Okinawa Conformal Bootstrap Lectures (Slava Rychkov)

Rough plan 1. Physical foundations 2. Conformal blocks 3. Bootstrap applications
Sources:
-About CFTs:
EPFL lectures at https://sites.google.com/site/slavarychkov/home,
Joshua Qualls 1511.04074
-About bootstrap:
Sheer El-Showks lectures at the last year's Asian Winter school
<http://home.kias.re.kr/MKG/upload/9thasianwinterschool/BootstrapLectures.pdf>
David Simmons-Duffin's TASI 2015 lectures <https://sites.google.com/a/colorado.edu/tasi-2015-wiki/lecture-topics/conformal-bootstrap>

Lecture 1

1 Physical foundations

An important unsolved problem in theoretical physics: *find an efficient algorithm to solve strongly coupled QFTs.*

Relevant for high-energy, stat., cond-mat.

E.g. lattice QCD is used to find spectrum of low lying hadrons - a year of supercomputer time. Can we do better?

QFTs come in various types:
- relativistic and nonrelativistic
- with mass gap (like pure Yang-Mills) or gapless

These lectures: relativistic QFTs which are fixed point of RG flow. Spectrum invariant under rescalings $x \to \lambda x$, $E \to \lambda^{-1} E$ (scale invariance). So two possibilities:
- massless particle (free)
- continuous spectrum (interacting theories)

We will be mostly working in Euclidean signature so for us relativistic = rotationally invariant.

Relativistic fixed points are more tractable because they have an emergent symmetry - conformal invariance (CFT).

More precisely **the following seems to be true:** Suppose the theory is (a) relativistic (b) scale invariant (c) local, i.e. has a local $T_{\mu\nu}$. Then *generically* the theory is conformally invariant. "Generically" because there are a few known counterexamples.

Remark: all known counterexamples are free theories and/or nonunitary. Can we remove "generic" by requiring that the theory is not free? (open problem) Or by requiring that it's unitary (known in 2d)

Notice that unitarity assumption is natural for high energy and quantum cond-mat, but not necessary for statistical physics. More on this later.

Physical reason for the emergence of conformal invariance.

First of all recall the definition of conformal transformations. These are coordinate transformations $x \to x' = f(x)$ which are locally compositions of rotations and dilatations, i.e. Jacobian

$$J^\mu{}_\nu = \partial f^\mu / \partial x^\nu = \lambda(x) M^\mu{}_\nu$$

where $M \in O(d)$ is an orthogonal matrix. <Figure>

We can view this transformation as a "local RG transformation" where the scale factor is x-dependent. It's natural to assume that locality + RG invariance $\to$ inv. under such transformations. Hard to make this argument rigorous (except for perturbative models).

The last word on "Scale invariance implies conformal invariance" has not been said yet. See the review by Nakayama 1302.0884, and section 4 of 1509.00008.

2 $d = 2$ vs $d > 2$

Many of you are familiar with CFT in $d = 2$.

Main differences for $d > 2$:

- no holomorphic-antiholomorphic factorization (since conformal group does not factorize)

- finite dimensional conformal algebra (Poincare+dilatations+SCTs). This is like using $SL(2, C)$ subalgebra of Virasoro (the one generated by $L_n, \bar{L}_n$ for $n = 0, \pm 1$) which generates global conformal transformations.

The absence of higher generators has a number of consequences:

- the primary in $d > 2$ is like a quasiprimary in $d = 2$. Conformal multiplets are thus "smaller" in $d > 2$ (recall that in 2d conformal multiplet contains infinitely many quasiprimary multiplets)

- in 2d the stress tensor is the Virasoro descendant of the identity: $T = L_{-2}1$, in $d > 2$ it will be a primary

- central charge c in 2d has an algebraic meaning (appears in Virasoro commutation relations). It can also be defined as the coefficient in TT two point function:

$$\langle TT \rangle = c/z^4$$

Only the second interpretation survives in $d > 2$.

- in $d > 2$ no way to construct degenerate fields (i.e. fields whose descedant is null and can be set to zero). E.g. in $d = 2$ primary field ϕ of weight h has a descendant

$$(L_{-2} - \frac{3}{2(2h+1)} L_{-1}^2)\phi$$

which is null provided that

$$h = \frac{5 - c \pm \sqrt{(c-25)(c-1)}}{16}$$

This condition and its higher-level analogues play a crucial role in the construction of the 2d minimal models. In d>2 we can't play similar tricks.

- as a consequence in $d > 2$ there are no known analogue of minimal models, i.e. models which have a finite number of primary fields. In fact any CFT in $d > 2$ contains an infinite number of primaries (just like in $d = 2$ we always have an infinite number of quasiprimaries). In $d > 2$ it's not known how to unite conformal primary multiplets into larger multiplets which would be few (or perhaps finite) in number.

In $d = 2$ there are two main classes of solved CFTs (apart from free theories):

(a) minimal models and their cousins (rational CFTs), which have a finite number of primaries

(b) a very special irrational theory - Liouville theory - and its cousins. Analysis of irrational 2d CFTs, which have infinitely many primaries, typically leads to difficulties which are somewhat similar to the ones for $d > 2$ CFTs.

In $d > 2$ there is not a single CFT (apart from free theories) which would be solved in the same sense.

($\mathcal{N} = 4$ SYM at large N is a very special case and even it has not yet been fully solved)

Properties common to $d > 2$ and $d = 2$:

- 3pt functions of primaries are fixed up to a finite number of constants
- 4pt functions of primaries are fixed up to a function of two conf. invariant cross ratios u, v
- convergent OPE
- radial quantization

3 Examples of $d > 2$ CFTs

There are many examples of CFTs in $d > 2$ which we think exist (with various degree of certianty) but cannot solve.

3d Ising model and $O(N)$ models

These models are defined on the lattice by writing a statistical partition function. We have N-component spins s_i, $|s_i| = 1$, at each lattice site interacts ferromagnetically with nearest neighbors: $E = -J \sum_{\langle ij \rangle} s_i s_j$. If $N = 1$ we have the Ising model, if $N > 1$ the $O(N)$ model. At low temperatures the model is ordered and the spin has a vev; at high temperatures it's disordered. There is a critical temperature $T = T_c$ at which the model has a 2nd order phase transition. This temperature is $O(J)$ but its precise value is not universal and not interesting (e.g. it depends on the lattice, on whether we have some small next-to-nearest coupling etc). On the other hand the theory at $T = T_c$ is universal - it's a CFT with global symmetry Z_2 ($N = 1$) and $O(N)$ ($N > 1$) which depends only on N but not on other microscopic details. This transition is observed experimentally in N-axial ferromagnets ($N = 1, 2, 3$). The superfluid phase transition in liquid He^4 belongs to the same universality class ($N = 2$) - this is related to the fact that the superfluid component wavefunction $\Psi(x)$ is complex. More interestingly, the liquid-vapor critical point in ordinary fluids such as water belongs to the same universality class ($N = 1$).

