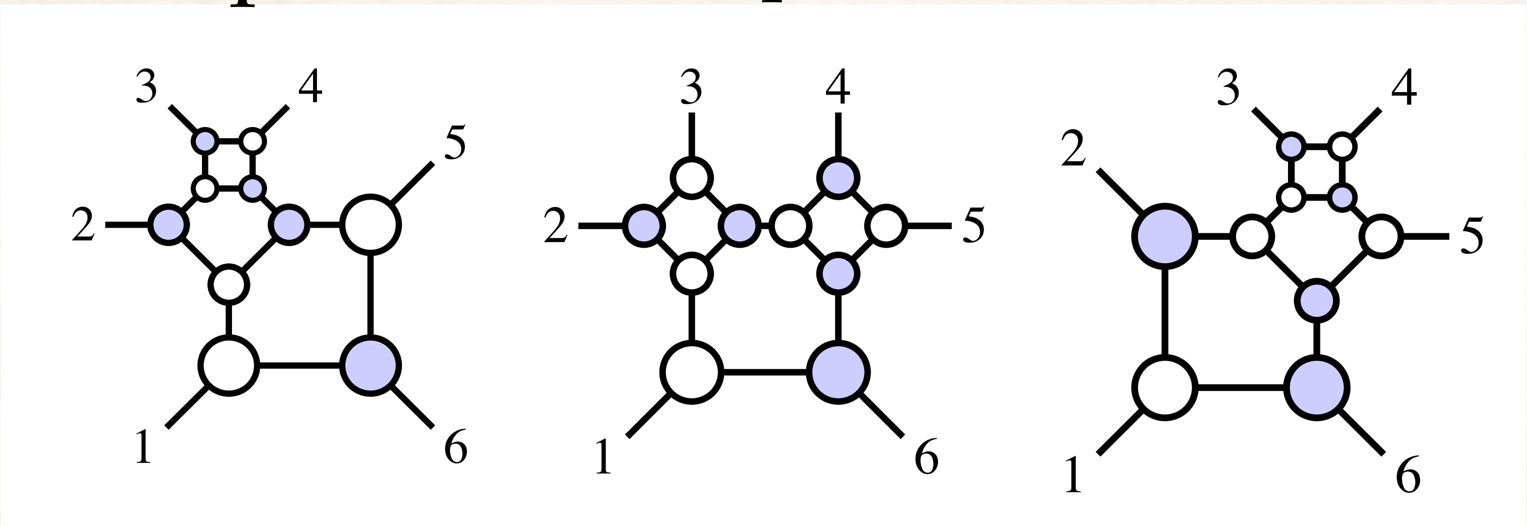
Product of three point on-shell amplitudes

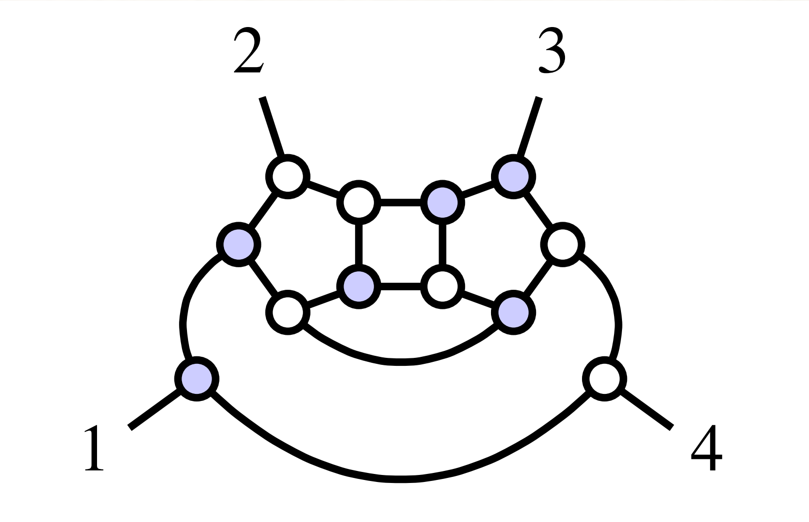
Recursion relations

- Recursion relations for ℓ-loop integrand



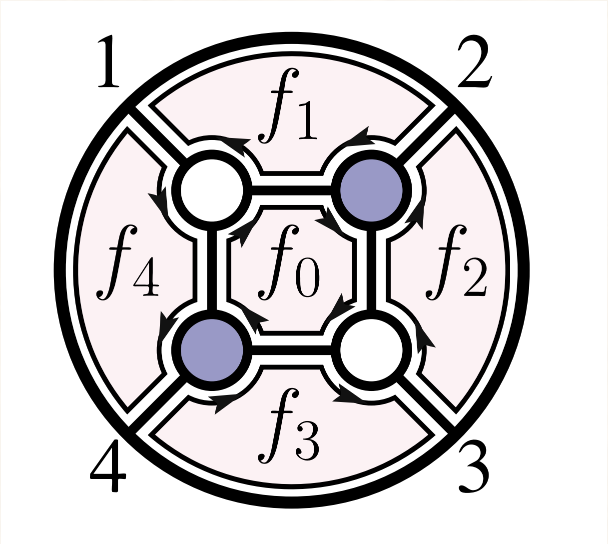
- Examples: 6pt tree 4pt 1-loop





Positive Grassmannian

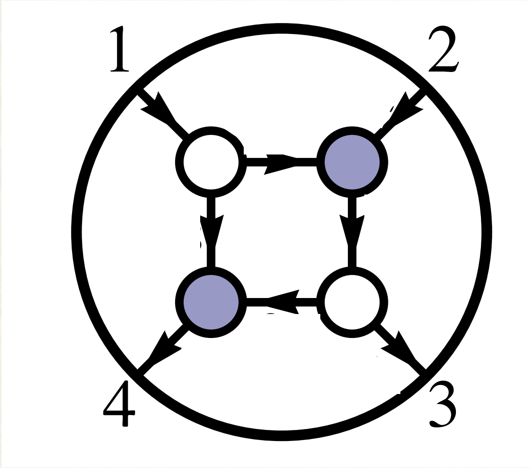
- Face variables



with the property

$$\prod_j f_j = -1$$

- Perfect orientation



Elements of $(k \times n)$ matrix

$$c_{ab} = -\sum_{\Gamma} \prod_{j} (-f_j)$$

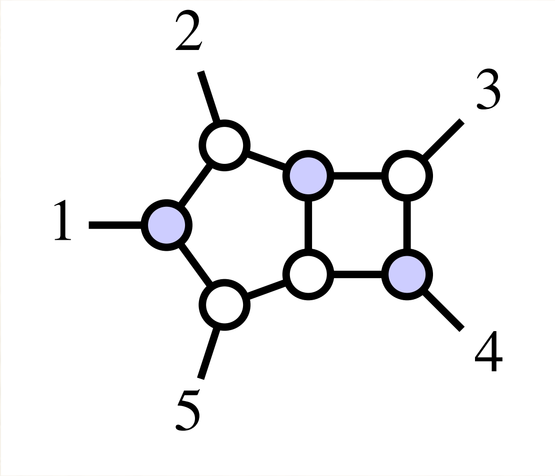
$$k \overset{n}{\begin{pmatrix} * & * & * & \dots & * \\ * & * & * & \dots & * \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ * & * & * & \dots & * \end{pmatrix}}$$

Cell in positive Grassmannian $G_+(k,n)$

$$k \overset{k}{\begin{vmatrix} * & * & \dots & * \\ * & * & \dots & * \\ \vdots & \vdots & \vdots & \vdots \\ * & * & \dots & * \end{vmatrix}} \geq 0$$

Logarithmic form

- For any on-shell diagram



only dependence on kinematics

$$R = \int \frac{df_1}{f_1}\frac{df_2}{f_2}\dots\frac{df_m}{f_m}\,\delta(C\cdot Z)$$

$$\Updownarrow$$

$$\mathcal{M}_1^{tree}\mathcal{M}_2^{tree}\dots\mathcal{M}_m^{tree}$$

- Amplitude still given by BCFW: sum of on-shell diagrams

Unitarity

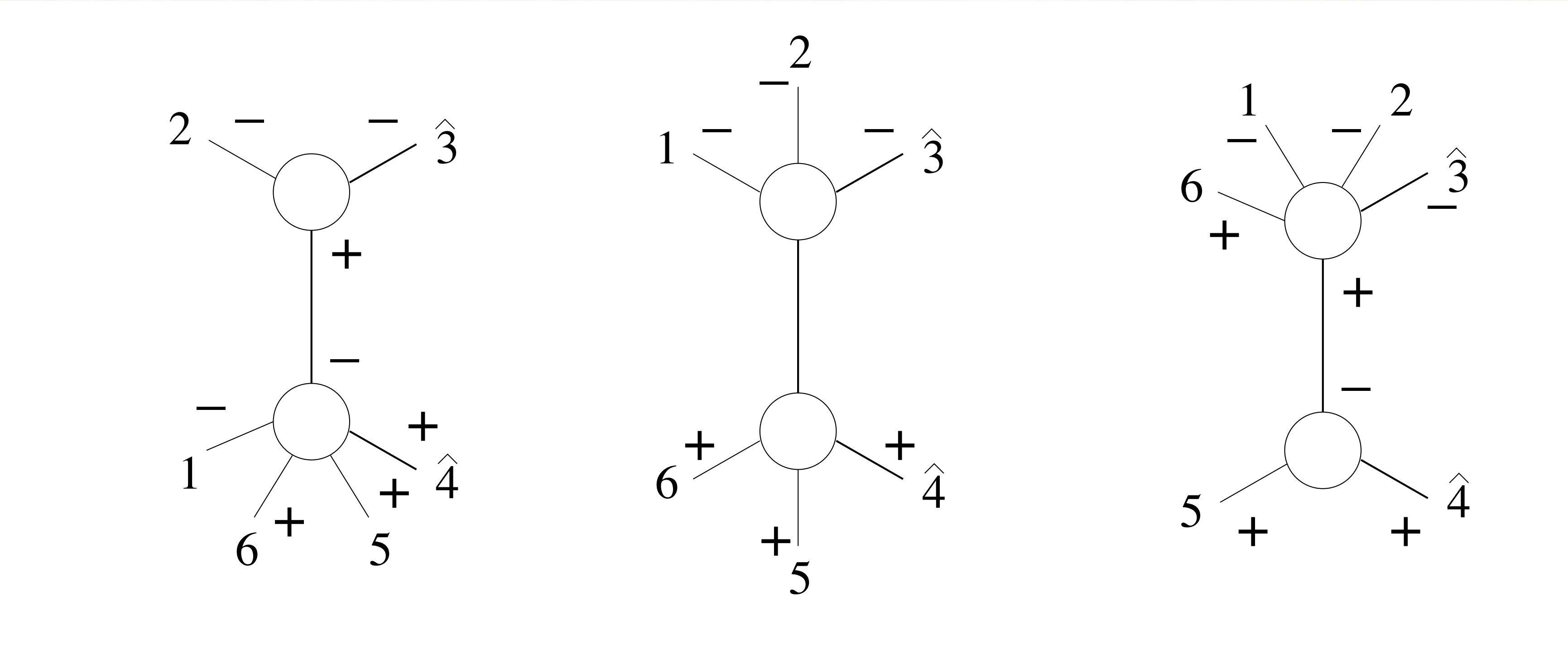
- New definition of amplitude wanted

Prelude

Volume of polyhedron

(Hodges 2009)

- Study tree-level scattering amplitude $A_6(1^-2^-3^-4^+5^+6^+)$
- BCFW recursion relations in momentum twistor space



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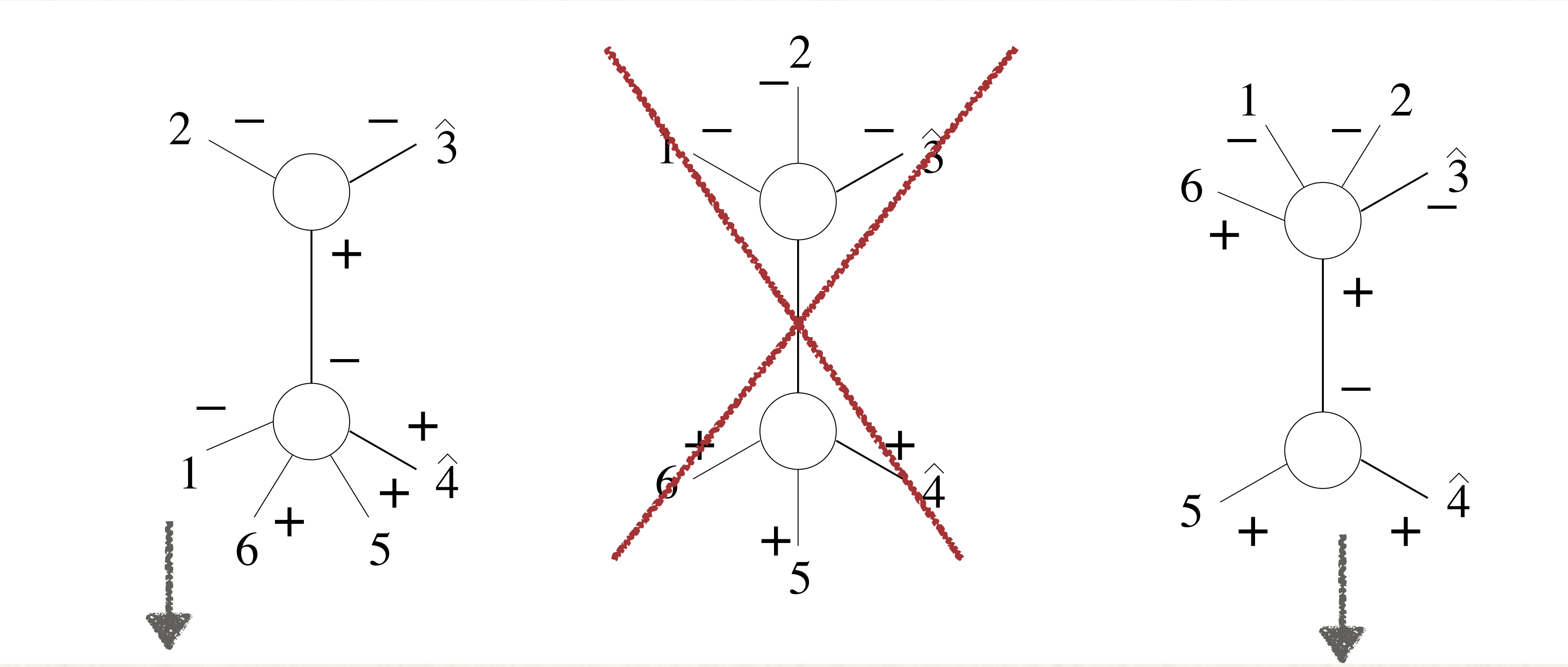

$$\frac{\langle 1345\rangle^3}{\langle 1234\rangle\langle 1245\rangle\langle 2345\rangle\langle 1235\rangle} \qquad \frac{\langle 1356\rangle^3}{\langle 1235\rangle\langle 1256\rangle\langle 2356\rangle\langle 1236\rangle}$$

Volume of polyhedron

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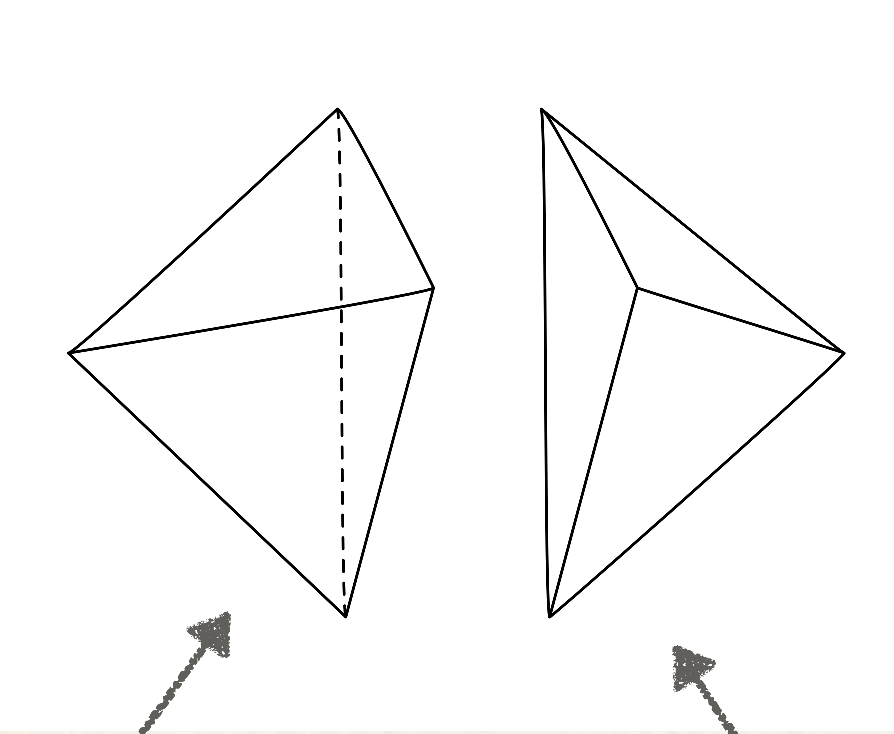
Volume of tetrahedron in momentum twistor space!



Each face labeled by $\langle abcd \rangle$

$$\frac{\langle 1345 \rangle^3}{\langle 1234 \rangle \langle 1245 \rangle \langle 2345 \rangle \langle 1235 \rangle} \qquad \frac{\langle 1356 \rangle^3}{\langle 1235 \rangle \langle 1256 \rangle \langle 2356 \rangle \langle 1236 \rangle}$$

Volume of polyhedron

(Hodges 2009)

- Study tree-level scattering amplitude $A_6(1^-2^-3^-4^+5^+6^+)$
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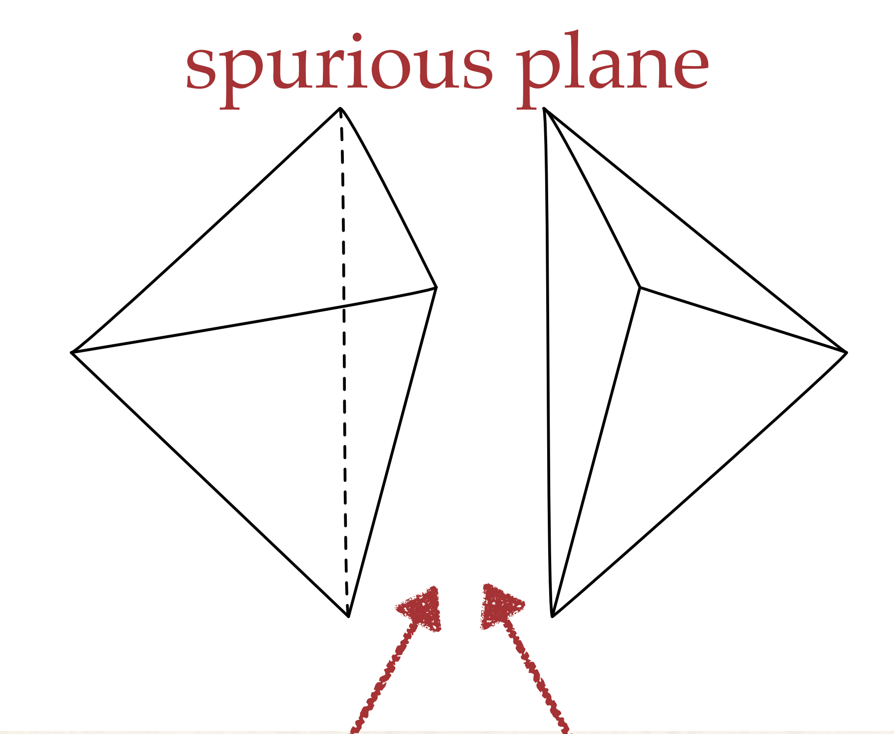
Volume of tetrahedron in momentum twistor space!



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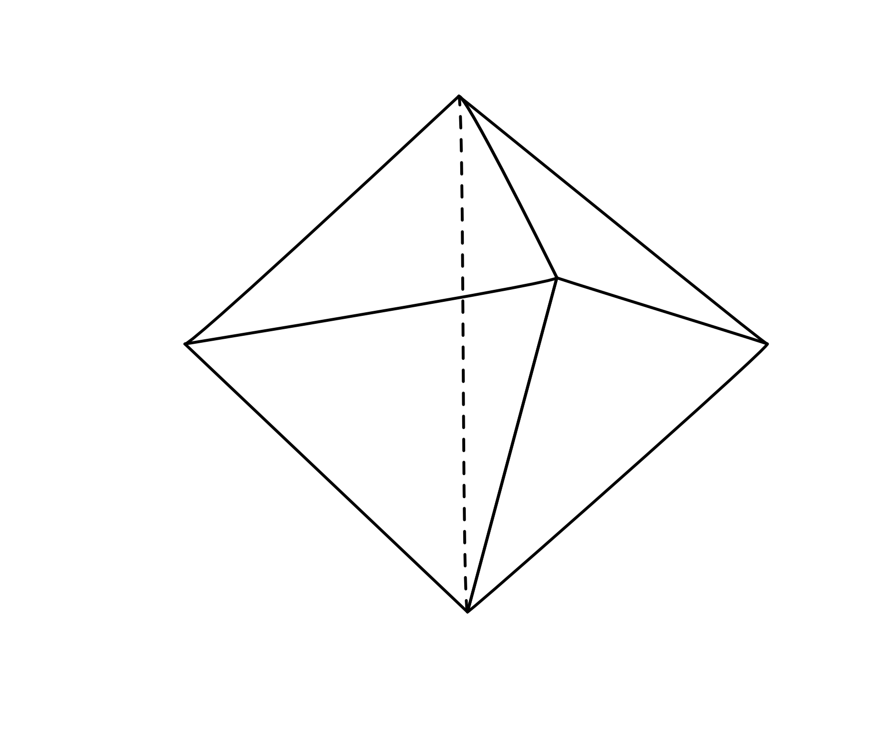
Volume of polyhedron

(Hodges 2009)

- Study tree-level scattering amplitude $A_6(1^-2^-3^-4^+5^+6^+)$
- BCFW recursion relations in momentum twistor space

Amplitude is a volume of polyhedron



Each face labeled by $\langle abcd \rangle$

$$\frac{\langle 1345\rangle^3}{\langle 1234\rangle\langle 1245\rangle\langle 2345\rangle\langle 1235\rangle} \qquad \frac{\langle 1356\rangle^3}{\langle 1235\rangle\langle 1256\rangle\langle 2356\rangle\langle 1236\rangle}$$

“Conjecture”

Amplitudes are volumes
of *some regions* in *some space*

“Conjecture”

Amplitudes are volumes
of *some regions* in *some space*

Must be related to
positive Grassmannian

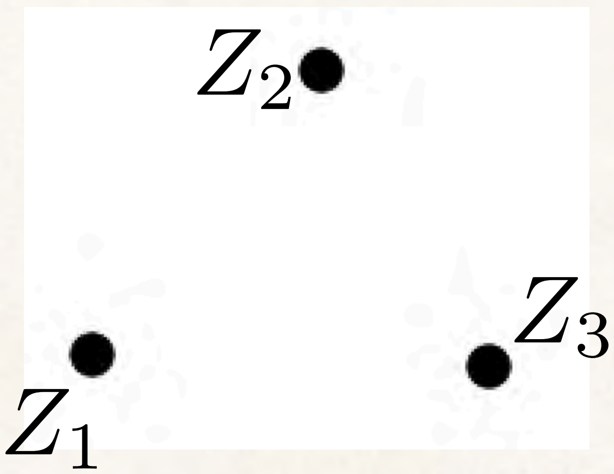
Strategy

- Simple intuitive geometric ideas
- Use suitable mathematical language to describe them
- Generalize to more complicated (non-intuitive) cases

Inside of the triangle

Inside of the triangle

- Let us consider three points in a projective plane

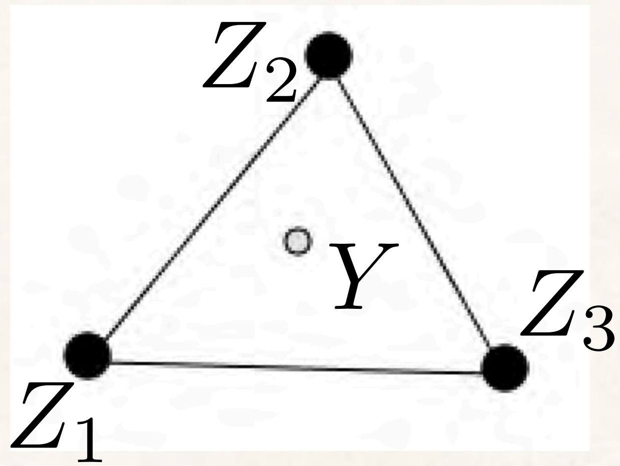


$$Z_j = \begin{pmatrix} * \\ * \\ * \end{pmatrix} \quad Z_j \sim tZ_j$$

We can also fix $Z_j = \begin{pmatrix} 1 \\ a_j \\ b_j \end{pmatrix}$

Inside of the triangle

- Point inside the triangle



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We can also fix $Z_j = \begin{pmatrix} 1 \\ a_j \\ b_j \end{pmatrix}$

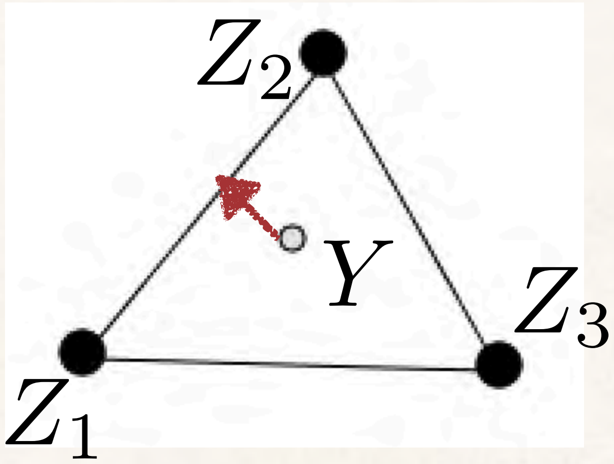
- Point inside the triangle

$$Y = c_1 Z_1 + c_2 Z_2 + c_3 Z_3 \qquad c_1, c_2, c_3 > 0$$

Projecte: one of c_j can be fixed to 1

Inside of the triangle

- Point inside the triangle



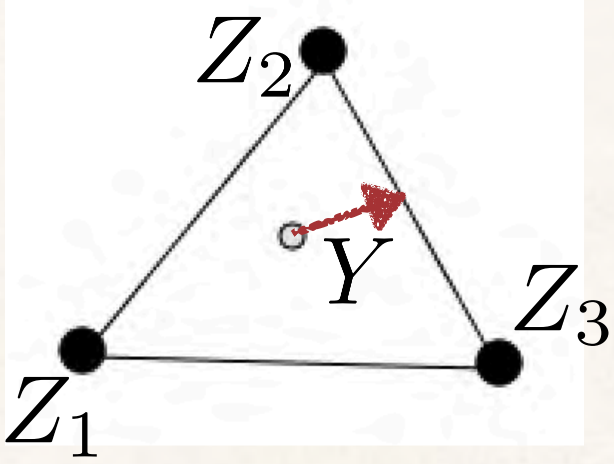
$$Y = c_1 Z_1 + c_2 Z_2 + c_3 Z_3$$

On the boundary

$$c_3 = 0$$

Inside of the triangle

- Point inside the triangle



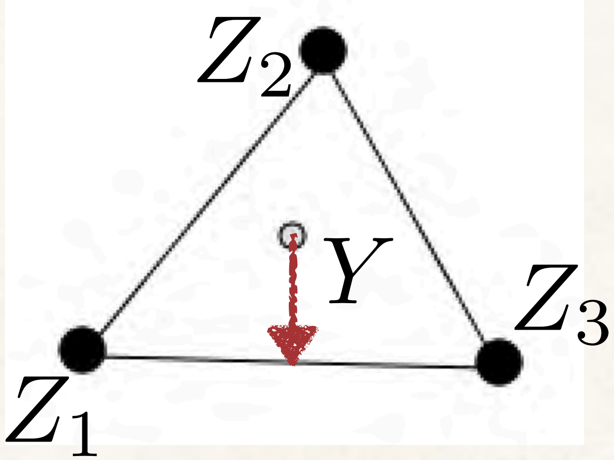
$$Y = c_1 Z_1 + c_2 Z_2 + c_3 Z_3$$

On the boundary

$$c_1 = 0$$

Inside of the triangle

- Point inside the triangle



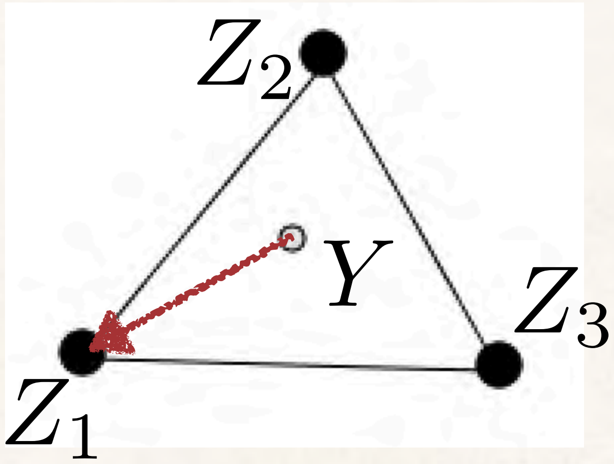
$$Y = c_1 Z_1 + c_2 Z_2 + c_3 Z_3$$

On the boundary

$$c_2 = 0$$

Inside of the triangle

- Point inside the triangle



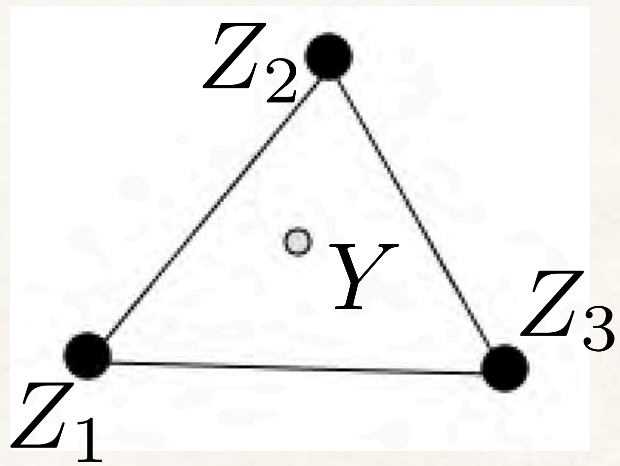
$$Y = c_1 Z_1 + c_2 Z_2 + c_3 Z_3$$

On the boundary

$$c_2 = c_3 = 0$$

Logarithmic form

- Point inside the triangle



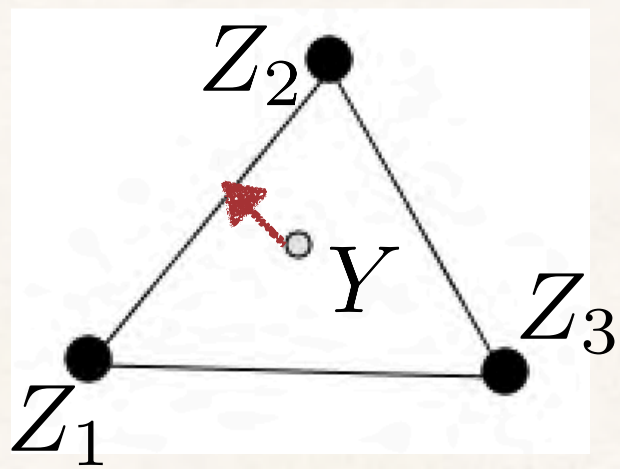
$$Y = Z_1 + c_2 Z_2 + c_3 Z_3$$

- Form with logarithmic singularities on boundaries

$$\Omega = \frac{dc_2}{c_2}\frac{dc_3}{c_3}$$

Logarithmic form

- Point inside the triangle



$$c_3 = 0$$

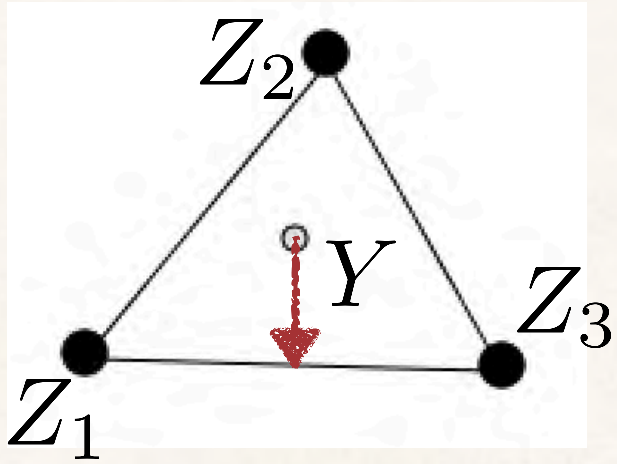
$$Y = Z_1 + c_2 Z_2 + \cancel{c_3 Z_3}$$