The above facts are best understood by Wilson's RG. RG in strongly coupled theories is a qualitative theory - it's great for mapping out the phase diagram and for constructing approximate fixed point solutions, but it is not a precision tool. The hope is that CFT can do better. We'll see.

RG logic can also be used to count the number of relevant scalar primary operators in the CFT, i.e. operators with dimension $\Delta < d$, by counting the number of microscopic parameters needed to finetune to get to the critical point. E.g. for the 3d Ising model one needs to finetune one parameter (temperature), and this means that there is exactly one relevant Z_2-even scalar: $n_+ = 1$. Turning on the magnetic field destroys the critical point, and so there is at least one relevant Z_2-odd scalar: $n_- \geq 1$. Finally, consider the liquid-vapor critical point. To get to it we have to finetune two microscopic parameters: P, T. The system does not have Z_2 at the microscopic level - this symmetry only emerges at the transition. This means that the total number of relevant scalars is exactly 2: $n_+ + n_- = 2$. To summarize, the 3d Ising model critical point has precisely one Z_2-even and one Z_2-odd relevant scalar operator. This will play a role later.

Notice that the Ising model ($N = 1$) critical point (but not for $N > 1$) exists also for $d = 2$ where it's exactly solvable (it's the minimal model CFT $M_{3,4}$).

Wilson-Fisher fixed point

Consider the quartic scalar $O(N)$ field theory in d-dimensions:

$$(\partial \vec{\phi})^2 + m^2 (\vec{\phi})^2 + \lambda (\vec{\phi})^4$$

Suppose $d < 4$ so that the quartic is relevant. It has dimension $4 - d$: $\lambda = \mu^{4-d}$. What is the low-energy physics of the model? It depends on the dimensionless ratio $G = m^2/\mu^2$. If $G \to +\infty$ (large positive mass) the theory is that of weakly interacting massive particles. If $G \to -\infty$ (large negative mass), the $O(N)$ symmetry will be spontaneously broken. One can expect that there will be a critical G separating these two phases where the theory will flow in IR to a conformal fixed point, called the Wilson-Fisher fixed point.

Wilson and Fisher showed that this is true in $d = 4 - \epsilon$ dimensions. However this fixed point extends to $2 \leq d < 4$ ($N = 1$) and $2 < d < 4$ ($N > 1$). By universality, for integer $d = 2, 3$, this CFT is identical to the d-dim $O(N)$ model critical point.

It's not 100% known if the Wilson-Fisher fixed point for non-integer d makes sense nonperturbatively. See 1512.00013 for a recent discussion.

Lee-Yang fixed points

Consider the cubic field theory with purely imaginary linear and cubic coupling:

$$(\partial \phi)^2 + ih\phi + ig\phi^3$$

Quadratic coupling is not included since in presence of a cubic one it can be removed by a shift in ϕ. Notice that if the cubic coupling is real it's hard (or impossible) to make sense of the cubic theory (unbounded potential). For the imaginary coupling it seems to exist (potential is oscillating, not unbounded). A fixed point can be reached by keeping g fixed and finetuning the value of h. The cubic coupling becomes marginal in $d = 6$. The Wilson-Fisher-like analysis gives a fixed point at $d = 6-\epsilon$ at a real value of g. It is believed that this fixed point extends for $2 \le d < 6$. Coming from a theory with imaginary couplings, it is a non-unitary CFT. So some operators won't satisfy unitarity bounds. One can show that anomalous dimensions of simple operators are real in perturbation theory around $d = 6$. For $d = 2$ this model concides with the non-unitary minimal model $M_{2,5}$ of negative central charge $c = -22/5$. The field ϕ is the only primary field (apart from 1), it has conformal weight $h = -1/5 < 0$. So all operators have real dimensions.

Why is this interesting (apart from formally)?

It's related to the dependence of magnetization M on magnetic field H in the d-dimensional Ising model. At the Ising critical point we have

$$M \sim H^\sigma, \quad \sigma = \frac{\Delta_\phi}{d - \Delta_\phi}$$

At $T > T_c$ we have an smooth dependence along the real axis:

$$M \sim H(H^2 + H_0^2)^{\sigma'}$$

where we have branch points at the imaginary magnetic field $H = iH_0$. The strength of the singularity is the new critical exponent which is given by the same formula as σ above except the ϕ dimension is at the Lee-Yang fixed point (Padua L2).

Fixed points of gauge theory with matter

Consider the 4d $SU(N_c)$ gauge theory with N_f massless fundamental fermions. If N_f is small this theory has chiral symmetry breaking and massless Goldstone bosons. If it's very large the beta function changes sign and the theory is not asymptotically free. There is an interval of N_f ("conformal window") where the theory instead flows to an IR fixed point (Banks-Zaks fixed point). This observation was behind some theories of BSM physics (technicolor), largely rendered irrelevant by the Higgs boson discovery. The size of conformal window is not exactly known (except in SUSY cases).

Here is a simpler 3d example. Consider QED3 - 3d $U(1)$ gauge theory coupled to N_f massless fermions. The 3d gauge coupling is relevant, so this theory is UV complete for any N_f. For small N_f there is chiral symmetry breaking. For N_f above some critical value (again unknown) this theory is believed to flow to a fixed point.

QED3 is relevant for condensed matter. Consider quantum antiferromagnet on a 2d kagome lattice. Because of frustration, such systems (of which the mineral herbertsmithit is an example) are believed to remain disordered (i.e. **not** to acquire Neel order) down to $T = 0$. This behavior is called "spin liquid". Moreover an effective theory is constructed which contains $N_f = 4$ two-component massless fermions interacting with a compact $U(1)$ gauge field. This gauge field is first introduced as a redundancy of description representing the Heisenberg spin operators on site i in terms of "slave fermions" $f_{i\alpha}$ (also called spinons) which carry spin index α:

$$\vec{S}_i = \frac{1}{2} f^\dagger_{i\alpha} \vec{\sigma}_{\alpha\beta} f_{i\beta}$$

This description is redundant under the local gauge transformation multiplying spinons by a site-depending phase, hence gauge field. However upon integrating high-energy degrees of freedom, gauge field acquires dynamics of its own. Moreover, the low-energy dynamics is believed to be RG-attracted to a 2+1 Lorentz-invariant QED3. Such spin liquids with long-range power-like correlations are called "algebraic spin liquids". See 0803.1150, 1508.06278.