- Form with logarithmic singularities on boundaries

$$\Omega = \frac{dc_2}{c_2}\frac{dc_3}{c_3} \;\to\; \frac{dc_2}{c_2}$$

Logarithmic form

- Point inside the triangle



$$c_2 = 0$$

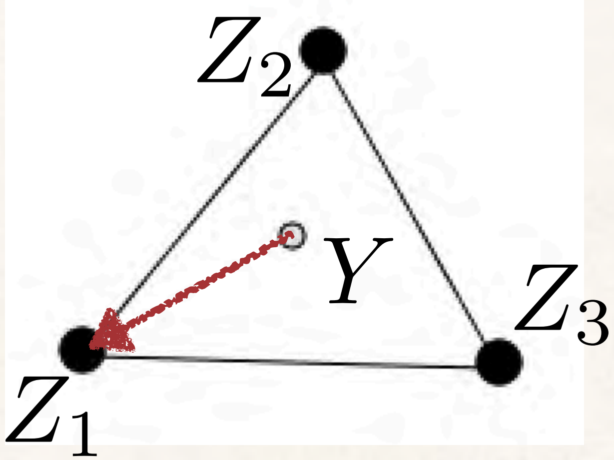
$$Y = Z_1 + c_2 Z_2 + c_3 Z_3$$

- Form with logarithmic singularities on boundaries

$$\Omega = \frac{dc_2}{c_2}\frac{dc_3}{c_3} \to \frac{dc_3}{c_3}$$

Logarithmic form

- Point inside the triangle



$$c_2 = c_3 = 0$$

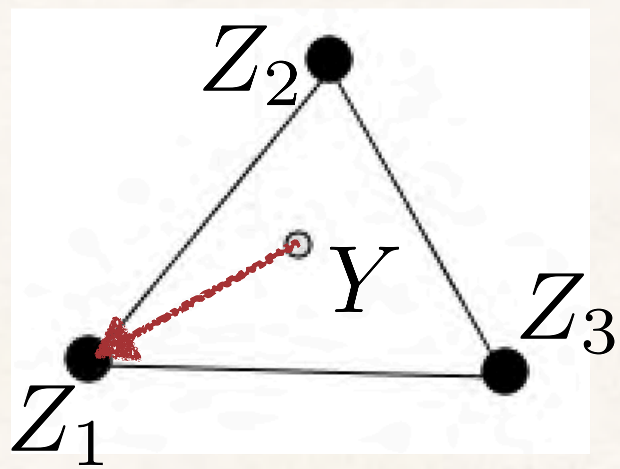
$$Y = Z_1 + c_2 Z_2 + c_3 Z_3$$

- Form with logarithmic singularities on boundaries

$$\Omega = \frac{dc_2}{c_2}\frac{dc_3}{c_3} \to \frac{dc_3}{c_3} \to 1$$

Logarithmic form

- Point inside the triangle



$$c_2 = c_3 = 0$$

$$Y = Z_1 + \cancel{c_2 Z_2} + \cancel{c_3 Z_3}$$

- Form with logarithmic singularities on boundaries

$$\Omega = \frac{dc_2}{c_2}\frac{dc_3}{c_3} \to \frac{dc_3}{c_3} \to 1$$

- Other boundaries can correspond to $c_2, c_3 \to \infty$

Logarithmic form

- Form with logarithmic singularities on boundaries

$$\Omega = \frac{dc_2}{c_2}\frac{dc_3}{c_3}$$

$$\langle X_1 X_2 X_3 \rangle = \epsilon_{abc} X_1^a X_2^b X_3^c$$

$$d^2Y = dc_2\, dc_3\, Z_2 Z_3$$

- Solve for c_2, c_3 from $Y = Z_1 + c_2 Z_2 + c_3 Z_3$

$$\downarrow$$

$$c_2 = \frac{\langle Y13 \rangle}{\langle Y23 \rangle} \quad c_3 = \frac{\langle Y12 \rangle}{\langle Y23 \rangle} \quad dc_2\, dc_3 = \frac{\langle Y\, d^2Y \rangle \langle 123 \rangle^2}{\langle Y23 \rangle^3}$$

Logarithmic form

- Form with logarithmic singularities on boundaries

$$\Omega = \frac{dc_2}{c_2}\frac{dc_3}{c_3}$$

$$\langle X_1 X_2 X_3 \rangle = \epsilon_{abc} X_1^a X_2^b X_3^c$$

$$d^2 Y = dc_2\, dc_3\, Z_2 Z_3$$

- Solve for c_2, c_3 from $Y = Z_1 + c_2 Z_2 + c_3 Z_3$

↓

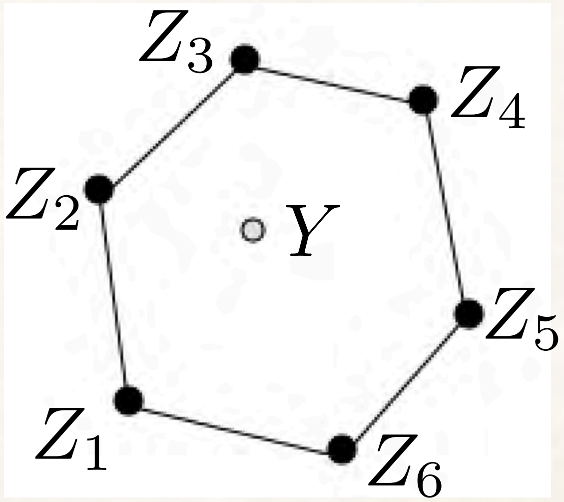
$$\Omega = \frac{\langle Y\, d^2 Y \rangle \langle 123 \rangle^2}{\langle Y12 \rangle \langle Y23 \rangle \langle Y31 \rangle}$$

Projective in all variables

Polygon

Point inside the polygon

- Consider a point inside a polygon in projective plane



$$Y = c_1 Z_1 + c_2 Z_2 + \dots c_n Z_n$$

$$c_j > 0$$

interior of the polygon

- Convex polygon: condition on points Z_i

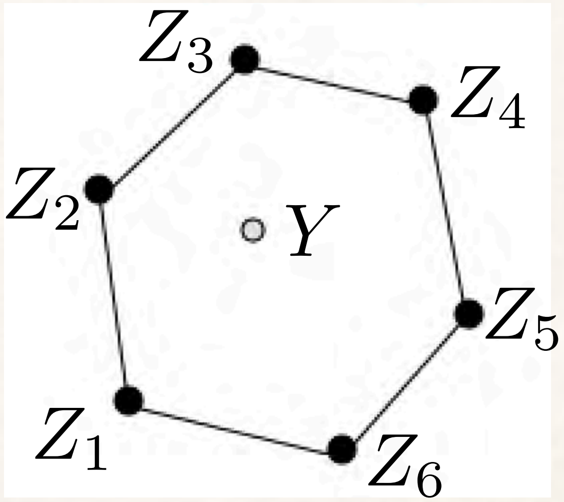
$$Z = \begin{pmatrix} \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\ Z_1 & Z_2 & Z_3 & \dots & Z_n \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \end{pmatrix}$$

All main minors positive

$$\begin{vmatrix} * & * & * \\ * & * & * \\ * & * & * \end{vmatrix} > 0$$

Point inside the polygon

- Consider a point inside a polygon in projective plane



$$Y = c_1 Z_1 + c_2 Z_2 + \ldots c_n Z_n$$

Space of all points
inside convex polygon

More formally:

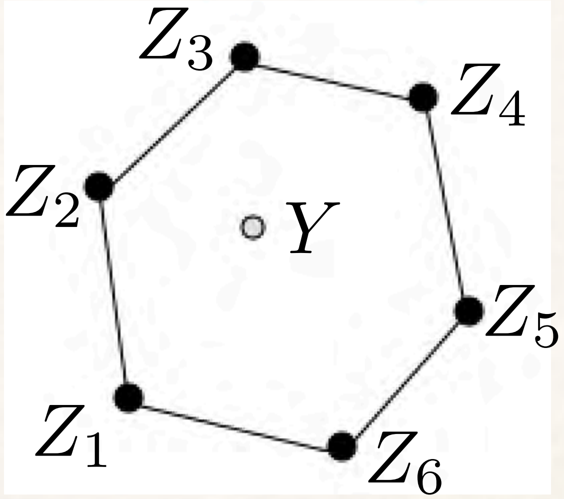
$$\boxed{Y = C \cdot Z}$$

$$C = \begin{pmatrix} c_1 & c_2 & c_3 & \ldots & c_n \end{pmatrix} \in G_+(1,n)$$

$$Z = \begin{pmatrix} \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\ Z_1 & Z_2 & Z_3 & \ldots & Z_n \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \end{pmatrix} \in M_+(3,n)$$

Logarithmic form

- Form with logarithmic singularities on boundaries



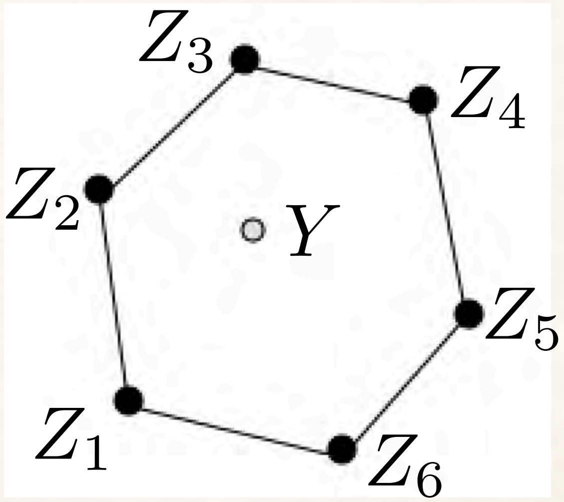
$$Y = c_1 Z_1 + c_2 Z_2 + \ldots c_n Z_n$$

First guess

$$\Omega = \frac{dc_1}{c_1} \frac{dc_2}{c_2} \ldots \frac{dc_n}{c_n}$$

Logarithmic form

- Form with logarithmic singularities on boundaries



$$Y = c_1 Z_1 + c_2 Z_2 + \ldots c_n Z_n$$

First guess

$$\Omega = \frac{dc_1}{c_1}\frac{dc_2}{c_2} \ldots \frac{dc_n}{c_n}$$

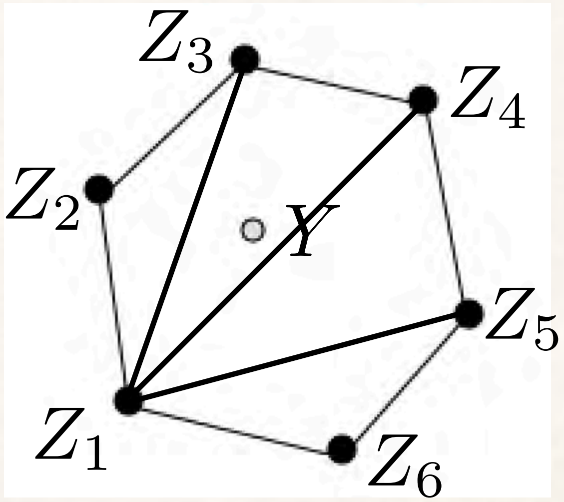
Space of all Y is only two-dimensional

- Two-form with n poles

$$\Omega \sim \frac{dc_1\, dc_2\, N(c_1, c_2)}{D(c_1, c_2)}$$

Logarithmic form

- Easiest way how to write the form is to triangulate



Logarithmic form

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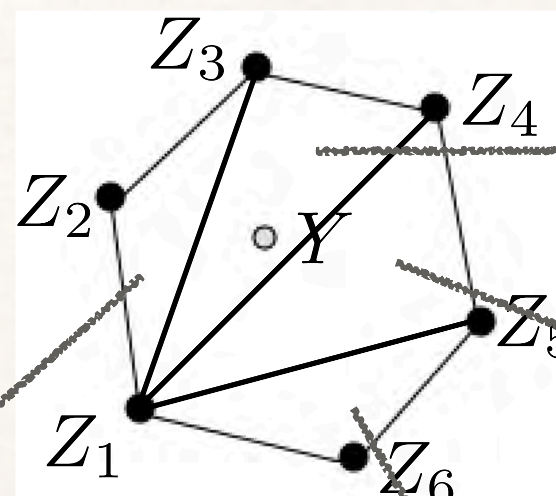


$$Y = Z_1 + c_3 Z_3 + c_4 Z_4$$
$$\Omega_2 = \frac{dc_3}{c_3}\frac{dc_4}{c_4} \quad c_3, c_4 \geq 0$$

$$Y = Z_1 + c_4 Z_4 + c_5 Z_5$$
$$\Omega_3 = \frac{dc_4}{c_4}\frac{dc_5}{c_5} \quad c_4, c_5 \geq 0$$

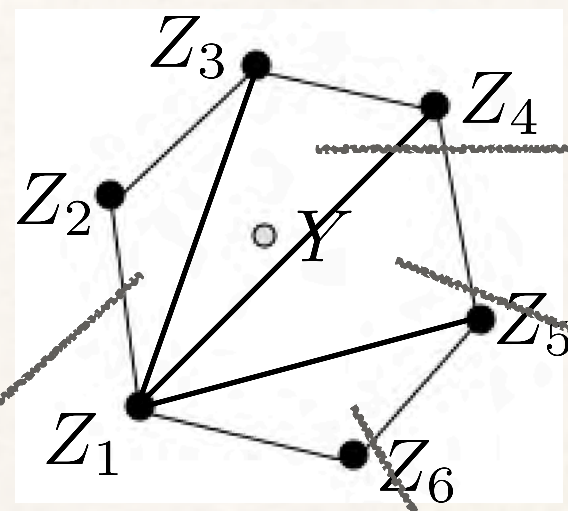
$$Y = Z_1 + c_2 Z_2 + c_3 Z_3$$
$$\Omega_1 = \frac{dc_2}{c_2}\frac{dc_3}{c_3} \quad c_2, c_3 \geq 0$$

$$Y = Z_1 + c_5 Z_5 + c_6 Z_6$$
$$\Omega_4 = \frac{dc_5}{c_5}\frac{dc_6}{c_6} \quad c_5, c_6 \geq 0$$

How to sum them?