Lecture 2

4 CFT observables

We will be studying CFT in $d > 2$ in flat Euclidean space.

The most familiar observable in CFT is a correlation function of local operators. There are other observables but they won't be studied here, e.g.:

-Defects and boundary conditions, see 1210.4258, 1310.5078,1502.07217

-Wilson loops in conformal gauge theories

-Hofman-Maldacena outgoing charge and energy fluxes at infinity, 0803.1467

Correlation functions can be all reduced to "CFT data" which consists of

(1) a list of local operators, each characterized by scaling dimensions Δ ($\Delta = h + \bar{h}$ in 2d) and transforming in an irrep of Lorentz and of the global symmetry group (if any). For example in 3d any operator will be a symmetric traceless tensor of the rotation group.

(2) their OPE coefficients

Remark: we will assume conformal symmetry is NOT spontaneously broken. This condition is satisfied in all known phenomenologically relevant CFTs. (A famous conjecture: one cannot have spontaneous conformal symmetry breaking without supersymmetry.) The only operator which has a vev is the unit operator: $\langle 1 \rangle = 1$. The other operators are normalized so that their two point function has unit coefficient. E.g. for scalars:$\langle \phi(x)\phi(0)\rangle = |x|^{-2\Delta}$.

Remark: there are CFTs called "logarithmic" where such diagonal normalization is impossible. We will not consider them in our lectures. See Cardy 1302.4279.

For the stress tensor and conserved currents one adopts a different normalization based on the Ward identities.

Remark: General constraints arising from putting CFTd in curved metric and on different topology are practically virgin territory. The easiest open case would be to study CFT on $R^{d-1} \times S^1$ which would correspond to CFT at finite temperature.

Remark: We will assume the spectrum is discrete. Unlike in 2d, in $d > 2$ there are no known CFTs with continuous spectrum (it would be nice to prove this). Naively, possibility of contunuous spectrum in 2d is intimately related to the fact that free scalar has dimension zero, and one can sigma-models with noncompact target spaces. This construction is impossible in $d > 2$.

5 OPE

One should distinguish between "leading OPE" and "conformal OPE". Leading OPE includes only the leading, most singular term, whose form can be fixed by rotation invariance and scaling. E.g. for a symmetric traceless rank l operator appearing in the OPE of two scalars it has the form:

$$O_1(x) \times O_2(0) = \sum_k \frac{f_{12}^{\ \ k}}{|x|^{\Delta_! + \Delta_2 - \Delta_k}} x^{\mu_1} \cdots x^{\mu_l} O_{k,\mu_1\cdots\mu_l}(0) + \dots$$

NB: For external nonscalar operators there can be several (but finitely many) leading OPEs consistent with Lorentz because one can create indices with x_μ and $\delta_{\mu\nu}$. E.g. for scalar in vector $\times$ vector there are two possibilities:

$$V_\mu(x) \times V_\nu(0) \supset \delta_{\mu\nu}\phi,\ x_\mu x_\nu \phi$$

Leading OPE is **not** conformally invariant. Acting on it with K_μ, in the LHS we get something nonzero as $[K_\mu, O_1(x)] \neq 0$ for $x \neq 0$, while the RHS is annihilated (primary O_k). The discrepancy can be corrected by adding to the RHS terms proportional to $\partial_\nu O_k$ and higher derivatives. Such terms are anyway expected to appear in the OPE if the leading term appears (e.g. if we insert O_k not at zero but at middle point $x/2$ and then Taylor-expand we will generate such terms). These terms are also not annihilated by K_μ. It's possible to add them, with precisely fixed coefficients, in a way that the whole OPE becomes conformally invariant. Conformally invariant OPE can thus be written in the form (omitting indices):

$$O_1(x)O_2(y) = \sum_k f_{12}^{\ k} P(x-y, \partial_y) O_k(y)$$

where P is an infinite series in derivatives. Conformal invariance can be expressed in the form:

$$[L_A, O_1(x)O_2(y)] = \sum_k f_{12}^{\ k} P(x-y, \partial_y)[L_A, O_k(y)]$$

where L_A is any conformal generator. This equation can be used to fix all higher coefficients in P. We don't give equations how conformal generators act on primaries but they can be found in any review.

Remark: There is another way to fix the subleading OPE terms, starting from the conformally invariant 3pt function $\langle O_1 O_2 O_k \rangle$. This is easier in practical computations, but conceptually the above way of thinking is important and it will appear below.

Remark: In 2d CFT courses one oftens pays particular attention to the leading, singular, OPE terms, omitting the subleading ones. E.g. one often sees the leading OPE

$$T(z)T(0) = \frac{c/2}{z^4} + \frac{2T(0)}{z^2} + \frac{\partial T(0)}{z} + \dots$$

which leads to the Virasoro algebra, but one rarely hears about the subleading terms. On the contrary, these subleading terms, and the fact they are fixed, are absolutely crucial for the conformal bootstrap.

6 Conformal blocks

As is well known the functional form of 3pt functions is fixed, and the overall coefficient is the same as the OPE coefficient f_{ijk}. New information may be expected to come from the 4pt functions. We can compute the 4pt functions using the CFT data and the OPE:

$$\langle O_1(x_1)O_2(x_2)O_3(x_3)O_4(x_4)\rangle = \sum_k f_{12k} f_{34k} P(x_{12}, \partial_2) P(x_{34}, \partial_4) \langle O_k(x_2) O_k(x_4)\rangle$$

$$= \sum_k f_{12k} f_{34k} G(x_1, x_2, x_3, x_4)$$

where G's are called conformal partial waves (CPWs), or conformal blocks (CBs). Notice that CPWs have the same conformal invariance properties as the correlators:

$$L_A G \equiv (\sum L_A^{(i)}) G = 0$$

Proof: by conformal invariance of the OPE, we can move the generators $L_A^{(1,2)}$ acting at 1,2 to the operator $O_k(x_2)$ and those at 3,4 to $O_k(x_4)$. The statement then follows by the conf. invariance of the two point functions. See 1109.6321.

For correlators of 4 scalars CPWs, as the 4pt functions themselves, will involve functions of conformal cross ratios u, v (up to a trivial kinematical factor). How to find those functions?

Method 1. Use the above definition (quite laborious and impractical)

Method 2. **Casimir differential equation.**

Consider conformal generators acting only on points 1,2: $L_A^{(1)} + L_A^{(2)}$, and form the quadratic Casimir:

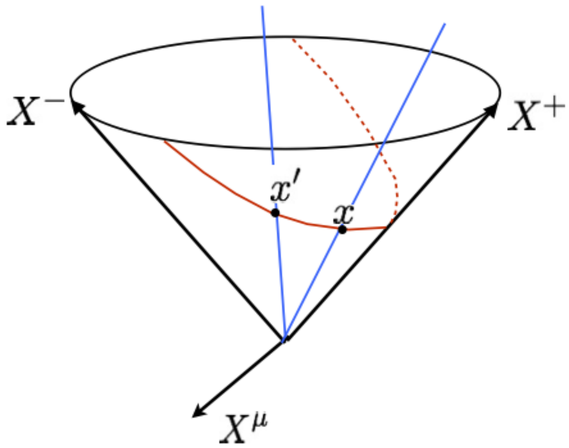
$$(L_A^{(1)} + L_A^{(2)})(L^{(1)A} + L^{(2)A})$$

This is a second order differential operator. On the other hand each factor can be pushed using the OPE on $O_k(x_2)$. When the quadratic Casimir acts on one operator it gives an eigenvalue. For symmetric traceless operators it's given by

$$C_{\Delta,l} = \Delta(\Delta - d) + l(l + d - 2)$$

We conclude that global conformal blocks satisfy 2nd order eigenvalue equations.

To actually work out the equation, it's useful to apply the "projective null cone" (aka "embedding") formalism (see 1107.3554). In this formalism the correlation functions are lifted from the Euclidean d-dim space to $(d+1)$-dim light cone $X^2 = 0$ in $d+2$ dim space of signature $(d+1, 1)$. In this space we will use the light cone coodinates (X^+, X^-, X^μ) so that $X^2 = X^\mu X^\mu - X^+ X^-$.



It's called projective light cone because the Eucldean points are identified with light rays, i.e. $X \sim \lambda X$. More precisely we can consider the Euclidean section of the cone:

$$(X^+, X^-, X^\mu) = (1, x^2, x^\mu)$$

One can check that the induced metric on this section is flat Euclidean.

Conformal group naturally acts as the group of isometries $SO(d+1, 1)$ on the light rays and this induces the action on Euclidean space. The fields are extended from the Euclidean section homogeneously:

$$\phi(\lambda X) = \lambda^{-\Delta}\phi(X)$$

The action on the fields is then as it should be. On the cone, the correlation functions should be written respecting homogeneity and $SO(d+1, 1)$ invariance. E.g. two point functions:

$$\langle \phi(X)\phi(Y) \rangle = \frac{1}{(X.Y)^\Delta}$$

Check that on the Euclidean section

$$(X.Y) = -\frac{1}{2}(x - y)^2$$

so it reduces to the usual formula up to a constant factor. The 3pt function is analogous (work it out). Finally the 4pt function is written as (four identical scalars for simplicity)

$$\langle \phi(X_2)\phi(X_3)\phi(X_4)\phi(X_5) \rangle = \frac{1}{(X_1.X_2)^\Delta (X_3.X_4)^\Delta} g(u, v)$$

$$u = \frac{(X_1 \cdot X_2)(X_3 \cdot X_4)}{(X_1 \cdot X_3)(X_2 \cdot X_4)}\,, \quad \text{and} \quad v = u|_{2\leftrightarrow 4} = \frac{(X_1 \cdot X_4)(X_2 \cdot X_3)}{(X_1 \cdot X_3)(X_2 \cdot X_4)}\,.$$

These have zero scaling and so an arbitrary function of u, v can appear. When projecting to the Euclidean section we get the usual conformally invariant cross ratios.

On the light cone conformal group generators are simply the $SO(d+1,1)$generators, acting as

$$L_{AB} = X_A \frac{\partial}{\partial X_B} - X_B \frac{\partial}{\partial X_A}$$

The Casimir equation takes the form

$$[\frac{1}{2}(L^{(1)}_{AB} + L^{(2)}_{AB})^2 + C_{\Delta,l}] \langle \phi(X_2)\phi(X_3)\phi(X_4)\phi(X_5)\rangle |_{\Delta,l} = 0$$

Working it out we get an equation for the function $g_{\Delta,l}(u,v)$. It's a 2nd order PDE, see Dolan, Osborn hep-th/0309180. This PDE has to be solved with particular boundary conditions coming from short-distance behavior for $x_2 - x_1 \to 0$ (i.e. consistency with leading OPE).

I will write down this equation shortly but first let us introduce some coordinates, better suited than u, v to imagine the relative positions of the four points. Conformal transformations allow to move these four points around, and we use this freedom to fix three points to $0, 1, \infty$ and use position of the fourth point as a coordinate. The polar angle does not matter, so just as in 2d we have two coordinates z and $\bar{z}$ (same number as the cross ratios u, v). In Euclidean space $\bar{z}$ and z are complex conjugates. <Figure>

Remark: In some applications it's interesting to consider analytic continuation to the Minkowski space when they become two independent real variables.

We have $u = |z|^2$, $v = |1-z|^2$. The Casimir equation in $z, \bar{z}$ coordinates takes the form:

$$[\mathcal{D} - \frac{1}{2}C_{\Delta,l}]g_{\Delta,l}(z,\bar{z}) = 0$$

$$\mathcal{D} = [z^2(1-z)\frac{\partial^2}{\partial z^2} - z^2\frac{\partial}{\partial z} + z \to \bar{z}] + (d-2)\frac{z\bar{z}}{z-\bar{z}}[(1-z)\frac{\partial}{\partial z} - z \to \bar{z}]$$

So we see that for $d = 2$ the Casimir equation factorizes and in fact its solutions are the products of hypergeometric functions:

$$g_{h,\bar{h}}(z\,\bar{z}) = k_h(z)k_{\bar{h}}(\bar{z}) + z \to \bar{z}, \quad k_h(z) = z^h\,{}_2F_1(h,h,2h,z)$$

For other even dimensions solutions in terms of hypergeometric functions can also be found, e.g. in $d = 4$:

$$g_{\Delta,l}(z,\bar{z}) = \frac{z\bar{z}}{z-\bar{z}}(k_{\frac{\Delta+l}{2}}(z)k_{\frac{\Delta-l-2}{2}}(\bar{z}) - z \to \bar{z})$$

In odd dimensions closed-form analytic expressions are unavailable, but one can always solve the Casimir differential equation order by order in a power series expansion with appropriate initial conditions. See below.