Logarithmic form

- Easiest way how to write the form is to triangulate



$$Y = Z_1 + c_3 Z_3 + c_4 Z_4$$
$$\Omega_2 = \frac{\langle Y\, d^2Y\rangle\langle 134\rangle^2}{\langle Y13\rangle\langle Y34\rangle\langle Y41\rangle}$$

$$Y = Z_1 + c_4 Z_4 + c_5 Z_5$$
$$\Omega_3 = \frac{\langle Y\, d^2Y\rangle\langle 145\rangle^2}{\langle Y14\rangle\langle Y45\rangle\langle Y51\rangle}$$

$$Y = Z_1 + c_2 Z_2 + c_3 Z_3$$
$$\Omega_1 = \frac{\langle Y\, d^2Y\rangle\langle 123\rangle^2}{\langle Y12\rangle\langle Y23\rangle\langle Y31\rangle}$$

$$Y = Z_1 + c_5 Z_5 + c_6 Z_6$$
$$\Omega_4 = \frac{\langle Y\, d^2Y\rangle\langle 156\rangle^2}{\langle Y15\rangle\langle Y56\rangle\langle Y61\rangle}$$

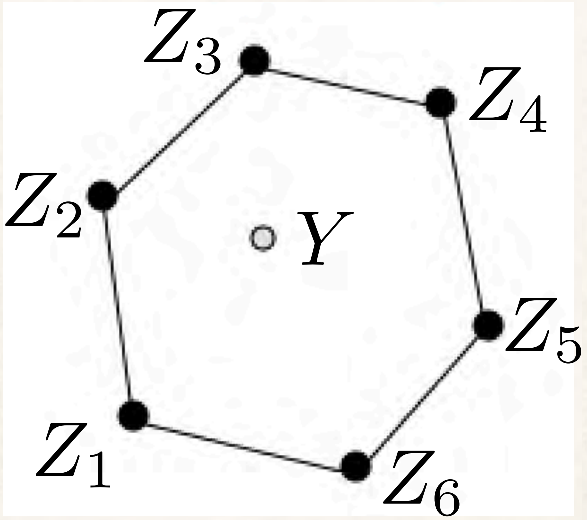
Write in projective form

Logarithmic form

- Now it makes sense to sum them

$$\Omega = \frac{\langle Y\, d^2Y\rangle\langle 123\rangle^2}{\langle Y12\rangle\langle Y23\rangle\langle Y31\rangle} + \frac{\langle Y\, d^2Y\rangle\langle 134\rangle^2}{\langle Y13\rangle\langle Y34\rangle\langle Y41\rangle} + \frac{\langle Y\, d^2Y\rangle\langle 145\rangle^2}{\langle Y14\rangle\langle Y45\rangle\langle Y51\rangle} + \frac{\langle Y\, d^2Y\rangle\langle 156\rangle^2}{\langle Y15\rangle\langle Y56\rangle\langle Y61\rangle}$$

- Boundaries of the polygon are $\langle Y\, i\, i+1\rangle = 0$



Logarithmic form

- Now it makes sense to sum them

$$\Omega = \frac{\langle Y\, d^2Y\rangle\langle 123\rangle^2}{\langle Y12\rangle\langle Y23\rangle\langle Y31\rangle} + \frac{\langle Y\, d^2Y\rangle\langle 134\rangle^2}{\langle Y13\rangle\langle Y34\rangle\langle Y41\rangle} + \frac{\langle Y\, d^2Y\rangle\langle 145\rangle^2}{\langle Y14\rangle\langle Y45\rangle\langle Y51\rangle} + \frac{\langle Y\, d^2Y\rangle\langle 156\rangle^2}{\langle Y15\rangle\langle Y56\rangle\langle Y61\rangle}$$

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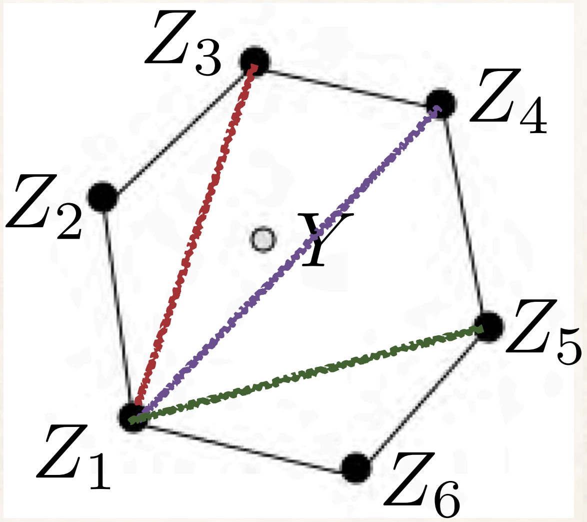


Spurious poles

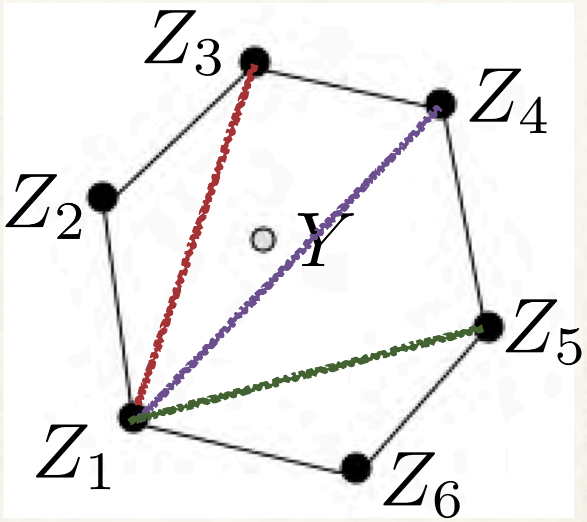
Cancel in the sum

Logarithmic form

- Now it makes sense to sum them

$$\Omega = \frac{\langle Y\, d^2Y\rangle\langle 123\rangle^2}{\langle Y12\rangle\langle Y23\rangle\langle Y31\rangle} + \frac{\langle Y\, d^2Y\rangle\langle 134\rangle^2}{\langle Y13\rangle\langle Y34\rangle\langle Y41\rangle} + \frac{\langle Y\, d^2Y\rangle\langle 145\rangle^2}{\langle Y14\rangle\langle Y45\rangle\langle Y51\rangle} + \frac{\langle Y\, d^2Y\rangle\langle 156\rangle^2}{\langle Y15\rangle\langle Y56\rangle\langle Y61\rangle}$$

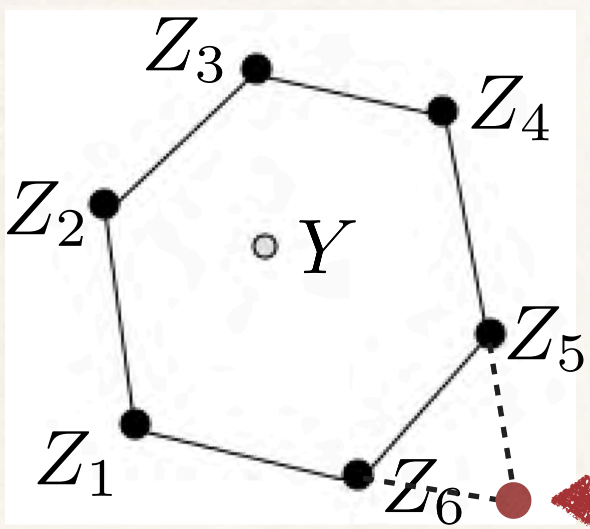
- Boundaries of the polygon are $\langle Y\, i\, i+1\rangle = 0$



$$\Omega = \frac{\langle Y\, d^2Y\rangle\, \mathcal{N}(Y, Z_j)}{\langle Y12\rangle\langle Y23\rangle\langle Y34\rangle\langle Y45\rangle\langle Y56\rangle\langle Y61\rangle}$$

Numerator of the form

- The numerator has special properties



$$\Omega = \frac{\langle Y\, d^2Y \rangle\, \mathcal{N}(Y, Z_j)}{\langle Y12\rangle\langle Y23\rangle\langle Y34\rangle\langle Y45\rangle\langle Y56\rangle\langle Y61\rangle}$$

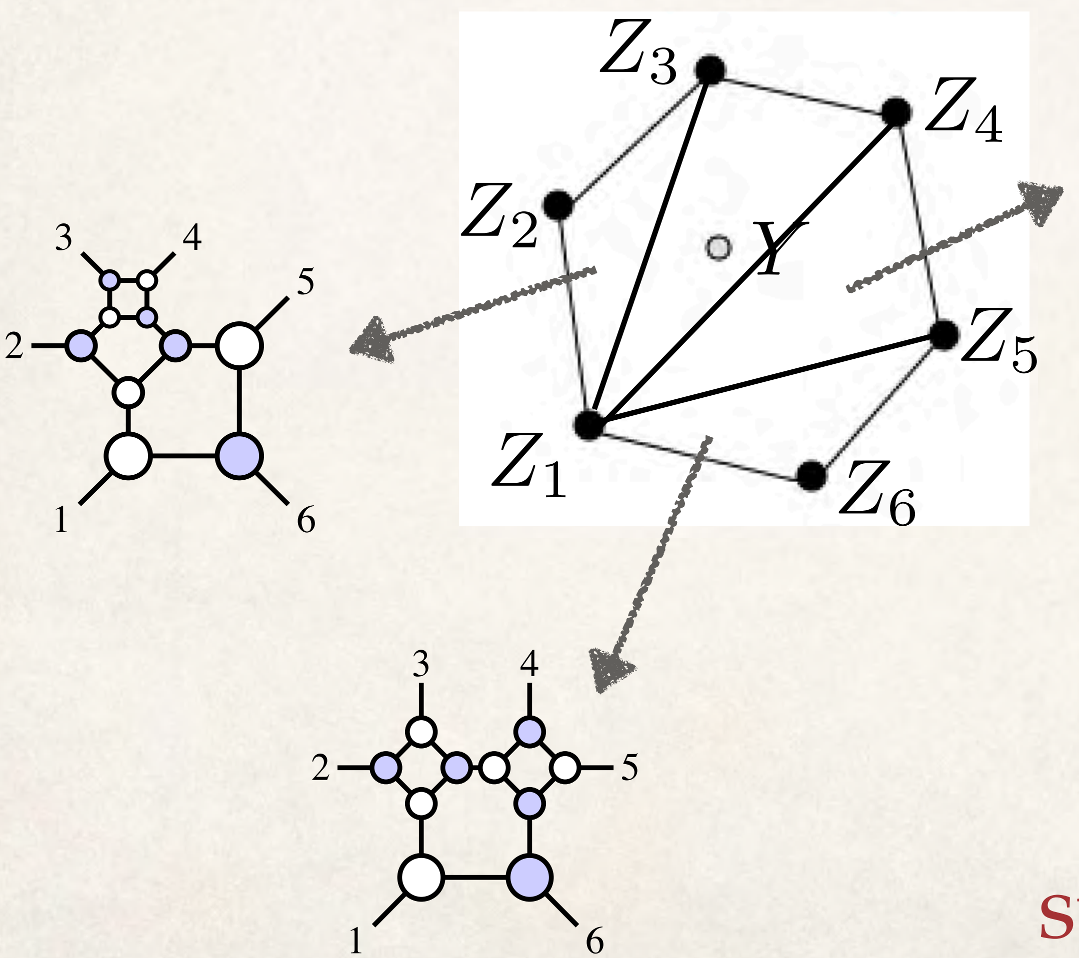
$\langle Y45\rangle \to 0$, $\langle Y61\rangle \to 0$

Denominator generates pole
Numerator must vanish at this point

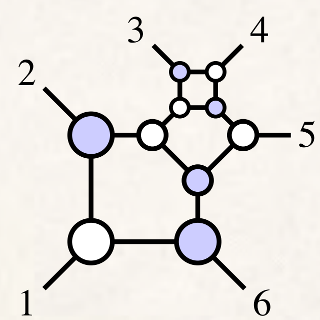
- These conditions fix the numerator without triangulation

Similarities with on-shell diagrams

- Notice some similarities with recursion relations and on-shell diagrams



$$C = \begin{pmatrix} 1 & 0 & 0 & c_4 & c_5 & 0 \end{pmatrix} \in G_+(1,6)$$



$$\Omega = \frac{dc_4}{c_4}\frac{dc_5}{c_5}$$ looks similar

$$Y = C \cdot Z \qquad \delta(C \cdot Z)$$

$$\Omega = \frac{\langle Y\, d^2Y\rangle \langle 145\rangle^2}{\langle Y14\rangle\langle Y45\rangle\langle Y51\rangle}$$

On-shell diagrams:

supersymmetric, no Y

From Y to supersymmetry

- Let us take the form for the triangle

$$\Omega = \frac{\langle Y\, d^2 Y\rangle \langle 123\rangle^2}{\langle Y 12\rangle\langle Y 23\rangle\langle Y 31\rangle}$$

- Rewrite external Z:

$$Z_j = \begin{pmatrix} z_j^{(1)} \\ z_j^{(2)} \\ (\phi \cdot \eta_j) \end{pmatrix}$$

$z_j \in \mathbb{P}^2$ bosonic

η_j^A ferimionic

ϕ^A auxiliary

$A = 1, 2$

- Also define

$$Y_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Consider $\int \Omega\, \delta(Y - Y_0)$

Effectively set: $Y = Y_0$

From Y to supersymmetry

Calculate: $\langle 123 \rangle \to \begin{vmatrix} z_1^{(1)} & z_2^{(1)} & z_3^{(1)} \\ z_1^{(2)} & z_2^{(2)} & z_3^{(2)} \\ (\phi \cdot \eta_1) & (\phi \cdot \eta_2) & (\phi \cdot \eta_3) \end{vmatrix} = \begin{matrix} \langle 12 \rangle (\phi \cdot \eta_3) \\ +\langle 23 \rangle (\phi \cdot \eta_1) \\ +\langle 31 \rangle (\phi \cdot \eta_2) \end{matrix}$

$$\langle Y12 \rangle \to \begin{vmatrix} 0 & z_1^{(1)} & z_2^{(1)} \\ 0 & z_1^{(2)} & z_2^{(2)} \\ 1 & (\phi \cdot \eta_1) & (\phi \cdot \eta_2) \end{vmatrix} = \langle 12 \rangle$$

$$\langle Y23 \rangle \to \begin{vmatrix} 0 & z_2^{(1)} & z_3^{(1)} \\ 0 & z_2^{(2)} & z_3^{(2)} \\ 1 & (\phi \cdot \eta_2) & (\phi \cdot \eta_3) \end{vmatrix} = \langle 23 \rangle$$

$$\langle Y31 \rangle \to \begin{vmatrix} 0 & z_3^{(1)} & z_1^{(1)} \\ 0 & z_3^{(2)} & z_1^{(2)} \\ 1 & (\phi \cdot \eta_3) & (\phi \cdot \eta_1) \end{vmatrix} = \langle 31 \rangle$$

Old variables

$$Z_j = \begin{pmatrix} z_j^{(1)} \\ z_j^{(2)} \\ (\phi \cdot \eta_j) \end{pmatrix}$$

where new invariants are

$$\langle ij \rangle = \epsilon_{ab} z_i^a z_j^b$$

From Y to supersymmetry

- We plug them into the form for triangle:

$$\frac{\langle Y\, d^2 Y\rangle\langle 123\rangle^2}{\langle Y12\rangle\langle Y23\rangle\langle Y31\rangle} \to \frac{(\langle 12\rangle(\phi\cdot\eta_3) + \langle 23\rangle(\phi\cdot\eta_1) + \langle 31\rangle(\phi\cdot\eta_2))^2}{\langle 12\rangle\langle 23\rangle\langle 31\rangle}$$

- Final step: integrate over ϕ :

$$\int d^2\phi \int \Omega\, \delta(Y - Y_0) = \boxed{\frac{(\langle 12\rangle\eta_3 + \langle 23\rangle\eta_1 + \langle 31\rangle\eta_2)^2}{\langle 12\rangle\langle 23\rangle\langle 31\rangle}}$$

From Y to supersymmetry

- We plug them into the form for triangle:

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This remind the Yangian invariant: $\dfrac{(\langle 1234\rangle\eta_5+\langle 2345\rangle\eta_1+\langle 3451\rangle\eta_2+\langle 4512\rangle\eta_3+\langle 5123\rangle\eta_4)^4}{\langle 1234\rangle\langle 2345\rangle\langle 3451\rangle\langle 4512\rangle\langle 5123\rangle}$

From Y to supersymmetry

- We plug them into the form for triangle:

$$\frac{\langle Y\, d^2Y\rangle\langle 123\rangle^2}{\langle Y12\rangle\langle Y23\rangle\langle Y31\rangle} \to \frac{(\langle 12\rangle(\phi\cdot\eta_3)+\langle 23\rangle(\phi\cdot\eta_1)+\langle 31\rangle(\phi\cdot\eta_2))^2}{\langle 12\rangle\langle 23\rangle\langle 31\rangle}$$