7 Series expansions

We will need to introduce radial quantization. Consider a 4pt function

$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle$$

and assume $|x_1|, |x_2| \leq 1 \leq |x_3|, |x_4|$. We can then decompose this correlator into matrix elements with states living on a sphere of radius 1:

$$\langle \ldots \rangle = \sum_{\Psi} \langle \phi(x_1)\phi(x_2)|\Psi\rangle \langle \Psi|(x_3)\phi(x_4)\rangle$$

We can choose a basis of states which have well-defined scaling dimension Δ and spin j. Every local operator of dimension Δ and spin j gives rise to a state by inserting it at the origin. This is the usual state-operator correspondence. Moreover the states will come in conformal multiplets, i.e. for each primary we will have states associated with O, $P_\mu O$, $P_\mu P_\nu O$ etc. We can introduce a projector on a given conformal multiplet:

$$|O| = \sum_{O, P_\mu O, P_\mu P_\nu O} |\alpha\rangle \mathcal{N}^{-1}_{\alpha\beta} \langle \beta|$$

where $\mathcal{N}_{\alpha\beta} = \langle \alpha|\beta\rangle$ is the Gram matrix (matrix of inner products). It's needed because the descendants states in the P_μ basis are not orthonormal. Then the CB of a given operator O is given by:

$$\langle \phi(x_1)\phi(x_2)|O|\phi(x_3)\phi(x_4)\rangle$$

We will now introduce a complex ρ coordinate by mapping four points to $\rho, -\rho, 1, -1$. This coordinate is related to z by

$$\rho = \frac{z}{(1+\sqrt{1-z})^2}$$

It's more convenient than z because points are inserted symmetrically and because $\rho < z$ leading to faster expansions. A given descendant of O at level n of dim $E = \Delta + n$ and spin j will contribute

$$\langle \phi(-\rho)\phi(\rho)|E, j\rangle \langle E, j|\phi(1)\phi(-1)\rangle$$

The first matrix element is

$$\langle 0|\phi(-\rho)\phi(\rho)|E, j\rangle = \frac{1}{\rho^{2\Delta_\phi}} \langle \phi(-\vec{n})\phi(\vec{n})|E, j\rangle \, r^E$$

$r = |\rho|, \vec{n} = \rho/r$, where we used how ϕ tranforms under dilatations ($\lambda^{-\Delta_\phi}\phi(\lambda x) = \lambda^{-D}\phi(x)\lambda^D$ where D is the dilatation generator). Furthermore we have

$$\langle \phi(-\vec{n})\phi(\vec{n})|E, j\rangle_{\mu_1\ldots\mu_j} = const(n_{\mu_1}\ldots n_{\mu_j} - traces)$$

Contracting two such tensors corresponding to vectors n_1 and n_2 we get Gegenbauer polynomials $C^{(\nu)}_j(n_1 \cdot n_2)$, $\nu = d/2 - 1$, which is the spherical harmonic in d dimensions (it's $\cos n\phi$ in $d = 2$, Legendre in $d = 3$ etc). Finally we get

$$g_{\Delta,l}(\rho = re^{i\phi}) = \sum_{n,j} B_{n,j} r^{\Delta+n} C^{(\nu)}_j(\cos\phi)$$

Conformal blocks are normalized by $B_{0,l} = 1$ which is the only spin present on level n. On higher levels we have spins up from $|l - n|$ up to $l + n$. In a unitary theory we have $B_{n,j} \geq 0$ at all levels. The precise numerical values of these coefficients can be plugged into the Casimir eq. and solving order by order.

In fact the leading term $r^\Delta C_l(\cos\phi)$ is already a very good approximation for CBs. But for precision calculations it's important to include higher order terms.

8 Recursion relations

Computing conformal blocks using one the above methods one finds that the subleading term coefficients are rational functions of Δ. So conformal blocks have poles in Δ. These poles come from inverting the Gran matrix $\mathcal{N}$. For these Δ these the Gram matrix at some level n has a zero eigenvalue: there a descendant of O which becomes null, i.e. orthogonal to all states (in particular it has zero norm).

Actually the norms of descendants can be all related to the norms of primaries by conformal algebra using $P_\mu^\dagger = K_\mu$, namely for

$$|\Psi\rangle = P_\mu P_\nu \dots |O\rangle$$

we have ($K_\mu = P_\mu^\dagger$)

$$\langle\Psi|\Psi\rangle = \langle O|\dots K_\nu K_\mu P_\mu P_\nu \dots |O\rangle$$

and we then commute all K_μ's past P's until they annihilate $O(0)$. In a unitary theory the norms of descendants are related to the norms of primaries by a positive factor. The conditions for this to happen take the form ("unitarity bounds")

$$\Delta \geq \Delta_{min}(l)$$

We have $\Delta_{min}(0) = d/2 - 1$ while for symmetric traceless $l \geq 1$:

$$\Delta_{min}(l) = l + d - 2$$

The fields which saturate these bounds are the free scalar and conserved currents. There are analogous bounds for fermions, antisymmetric tensors etc. See Minwalla hep-th/9712074 and EPFL Lectures.

Here are some simple properties of null descendants:

(1) If ψ is a null descendant then its descendants are also null. *Proof:* Consider the overlap of $\psi' = (P)^N \psi$ with some other state on the same level:

$$\langle\psi''|\psi'\rangle = \langle\psi''|P^N|\psi\rangle \,.$$

This equals the overlap of ψ with the state $K^N|\psi''\rangle$. Since K lowers the dimension, the latter state is a descendant on the same level as ψ, so this overlap zero since by assumption that ψ is the zero eigenstate of the Gram matrix. Q.E.D.

(2) If ψ is a null descendant then either $K_\mu|\psi\rangle = 0$ or it's null. *Proof:* consider the overlap

$$\langle\psi'|K_\mu|\psi\rangle$$

where ψ' is at level $n-1$. It's the same as overlap of ψ with $P_\mu \psi'$ and so it's zero.

(1),(2) imply that null states come in multiplets of which the lowest state is a null descendant which is also a primary.