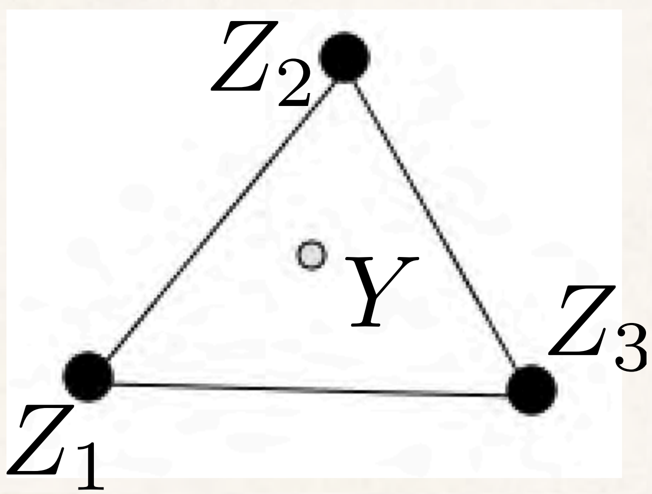
- Final step: integrate over ϕ:

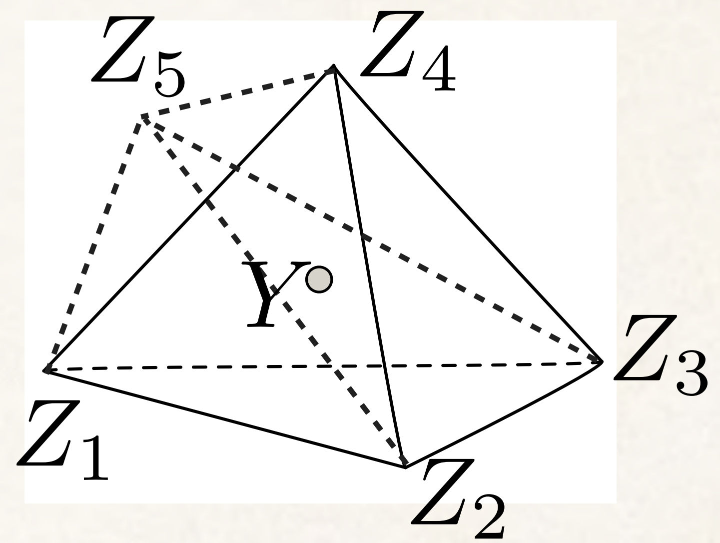
$$\int d^2\phi \int \Omega\,\delta(Y-Y_0) = \boxed{\frac{(\langle 12\rangle\eta_3+\langle 23\rangle\eta_1+\langle 31\rangle\eta_2)^2}{\langle 12\rangle\langle 23\rangle\langle 31\rangle}}$$

Yangian: $SL(4|4) \to SL(2|2)$

Repeating exercise

- We can repeat the exercise:





Point inside triangle → Point inside simplex

$$\frac{(\langle 12\rangle\eta_3 + \langle 23\rangle\eta_1 + \langle 31\rangle\eta_2)^2}{\langle 12\rangle\langle 23\rangle\langle 31\rangle} \rightarrow \frac{(\langle 1234\rangle\eta_5 + \langle 2345\rangle\eta_1 + \langle 3451\rangle\eta_2 + \langle 4512\rangle\eta_3 + \langle 5123\rangle\eta_4)^4}{\langle 1234\rangle\langle 2345\rangle\langle 3451\rangle\langle 4512\rangle\langle 5123\rangle}$$

Amplituhedron

Polygon

- Inside of polygon:

$$\boxed{Y = C \cdot Z} \qquad C = \begin{pmatrix} * & * & \dots & * \end{pmatrix} \in G_+(1,n)$$

$$Z = \begin{pmatrix} Z_1 & Z_2 & \dots & Z_n \end{pmatrix} \in M_+(3,n)$$

- Form with logarithmic singularities on boundaries Ω

- Kinematics

Final result

$$Z_j = \begin{pmatrix} z_j \\ (\phi \cdot \eta_j) \end{pmatrix} \quad Y_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\boxed{A = \int d^2\phi \int \Omega \, \delta(Y - Y_0)}$$

Tree-level amplitudes

- Inside of polygon:

$$\boxed{Y = C \cdot Z}$$

$$C = \begin{pmatrix} * & * & \dots & * \\ \vdots & \vdots & \vdots & \vdots \\ * & * & \dots & * \end{pmatrix} \in G_+(k,n)$$

$$Z = \begin{pmatrix} Z_1 & Z_2 & \dots & Z_n \end{pmatrix} \in M_+(k+4,n)$$

- Form with logarithmic singularities on boundaries Ω

- Kinematics

$$Z_j = \begin{pmatrix} z_j \\ (\phi_1 \cdot \eta_j) \\ \vdots \\ (\phi_k \cdot \eta_j) \end{pmatrix} \qquad Y_0 = \begin{pmatrix} \mathbf{0}_{4\times 4} \\ \mathbf{1}_{k\times k} \end{pmatrix}$$

<u>Final result</u>

$$\boxed{A = \int d^4\phi_1 \dots d^4\phi_k \int \Omega\, \delta(Y - Y_0)}$$

Tree-level amplitude in N=4 SYM $\mathcal{A}_{n,k}$

Loop amplitudes

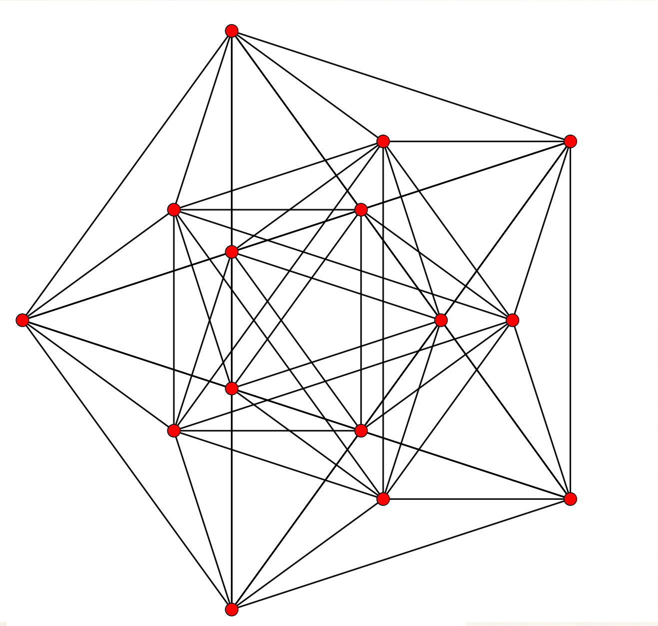
- Generalization to loops

$$\boxed{\mathcal{Y} = \mathcal{C} \cdot Z}$$

$$C = \begin{pmatrix} * & * & \dots & * \\ \vdots & \vdots & \vdots & \vdots \\ * & * & \dots & * \end{pmatrix} \longrightarrow \mathcal{C} = \begin{pmatrix} C \\ D_1 \\ \vdots \\ D_\ell \end{pmatrix} \text{ such that } \begin{pmatrix} C \\ D_{i_1} \\ \vdots \\ D_{i_m} \end{pmatrix} \in G_+(k+2m,n) \text{ for all } 0 \le m \le \ell$$

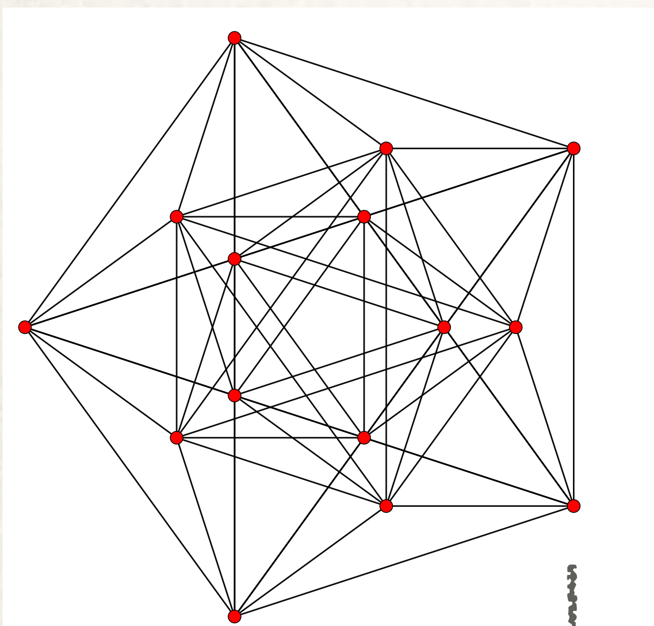
Integrand of amplitudes in planar N=4 SYM $\mathcal{I}_{n,k}^{\ell-loop}$

Triangulation



space specified by a set of inequalities → Ω logarithmic singularities → $\mathcal{M}_{n,k}^{\ell-loop}$

Triangulation



space specified by a set of inequalities → Ω logarithmic singularities → $\mathcal{M}_{n,k}^{\ell-loop}$

triangulate in terms of "simplices"

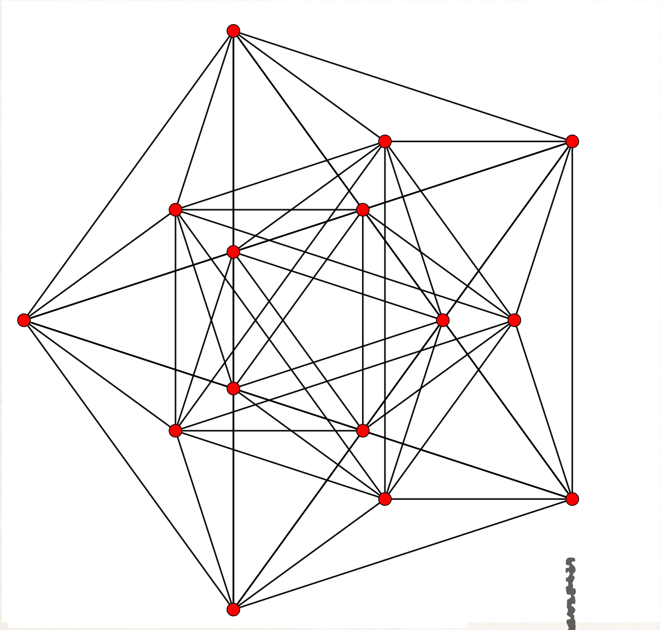
$\Omega_0 \sim \dfrac{dx}{x}$ for each

→ Set of regions:

- cover the whole space
- each region specified by $f_j \in (0, \infty)$

→ sum them

Triangulation



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↓

triangulate in terms of "simplices"

$\Omega_0 \sim \frac{dx}{x}$ for each

→ Set of regions:

- cover the whole space
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→ sum them

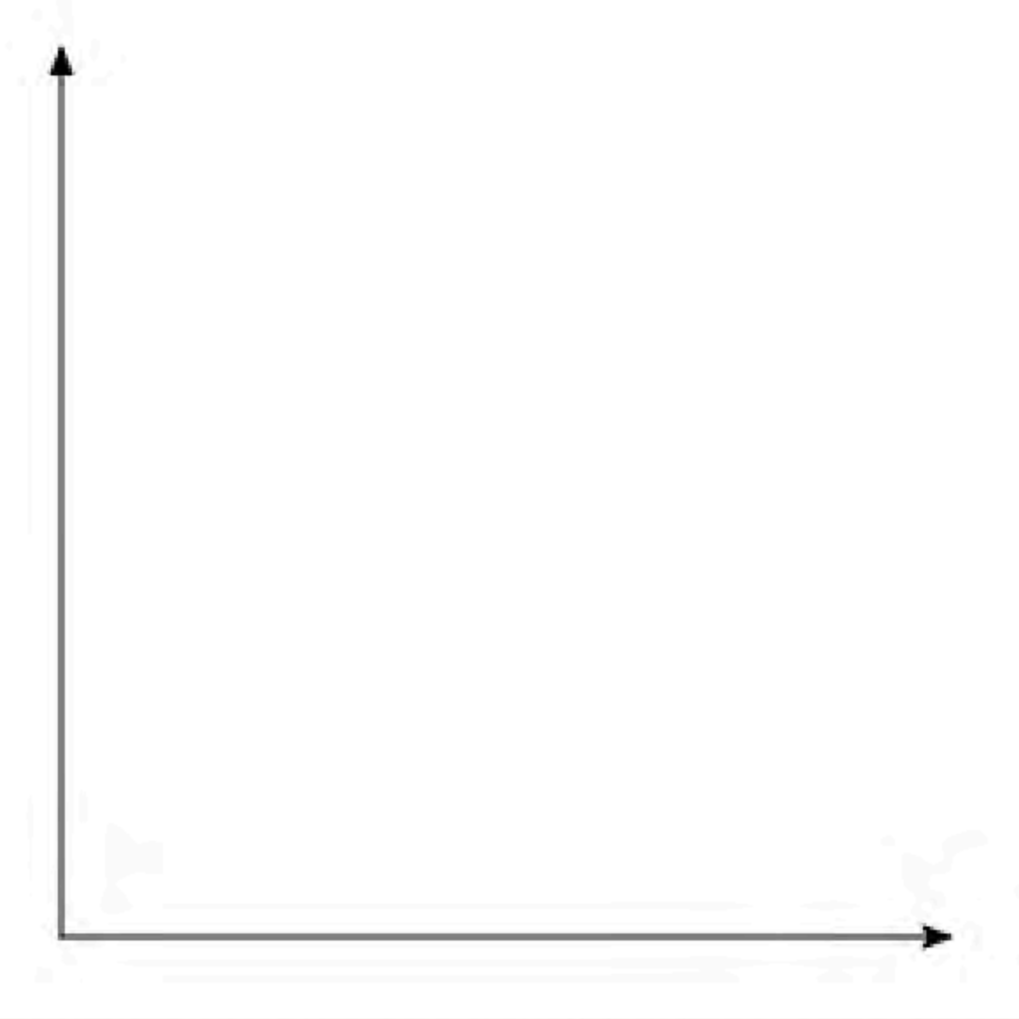
One explicit example:

4pt scattering to all loops

High school problem $gg \to gg$

High school problem $gg \to gg$

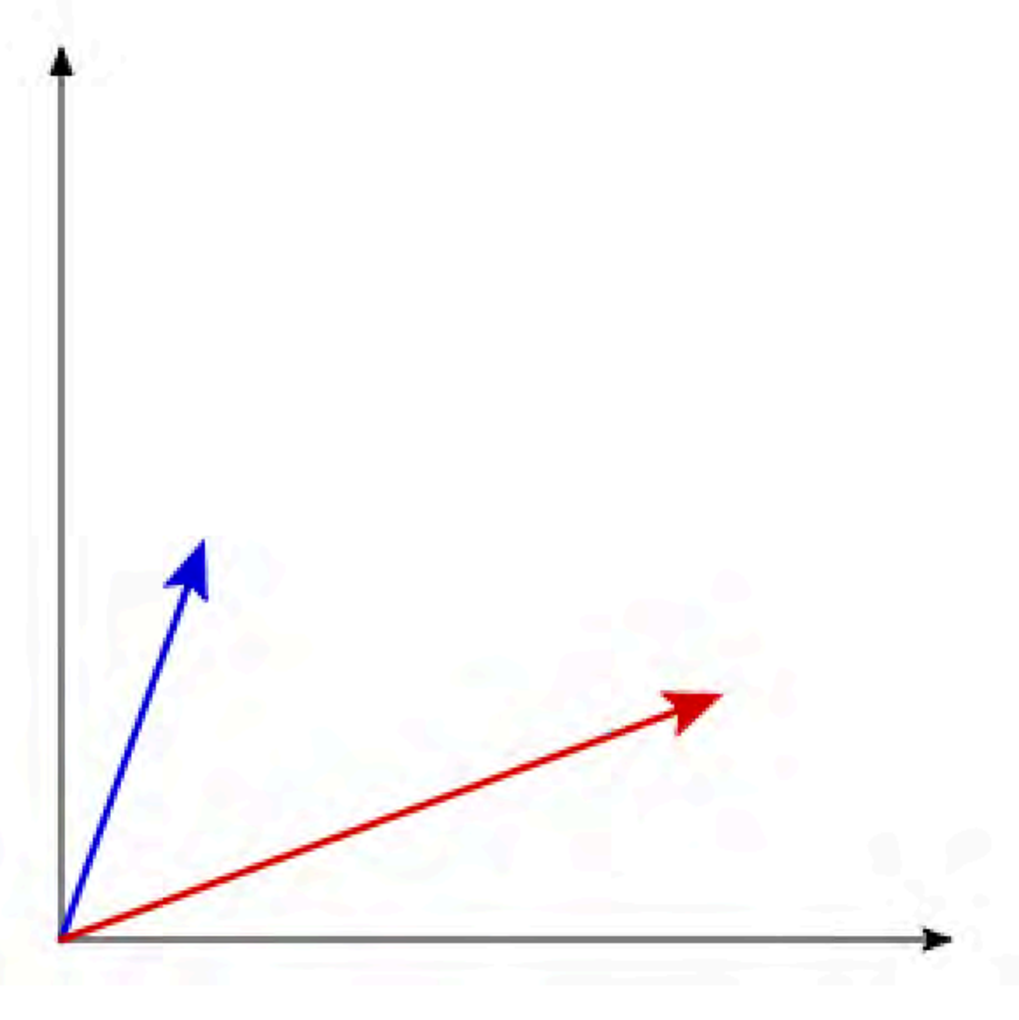
- Positive quadrant



High school problem $gg \to gg$

- Positive quadrant
- Vectors

$$\vec{a}_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \quad \vec{b}_1 = \begin{pmatrix} z_1 \\ w_1 \end{pmatrix}$$



$$\mathrm{Vol}\,(1) = \frac{dx_1}{x_1}\frac{dy_1}{y_1}\frac{dz_1}{z_1}\frac{dw_1}{w_1}$$

High school problem

- Change of variables

where $\ell^* = \frac{[12]}{[24]}\lambda_1\widetilde{\lambda}_4$

$$x_1 = \frac{\ell^2}{(\ell - \ell^*)^2} = \frac{\langle AB41\rangle}{\langle AB13\rangle} \qquad z_1 = \frac{(\ell + k_1 + k_2)^2}{(\ell - \ell^*)^2} = \frac{\langle AB23\rangle}{\langle AB13\rangle}$$

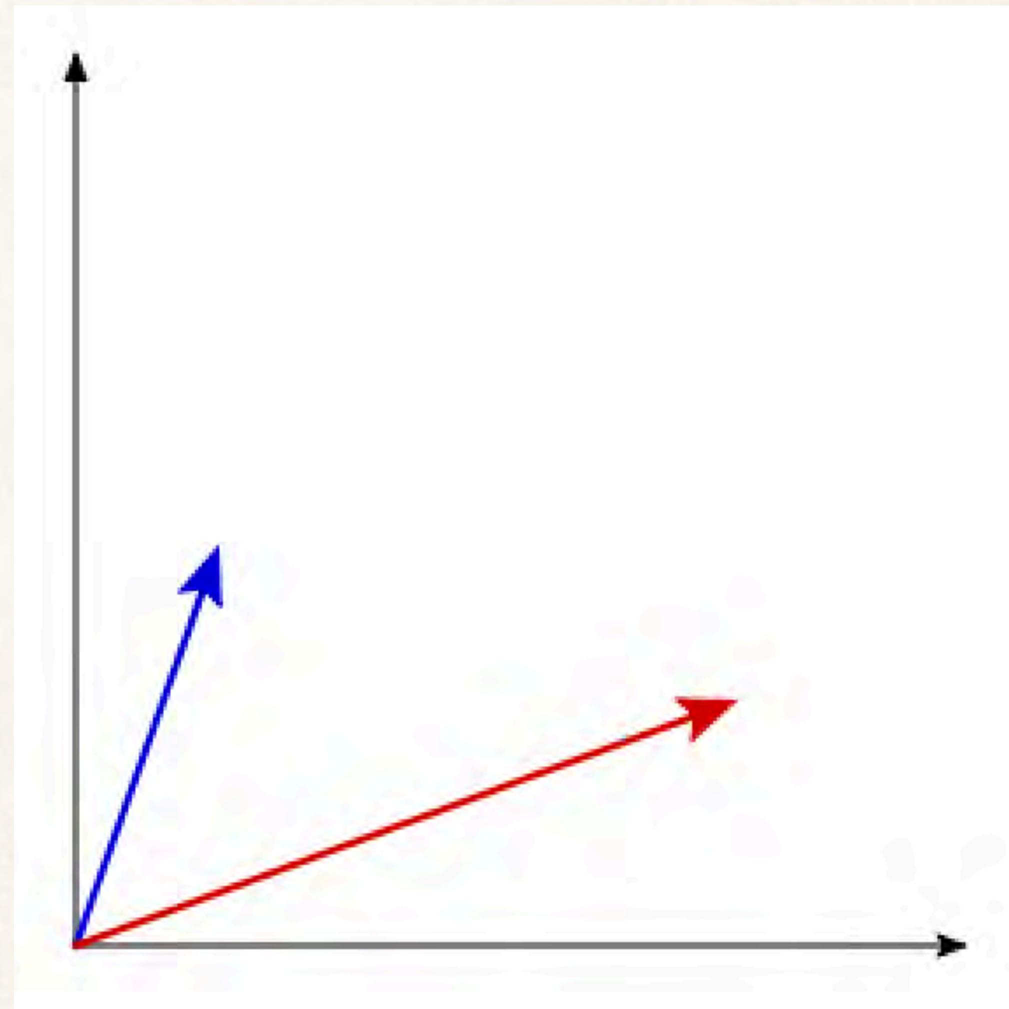
$$y_1 = \frac{(\ell + k_1)^2}{(\ell - \ell^*)^2} = \frac{\langle AB12\rangle}{\langle AB13\rangle} \qquad w_1 = \frac{(\ell - k_4)^2}{(\ell - \ell^*)^2} = \frac{\langle AB34\rangle}{\langle AB13\rangle}$$

Exactly the same expressions
true for all loops

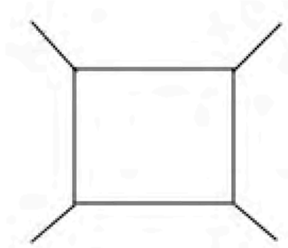
High school problem $gg \to gg$

- Positive quadrant
- Vectors

$$\vec{a}_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \quad \vec{b}_1 = \begin{pmatrix} z_1 \\ w_1 \end{pmatrix}$$



$$\text{Vol}\,(1) = \frac{dx_1}{x_1}\frac{dy_1}{y_1}\frac{dz_1}{z_1}\frac{dw_1}{w_1} =$$

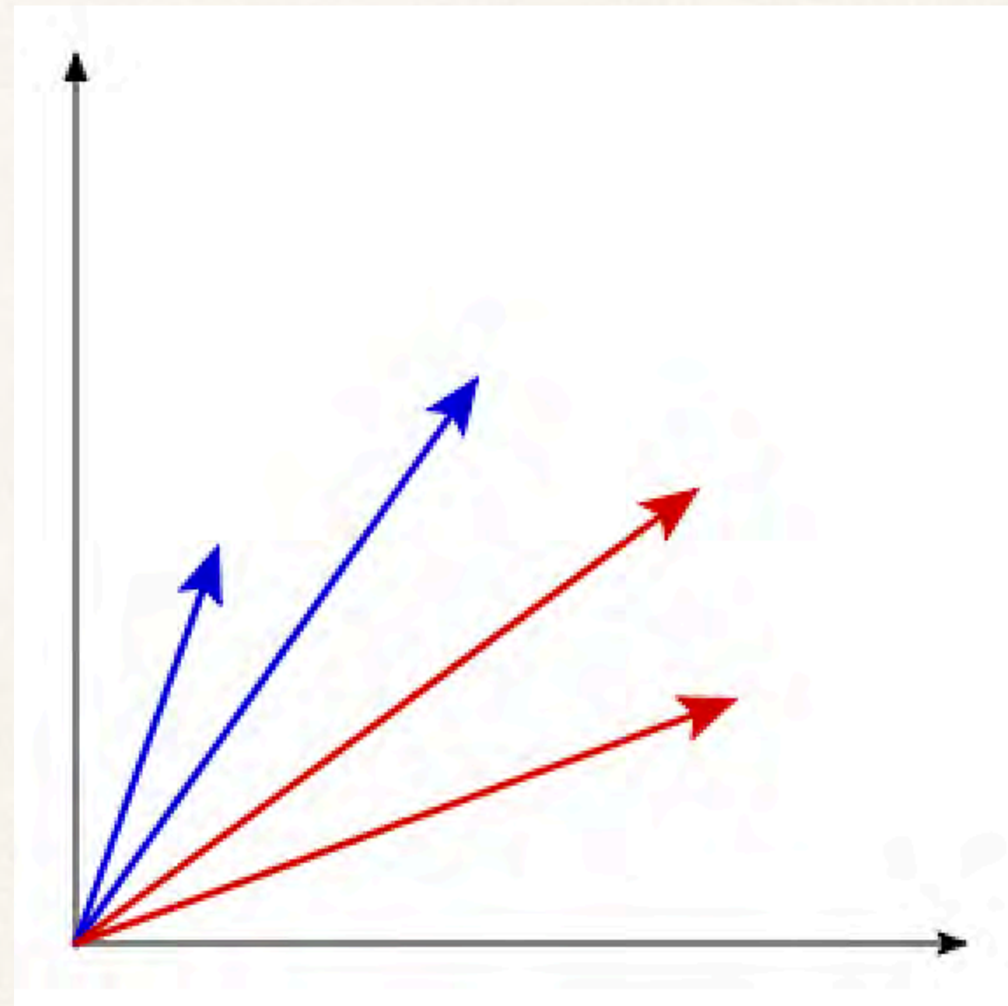


High school problem $\quad gg \to gg$

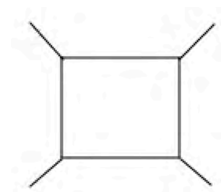
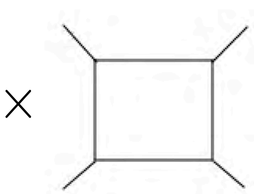
- Positive quadrant
- Vectors

$$\vec{a}_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \quad \vec{b}_1 = \begin{pmatrix} z_1 \\ w_1 \end{pmatrix}$$

$$\vec{a}_2 = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \quad \vec{b}_2 = \begin{pmatrix} z_2 \\ w_2 \end{pmatrix}$$



$$[\mathrm{Vol}\,(1)]^2 = \frac{dx_1}{x_1}\frac{dy_1}{y_1}\frac{dz_1}{z_1}\frac{dw_1}{w_1}\frac{dx_2}{x_2}\frac{dy_2}{y_2}\frac{dz_2}{z_2}\frac{dw_2}{w_2} =$$

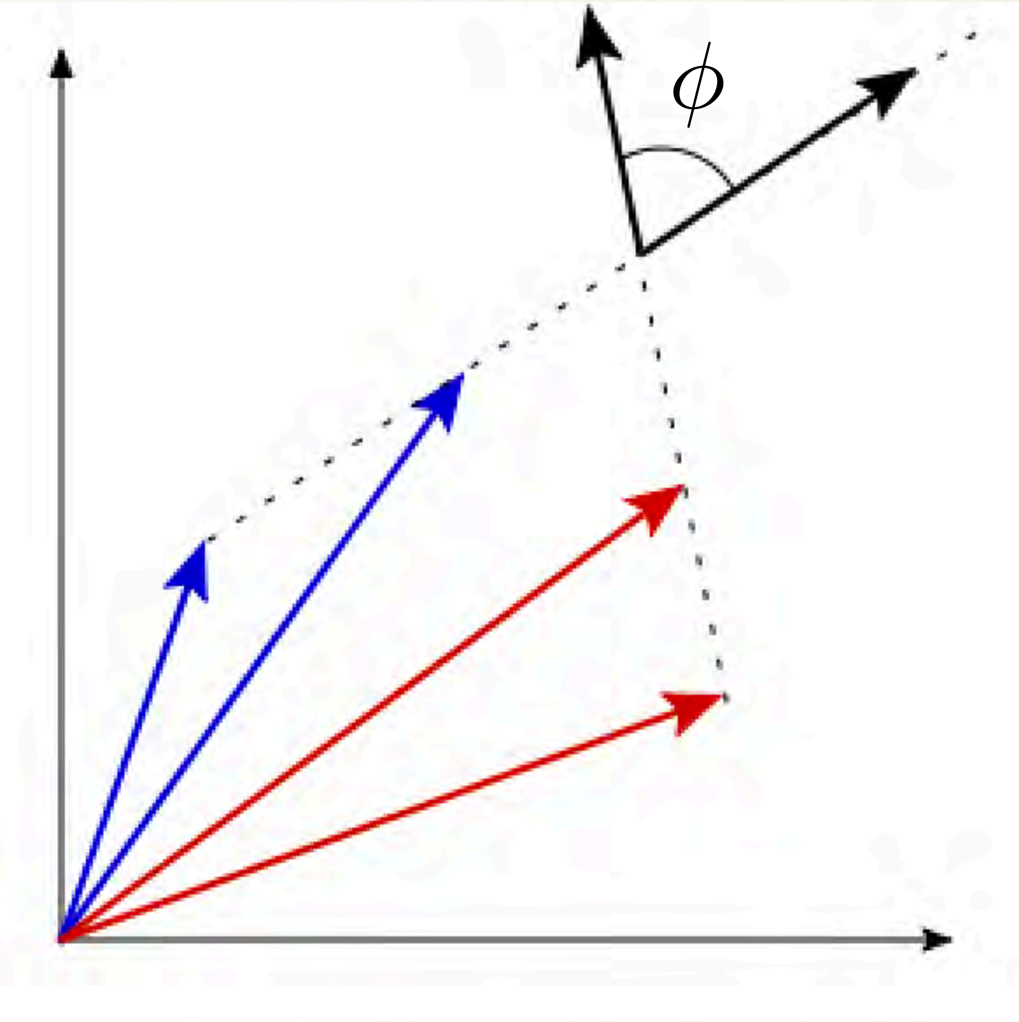
 $\times$ 

High school problem $gg \to gg$

- Positive quadrant
- Vectors

$$\vec{a}_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \quad \vec{b}_1 = \begin{pmatrix} z_1 \\ w_1 \end{pmatrix}$$

$$\vec{a}_2 = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \quad \vec{b}_2 = \begin{pmatrix} z_2 \\ w_2 \end{pmatrix}$$



- Impose: $(\vec{a}_2 - \vec{a}_1) \cdot (\vec{b}_2 - \vec{b}_1) \leq 0$ $\qquad \phi > 90^o$