The converse is also true: a primary descendant is null. *Proof:* Let $\psi = P^N O$ be a primary descendant. Then the overlap with any other descendant on the same level is:

$$\langle O|K^N P^N|O\rangle = \langle O|K^N|\psi\rangle = 0$$

Coming back to the above-mentioned poles associated with null descendants, it's clear that all poles will be at the unitarity bound or below it. Suppose that at $\Delta = \Delta_*$ the block has a pole and it's associated with a null descendant at level n, i.e. of dimension $\Delta_* + n$ and spin j. We have

$$g_{\Delta,l}(\rho,\bar\rho) \supset \frac{c}{\Delta - \Delta_*} g_{\Delta_*+n,j}(\rho,\bar\rho) \tag{1}$$

i.e. the coefficient of the pole is proportional to the conformal block of a dimension $\Delta_* + n$ and spin j. This follows from the Cassimir equation which near $\Delta = \Delta_*$ reduces to the Casimir equation for the coefficient block. Notice that the Casimir eigenvalue of the null descendant (since it's a descandant) is the same as for the original block:

$$C_{\Delta_*,l} = C_{\Delta_*+n,j}$$

For the conformal block of symmetric traceless primaries in the OPE of two scalars we have three series of poles (see 1307.6856, 1406.4858, 1509.00428:

$$(1)\ \Delta = 1 - l - n,\ j = l + n,\ n = 1, 2, \ldots$$

$$(2)\ \Delta = d/2 - n/2,\ l \text{ any},\ n \geq 2 \text{ even}$$

$$(3)\ \Delta = l + d - 1 - n,\ j = l - n,\ n = 1, \ldots l$$

All the poles are at or below the unitarity bounds. Coefficients c in 1 corresponding to each pole can be worked out comparing with the Casimir equation (1307.6856, 1406.4858) or directly 1509.00428. Notice that in even d some simple poles coalesce into double poles. Conformal blocks in even d are thus best computed by analytic continuation away from integer dimensions.

Finally we have an expansion

$$g_{\Delta,l}(r, \cos\phi) = r^{\Delta} h_{\Delta,l}(r, \cos\phi), \quad h_{\Delta,l} = C_l^{(\nu)}(\cos\phi) + O(r)$$

$$h_{\Delta,l} = h_{\infty,l} + \sum \frac{c_i}{\Delta - \Delta_i} r^{n_i} h_{\Delta_i+n_i,l_i}$$

Here $h_{\infty,l}$ is the conformal block at infinite dimension which can be evaluated by taking infinite Δ limit of the Cas. eq.. For equal external dimensions:

$$h_{\infty,l} = \frac{C_l^{(\nu)}(\cos\phi)}{(1 - r^2)^{\nu}\sqrt{(1 + r^2)^2 - 4r^2\cos^2\phi}}.$$

This is a recursion relation which is currently the most efficient way to compute conformal blocks in 3d.

In practice we evaluate it it some large fixed order in r, e.g. r^{100}. This representation converges for $r < 1$. As we will see in numerical analysis the relevant value is $r = 3 - 2\sqrt{2} \approx 0.17$. At this point the convergence is very rapid.

9 Bootstrap philosophy

So, given conformal blocks, any 4pt function can be expanded into them as

$$g(u, v) = \sum c_{12}^k c_{34}^k g_{\Delta,l}(u, v)$$

This representation will converge fast. One can work out examples of this decomposition in free theory and check convergence. One can also prove theorems about it (in unitary theories). The rate of convergence is roughly $r^{\Delta_{max}}$ where $r = |\rho|$ and Δ_{max} is the cutoff on operator dimension included. See 1208.6449.

Consider now the decomposition in the channels (12)(34) ("direct") and (14)(23) ("crossed"). They have to agree, which gives rise to the "bootstrap equation":

$$\frac{1}{x_{12}^{2\Delta_\phi} x_{34}^{2\Delta_\phi}} \sum c_k^2 g_{\Delta,l}(u, v) = 2 \leftrightarrow 4,$$

Permuting 2 and 4 ("crossing") we have $u \leftrightarrow v$ and so we get

$$\sum_k c_k^2 [v^{\Delta_\phi} g_{\Delta,l}(u, v) - u \leftrightarrow v] = 0$$

In terms of $z$the crossing is $z \to 1-z, \bar{z} \to 1-\bar{z}$. The conf. blocks are best computed in the ρ coordinate, but the bootstrap eqn. is analyzed in the z coordinate.

Notice as well that if we consider the 4pt function of identical scalars then the unit operator is present in the OPE with $c_0^2 = 1$ which is just the primary normalization.

The 4pt functions satisfying the bootstrap eqn are sometimes called "crossing-symmetric".

Up to now we were discussing CFT data as given. However, CFT data is constrained by the bootstrap eqn. These are not a full set of constraints to have a consistent CFT (such a set is not known in $d > 2$).

A simple example of bootstrap limitation: in any CFT invariant under a global symmetry the set of singlet operators closes under OPE, but these are not the full theory and presumably should not be considered a consistent theory. In 2d we would rule out such spectrum truncations using modular invariance, but no simple analogue in $d > 2$.

Still, in 1970's Polyakov and others conjectured that bootstrap constraints are sufficiently important as to allow only a discrete set of solutions, thus allowing determination of CFT data up to a finite number of ambiguities. This dream was achieved for the rational and some other special theories in 2d, but only recently people started making progress in d>2.

BTW once we solved 4pt functions constraints for all operators no new constraints will arise from higher-pt fns. This is because bootstrap equations can be expressed as a condition for OPE associativity:

$$(AB)C = A(BC)$$

Taking overlap with a fourth operator D we get back the bootstrap eqn.

Suppose we solved a bootstrap equation, how do we know if it corresponds to some known theory obtained as an IR fixed point of a Lagrangian theory or a critical point of a lattice model?

First of all we have to compare protected data, such as

- the global symmetry group (usually the same in UV and for CFT although enhancement can occur)

- in even dimensions, anomalies of global currents, which should coincide in UV and IR via 't Hooft anomaly matching

Beyond that, we have to resort to experiment or to lattice simulations. E.g. compare the number of relevant scalars. Or compare roughly the dimension of the lowest scalar. Due to universality, once we see that the few parameters more or less agree, we can make a reasonable conjecture that the theory is the same (and test it for more digits and more operators). This is actually standard in 2d where many exactly solved CFTs has to be identified with lattice models which are not always exactly solvable (although the 2d Ising model is exactly solvable also on the lattice).

We don't have yet any full exact bootstrap solutions in $d > 2$ (apart from gaussian theories). Actually, we don't even know fully any conformal non-gaussian 4pt function which satisfies crossing. (i.e. operator dimensions and ope coeffs of appearing operators). I will describe a couple of cases where partial answers could be obtained.

Lecture 4

10 Low-dimension operator results - rigorous

Take the bootstrap equation for the 4pt function of scalar ϕ. The OPE $\phi \times \phi$ contains 1 and infinitely many operators. However phenomenologically most relevant are the low-dimension operators. Can we say something about low-dimension operators *marginalizing* over the high-dimension operators?