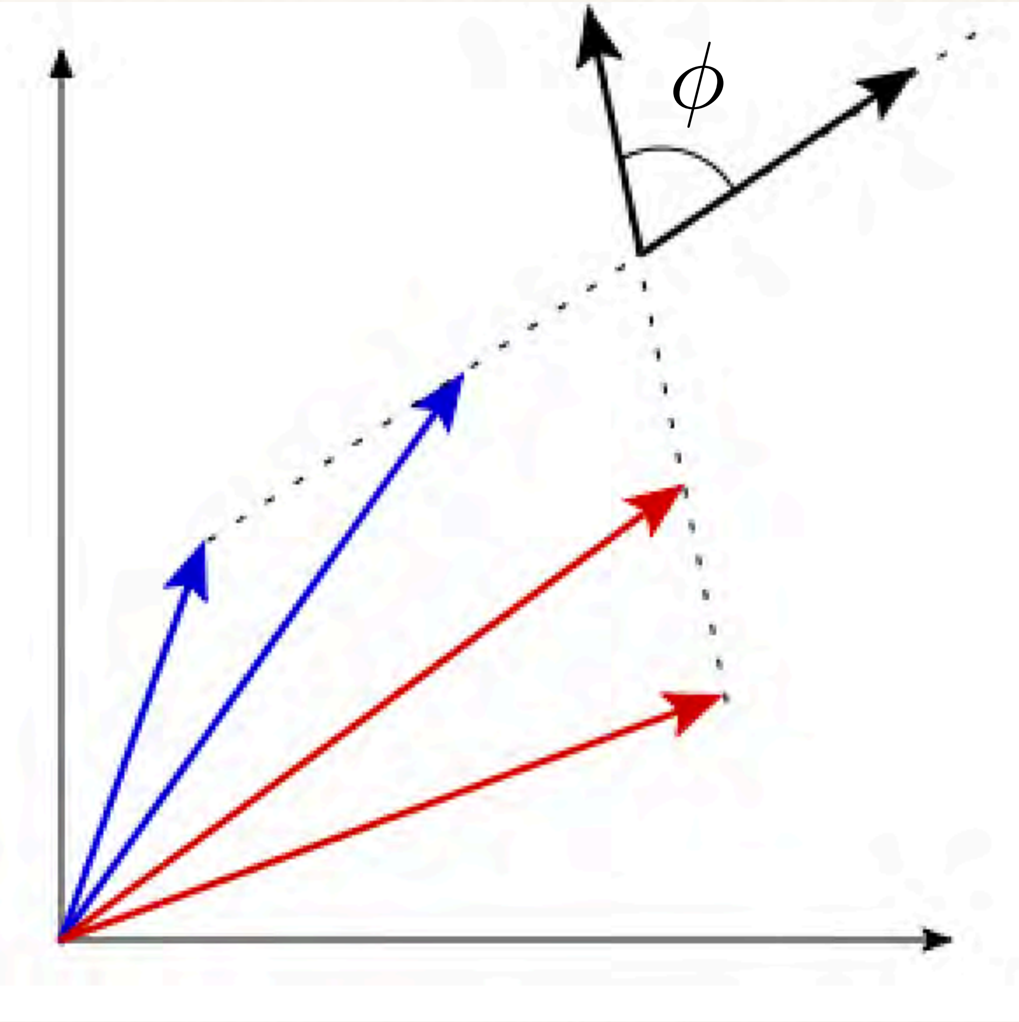
Subset of configurations allowed: triangulate

High school problem $gg \to gg$

- Positive quadrant
- Vectors

$$\vec{a}_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \quad \vec{b}_1 = \begin{pmatrix} z_1 \\ w_1 \end{pmatrix}$$

$$\vec{a}_2 = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \quad \vec{b}_2 = \begin{pmatrix} z_2 \\ w_2 \end{pmatrix}$$



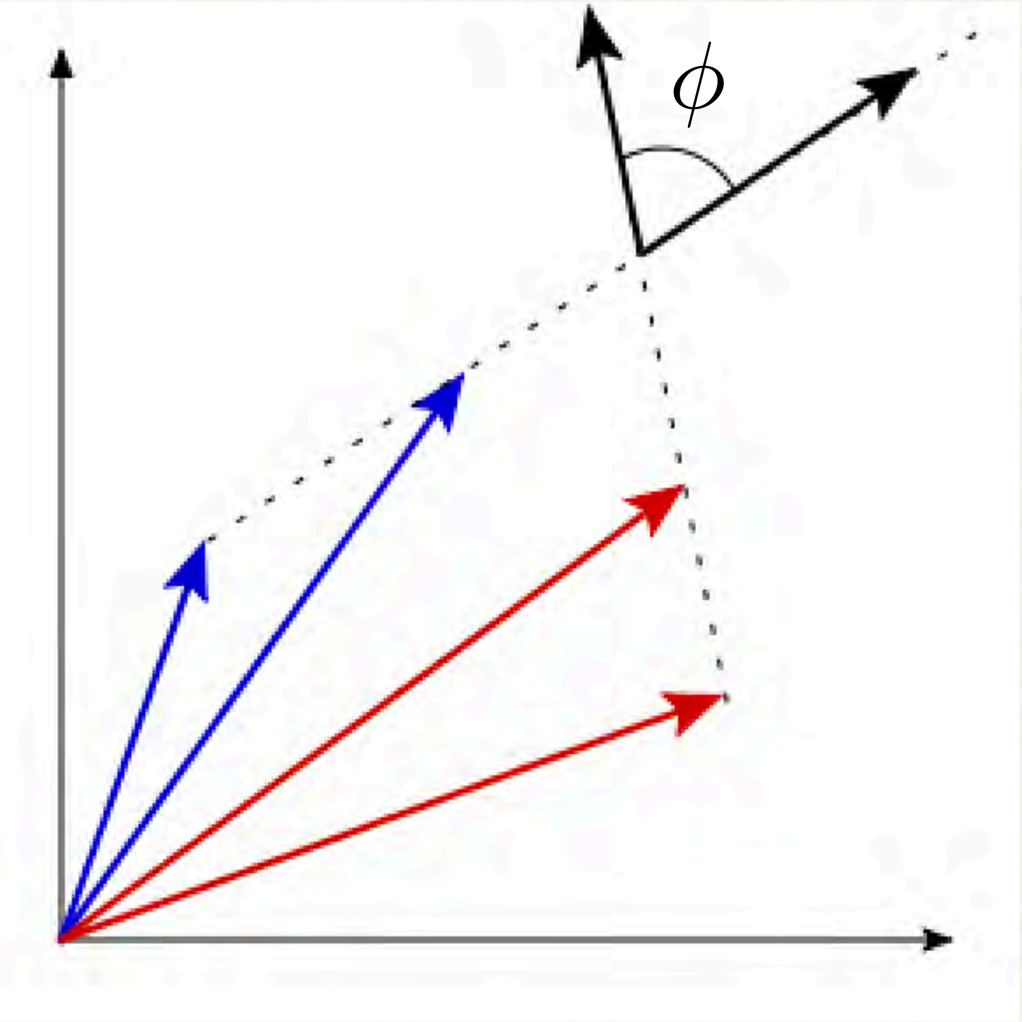
$$\mathrm{Vol}\,(2) = \frac{dx_1}{x_1}\frac{dy_1}{y_1}\frac{dz_1}{z_1}\frac{dw_1}{w_1}\frac{dx_2}{x_2}\frac{dy_2}{y_2}\frac{dz_2}{z_2}\frac{dw_2}{w_2}\left[\frac{\vec{a}_1 \cdot \vec{b}_2 + \vec{a}_2 \cdot \vec{b}_1}{(\vec{a}_2 - \vec{a}_1) \cdot (\vec{b}_2 - \vec{b}_1)}\right]$$

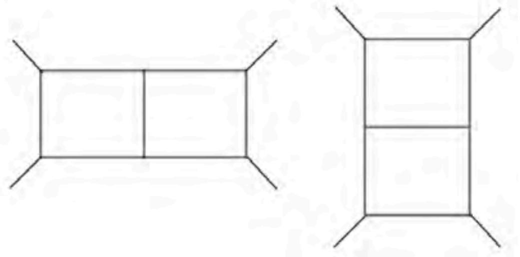
High school problem $gg \to gg$

- Positive quadrant
- Vectors

$$\vec{a}_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \quad \vec{b}_1 = \begin{pmatrix} z_1 \\ w_1 \end{pmatrix}$$

$$\vec{a}_2 = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \quad \vec{b}_2 = \begin{pmatrix} z_2 \\ w_2 \end{pmatrix}$$



$\mathrm{Vol}\,(2) =$ 

High school problem $gg \to gg$

- Positive quadrant
- Vectors

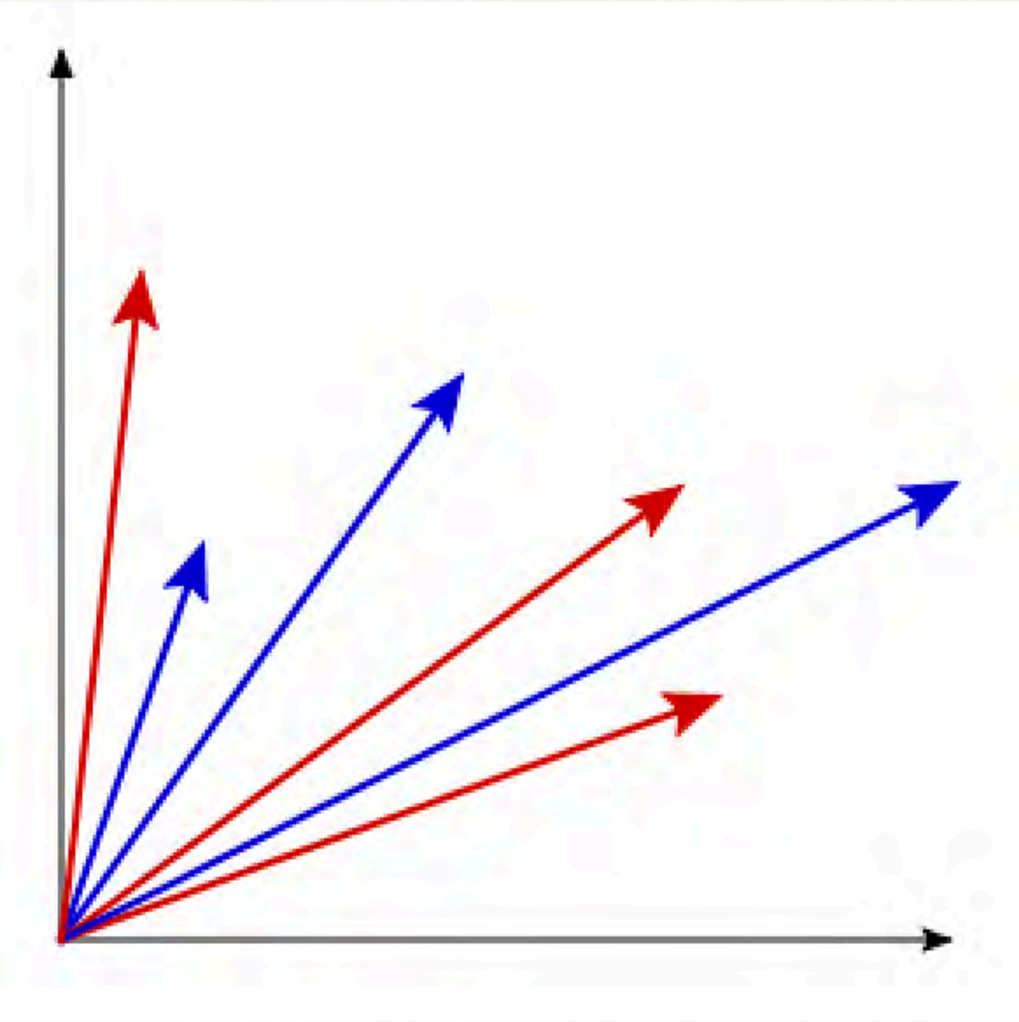
$$\vec{a}_1, \vec{a}_2, \vec{a}_3 \qquad \vec{b}_1, \vec{b}_2, \vec{b}_3$$

- Conditions

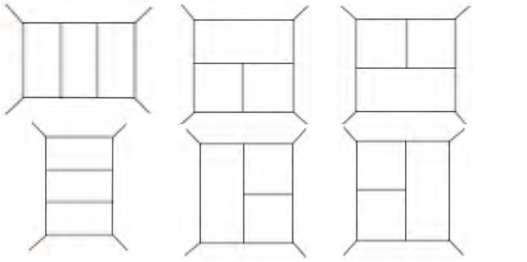
$$(\vec{a}_1 - \vec{a}_2) \cdot (\vec{b}_1 - \vec{b}_2) \leq 0$$

$$(\vec{a}_1 - \vec{a}_3) \cdot (\vec{b}_1 - \vec{b}_3) \leq 0$$

$$(\vec{a}_2 - \vec{a}_3) \cdot (\vec{b}_2 - \vec{b}_3) \leq 0$$



$$\text{Vol}\,(3) =$$



High school problem $gg \to gg$

- Positive quadrant
- Vectors

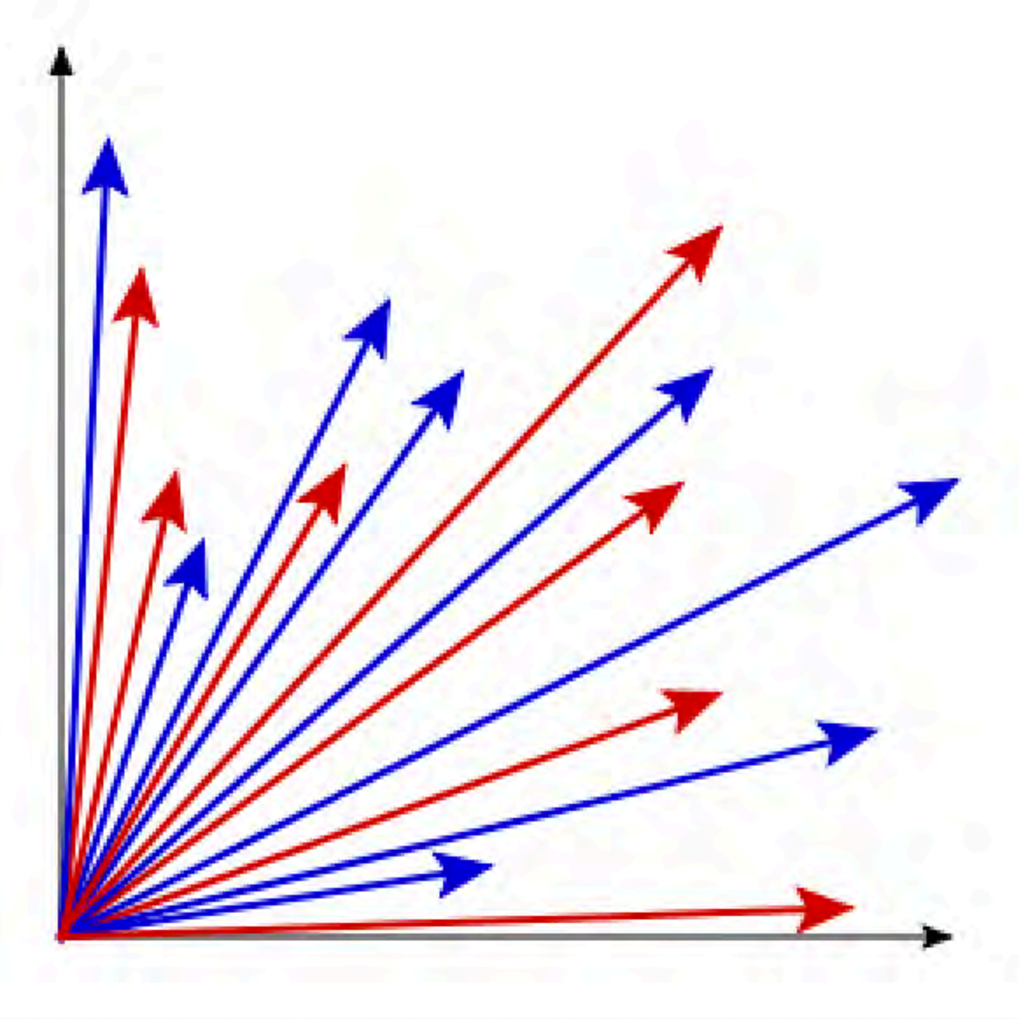
$\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_\ell \quad \vec{b}_1, \vec{b}_2, \ldots, \vec{b}_\ell$

- Conditions

$$(\vec{a}_i - \vec{a}_j) \cdot (\vec{b}_i - \vec{b}_j) \leq 0$$

for all pairs i, j

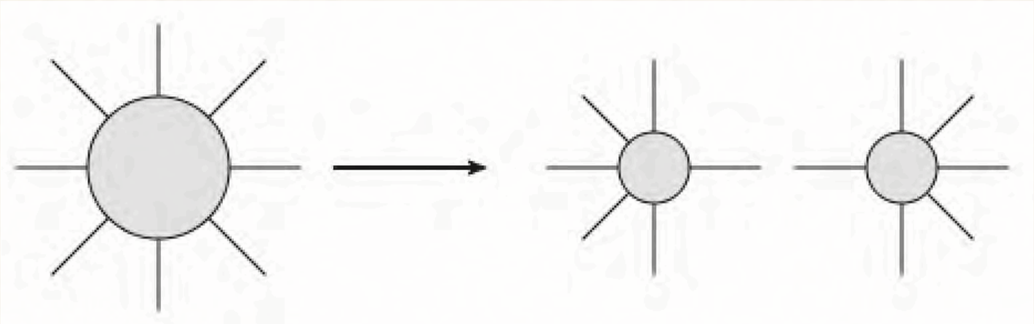
Let me know if you solve it!

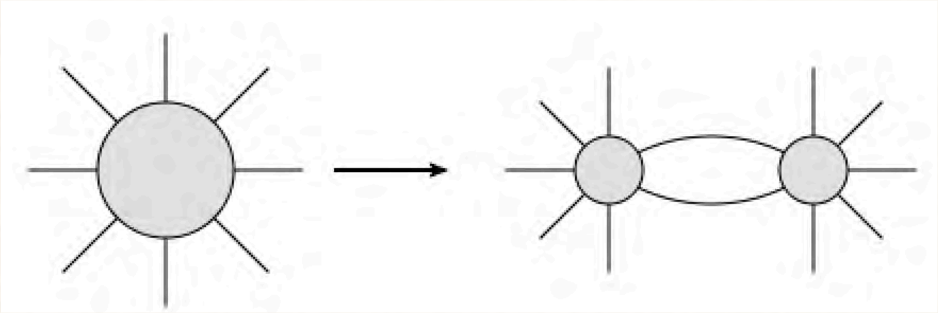


$$\text{Vol}\,(\ell) = \ldots\ldots$$

Why true?

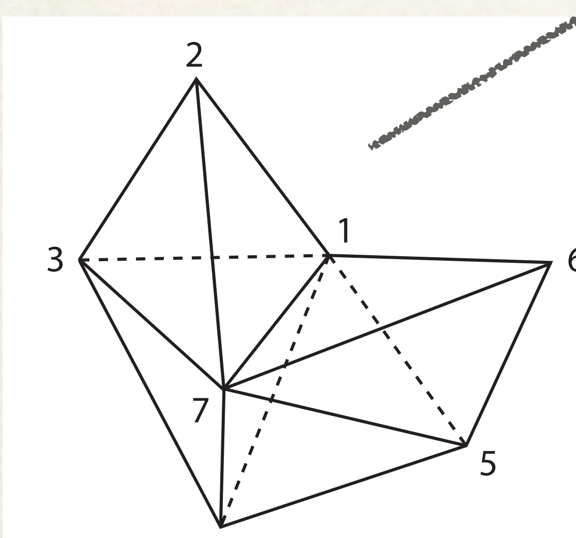
- No QFT proof because it is not QFT but geometry
- It is correct: the result satisfies locality and unitarity

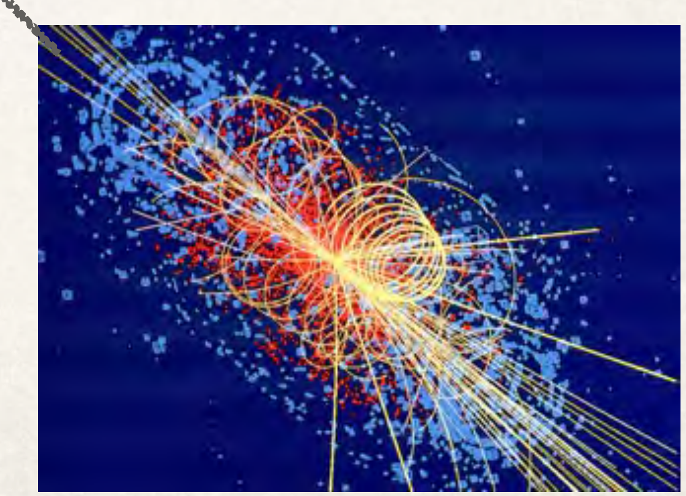




- Also check against results in the literature

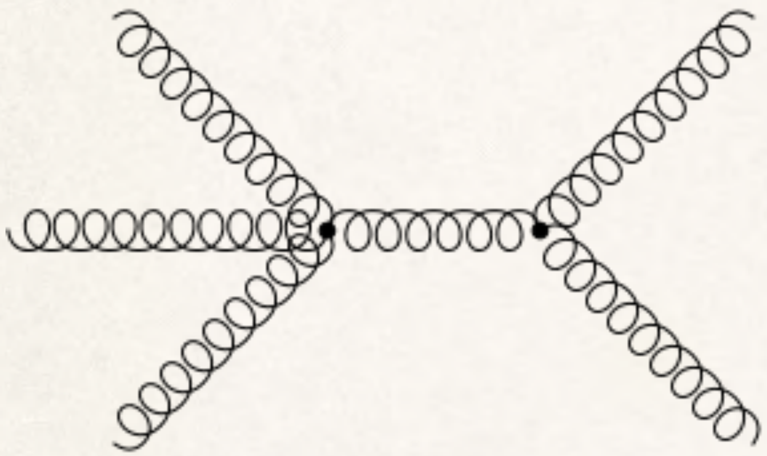
Amplituhedron

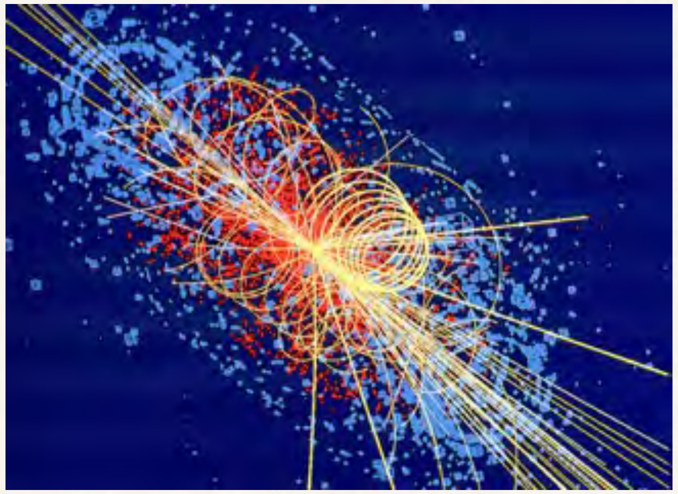




Physics vs geometry

- Dynamical particle interactions in 4-dimensions

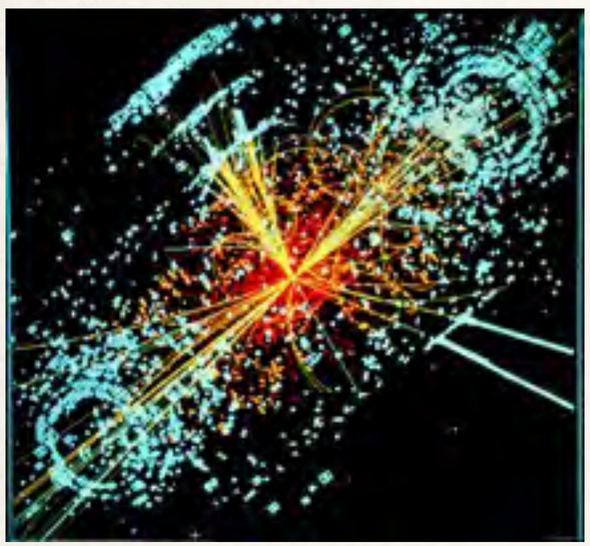




- Static geometry in high dimensional space

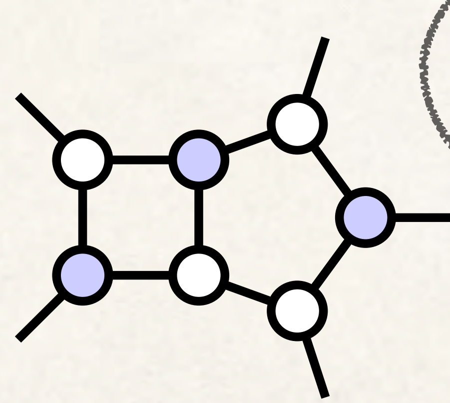
At the intersection

- Fascinating connection between fields which have never interacted so far



Predictions of particle collisions for experiments

Combinatorics
Algebraic geometry



$$\begin{pmatrix} * & * & * & * & * \\ * & * & * & * & * \end{pmatrix} \in G_+(2,5)$$

Amplitudes

- This is one of the directions in fast developing field
- Scattering equations, BCJ duality, string amplitudes, supergravity finiteness, hexagon bootstrap, cluster polylogarithms, LHC calculations,…..

Amplitudes

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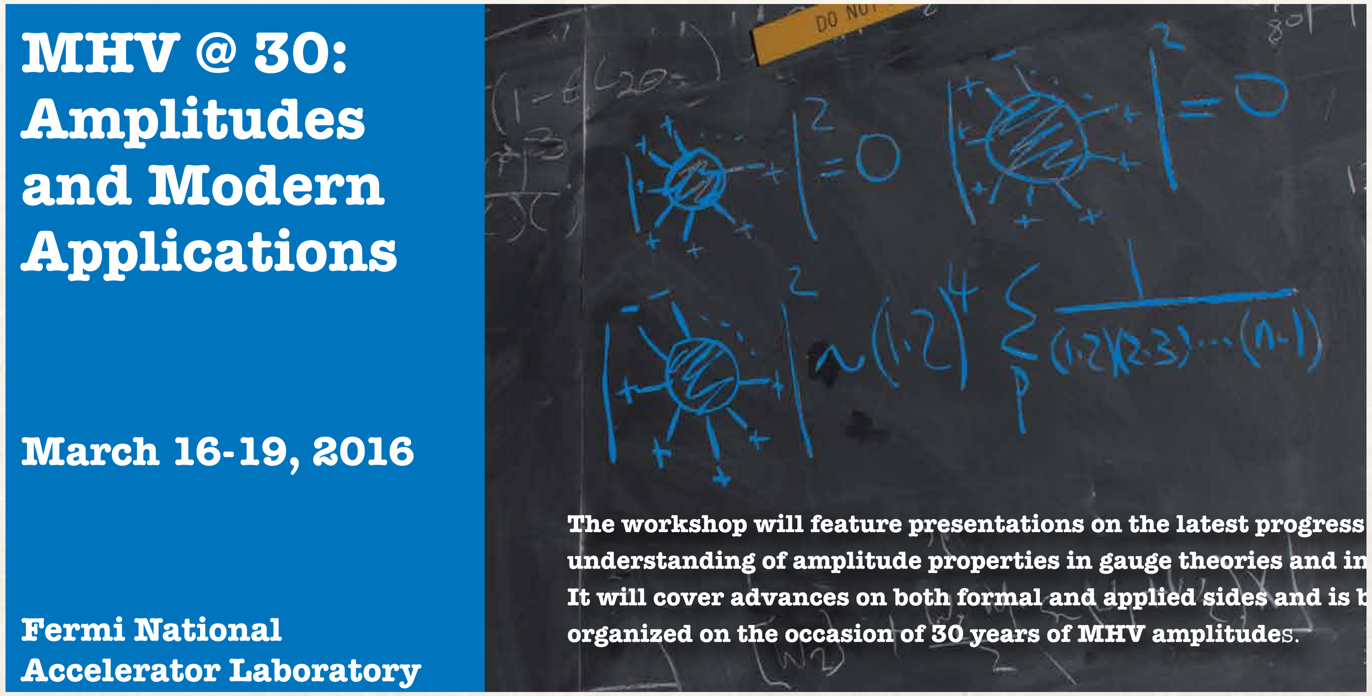
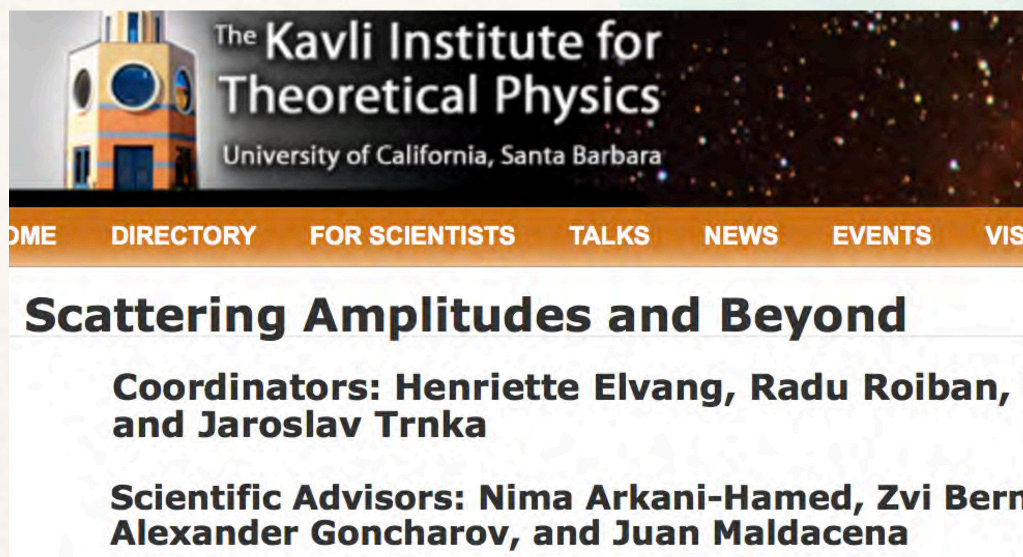


Amplitudes


6–10 july
amplitudes 2015
INTERNATIONAL CONFERENCE
ZURICH


Amplitudes 2016
Nordita, Stockholm
July 4-8, 2016
NORDITA
www.nordita.org/amplitudes2016/

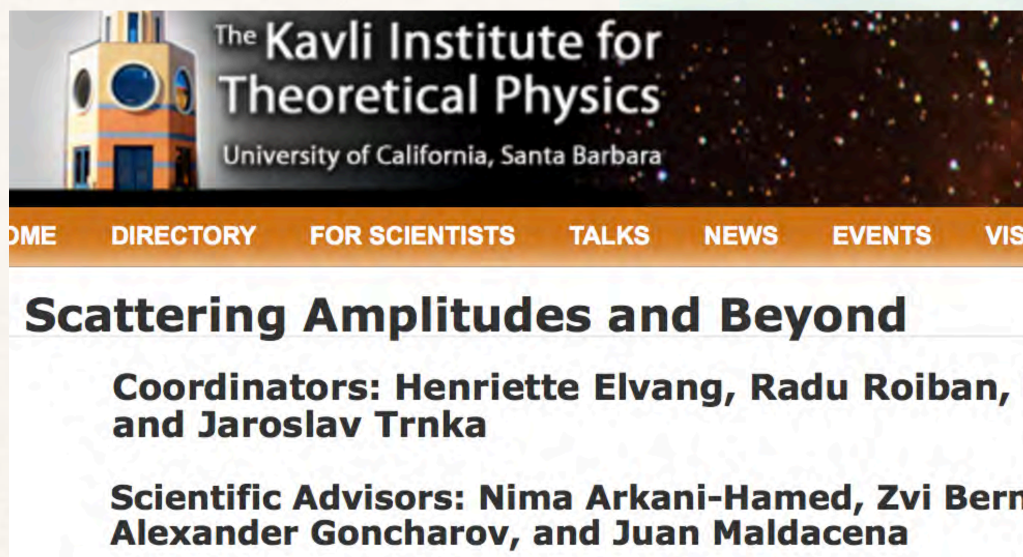

The Kavli Institute for Theoretical Physics
University of California, Santa Barbara
DIRECTORY FOR SCIENTISTS TALKS NEWS EVENTS
Scattering Amplitudes and Beyond
Coordinators: Henriette Elvang, Radu Roiban,
and Jaroslav Trnka
Scientific Advisors: Nima Arkani-Hamed, Zvi Bern,
Alexander Goncharov, and Juan Maldacena


HKUST Jockey Club INSTITUTE FOR ADVANCED STUDY
IAS Focused Program on
Scattering Amplitudes
in Hong Kong
17 – 21 November 2014


Amplitudes in Asia 2015
Welcome to Physics Division,
National Center for Theoretical Science


School and Workshop on Amplitudes in Beijing 2016
Date : From 2016-10-10 To 2016-10-21
开放交融
求真创新

Amplitudes



See you there!

Thank you for your attention