This can be formulated as follows. Suppose that we fix Δ_ϕ and the low dimension part of the OPE:

$$\phi \times \phi = \text{low} + \text{high}, \quad \text{low} = 1 + \sum_i c_i \mathcal{O}_{\Delta_i, \ell_i}$$

in the "low" part the Δ_i, ℓ_i are fixed, c_i can also be fixed or allowed to vary. The "high" part can contain any operators above a gap Δ_* (which can also be spin-dependent), and their OPE coefficients can by arbitrary. *Question:* given the "low" part is it always possible to find the "high" operators and OPE coefficients so that the four point function is crossing-symmetric?

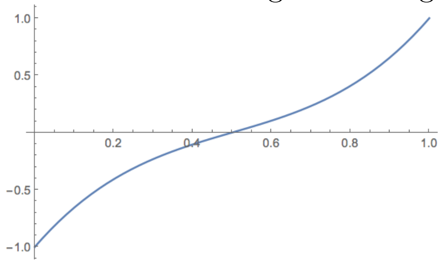
No, not always!

10.1 Toy example

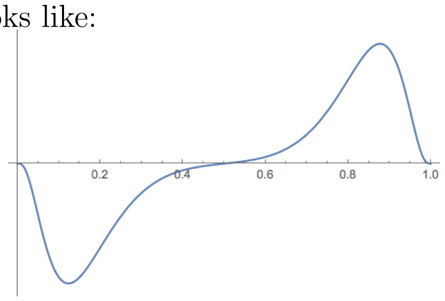
Let us approximate the conformal blocks by the leading term r^Δ times the Gegenbauer. Moreover let us analyze the bootstrap eq. only for real $0 < z = \bar{z} < 1$. On the real line the Gegenbauer is constant and we lose sensitivity to spin. Denoting $\delta = \Delta_\phi$ we have

$$-F_0(z) = \sum c_\Delta^2 F_\Delta(z), \quad F_\Delta = (1-z)^{2\delta}[\rho(z)]^\Delta - z^{2\delta}[\rho(1-z)]^\Delta$$

Let us consider the case low=1 and check how high we can push the gap Δ_*. The function $-F_0(z)$ is just a normal-looking decreasing function



For large Δ, $F_\Delta(z)$ is dominated by the ρ terms except near $z = 0, 1$ <graph of $\rho(z)$>, and it looks like:



For large Δ it becomes more and more concave near $z = 1/2$: $F'''/F' \to \infty$. Expanding around $z = 1/2$ we have ($x = z - 1/2$):

$$-F_0(x) = C_\delta(x + \frac{4}{3}(\delta-1)(2\delta-1)x^3 + \mathcal{O}(x^5)), \quad C_\delta > 0$$

Suppose the gap can be pushed to $\Delta \gg \delta$ then

$$F_\Delta(x) \approx (1/2)^{2\delta}\{[\rho(z)]^\Delta - [\rho(1-z)]^\Delta\} \simeq B_\Delta \left(x + \frac{4}{3}\Delta^2 x^3 + \dots\right), B_\Delta > 0 (\Delta \gg \delta)$$

Let's change conf. blocks normalization so that $B_\Delta = 1$ (incorporating it into the OPE coeffs). We then have two equations from the coefficients of x and of x^3:

$$C_\delta = \sum c_\Delta^2$$

$$C_\delta \frac{4}{3}(\delta - 1)(2\delta - 1) = \sum c_\Delta^2 \frac{4}{3}\Delta^2 \geq \frac{4}{3}\Delta_*^2 \sum c_\Delta^2 = \frac{4}{3}\Delta_*^2 C_\delta$$

Canceling C_δ we get

$$\Delta_* \leq \sqrt{(\delta - 1)(2\delta - 1)} = O(\delta)$$

which is a contradiction since we assumed $\Delta_* \gg \delta$. So not everything is possible.

10.2 Linear programming

Let us try to formalize this exercise to see if it can be made systematic. What did we do?

1) we replaced the bootstrap equation, which in principle is a functional equation which has to be satusfied for any $z, \bar{z}$, by a finite number of equations expanding in a power series around $z = \bar{z} = 1/2$. We worked for real z but it's not hard to also include transverse derivatives which will give us sensitivity to spin. Including more and more derivatives we will approach the full equation.

2) we worked with approximate conformal blocks but no problem to include full conformal blocks as long as we are happy to compute them numerically

3) To formalize the inequality step, consider vectors of taylor coeffs:

$$-F_0 \to \vec{b} = (1, \frac{4}{3}(\delta - 1)(2\delta - 1))$$

$$F_\Delta \to \vec{F}_\Delta = (1, \frac{4}{3}\Delta^2)$$

If $\Delta_* > \sqrt{(\delta - 1)(2\delta - 1)}$ there exists a "separating line" $\lambda.b < 0, \lambda.F \geq 0$ such that all F vectors and b lie on two different sides of this line. Then no solution of

$$\sum x_\Delta \vec{F}_\Delta = \vec{b}$$

with positive coeffs $x_\Delta = c_\Delta^2$ can exist. In case of more coordinates we call this a "linear functional" So need a systematic method to look for a separating linear functional given a set of vectors. More generally, consider the following "problems".

Problem 1 (direct).

Given a matrix $A \in \mathbb{R}^{m \times n}$ and a vector $b \in \mathbb{R}^m$ find a vector $x \in \mathbb{R}^n$ such that

- $x_j \geq 0$ (all components positive)
- $Ax = b$.

Problem 2 (dual)

Find a "linear functional" $\Lambda \in \mathbb{R}^m$ such that $\Lambda.b < 0$ and $\Lambda.A \geq 0$ (all components positive).

If Problem 2 has a solution, then Problem 1 does not have, and viceversa. They are equivalent.

These are called "linear programming" problems. In our applications the index j runs over all allowed dimensions and spins, so it's infinite (actually continuously infinite). There are ways to work around this, e.g. via truncation and discretization.

There are several algorithms to solve linear programming problems, the most famous being Dantzig's "symplex method "(realized by the LinearProgramming function of Mathematica). See 1403.4545 for a review of how these algorithms work.

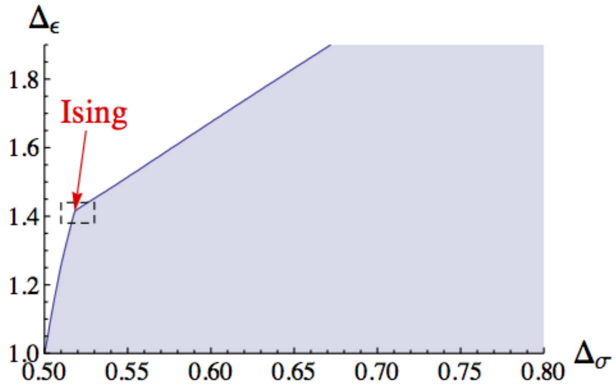
The basic idea for Problem 1. Consider an extended problem by including the "slack variables" $y \in \mathbb{R}^m$:

- $x_j \geq 0$, $y_i \geq 0$
- $(Ax)_i + b_i y_i$(no sum) $= b_i$.
- $f(y) = \sum y_i \to \min$

The point is that now there is a point $x_j = 0$, $y_i = 1$ which satisfies all constraints. The set of all points satisfying the constraint formes a convex polytope, and the minumum of a linear function must be attained on its vertex. We thus move from vertex to vertex reducing the cost function $f(y)$ in each step (no trapping in a local minimum is possible because of convexity), until we reach the global minimum.

Using the linear programming techniques, we can explore, numerically but rigorously, when the bootstrap equation has no solutions. And thus we can exclude, rigorously, subsets of low-dimension CFT data entering a single four-point function.

Example. In 3d we found the following bound on the gap as a function of Δ_ϕ :



10.3 Semidefinite programming

The 3d Ising model contains two low dimension operators, called σ ($\mathbb{Z}_2$ odd) and ϵ ($\mathbb{Z}_2$ even). In fact these are the only relevant scalars. Natural to study together 4pt functions $\langle\sigma\sigma\sigma\sigma\rangle$, $\langle\epsilon\epsilon\epsilon\epsilon\rangle$, $\langle\sigma\sigma\epsilon\epsilon\rangle$. This brings about several new issues compared to a single 4pt function case:

- We will now be sensitive to several OPEs (notice that $f_{\sigma\sigma\epsilon} = f_{\sigma\epsilon\sigma}$).

$$\begin{aligned} \sigma \times \sigma &= 1 + f_{\sigma\sigma\epsilon}\epsilon + \dots \\ \sigma \times \epsilon &= f_{\sigma\epsilon\sigma}\sigma + \dots \\ \epsilon \times \epsilon &= 1 + f_{\epsilon\epsilon\epsilon}\epsilon + \dots \end{aligned}$$

- In the analysis of 4pt function $\langle\sigma\sigma\epsilon\epsilon\rangle$ we will have two equations. Crossing for

$$\left\langle \begin{matrix} \sigma & \epsilon \\ \epsilon & \sigma \end{matrix} \right\rangle$$

will involve positive numbers $f^2_{\sigma\epsilon O}$. This is suitable for linear programming. But

$$\left\langle \begin{matrix} \sigma & \sigma \\ \epsilon & \epsilon \end{matrix} \right\rangle$$

will involve positive numbers $f^2_{\sigma\epsilon O}$ in the direct channel, but not sign-definite products $f_{\sigma\sigma O} f_{\epsilon\epsilon O}$ in the crossed channel. The squares of these numbers $f^2_{\sigma\sigma O}$, $f^2_{\epsilon\epsilon O}$ appear in the analysis of $\langle\sigma\sigma\sigma\sigma\rangle$, $\langle\epsilon\epsilon\epsilon\epsilon\rangle$. The idea is to group the unknowns into 2×2 matrices:

$$M_O = \sum_O \begin{pmatrix} f^2_{\sigma\sigma O} & f_{\sigma\sigma O} f_{\epsilon\epsilon O} \\ f_{\sigma\sigma O} f_{\epsilon\epsilon O} & f^2_{\epsilon\epsilon O} \end{pmatrix}$$

where the sum is over all operators occuring at a given dimension and spin (if there is degeneracy). The crucial property of the matrices M_O is that they are positive-semidefinite:

$$M_O \succeq 0$$

Indeed

$$\xi.M_O.\xi = \sum_O (\xi.f_O)^2,$$

where $f_O = (f_{\sigma\sigma O}, f_{\epsilon\epsilon O})$. The unknowns thus become the positive definite M_O's and the positive $p_O = f^2_{\sigma\epsilon O}$. The bootstrap equations are expressed as linear equations involving p_O's and matrix elements of M's. E.g. the equation for

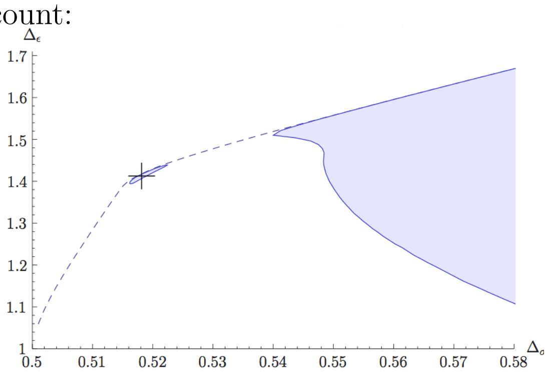
$$\left\langle \begin{matrix} \sigma & \sigma \\ \epsilon & \epsilon \end{matrix} \right\rangle$$

will take the form

$$\sum p_O g_O^{\sigma\epsilon,\sigma\epsilon}(x) = 1 + \sum M_{O12} g_O^{\sigma\sigma,\epsilon\epsilon}$$

The problem of resolving systems of linear equations with coefficients constrained to live in positive semidefinite matrices is called "semidefinite programming". This problem shares one common property with linear programming - convexity. The constraint of positive semidefiniteness is convex. This excludes "trapping". For this reason efficient algorithms exist. Reviewing these algorithms would take us too far. See 1502.02033.

This is how the above plot changes provided the three correlator constraints are taken into account:



With further numerical work (possible thanks to developing a custom-made SDP solver) this was perfected to the point that the following estimates were obtained (1502.02033):

$$\Delta_\sigma = 0.518151(6), \Delta_\epsilon = 1.41264(6).$$

These are one order of magnitude better than the best available Monte Carlo results and some three orders of magnitude better than the resummed perturbation theory (from 7 loops).

The 3d Ising model has been the most striking application of the linear/semidefinite programming methods in bootstrap. Why is it possible to constrain this model so much just from three correlation functions? Does it hint at the exact solvability? Currently these questions have no answers. Very similar, although somewhat less precise results were obtained for $O(N)$ models (see 1307.6856,1504.07997)

Polyakov many years ago conjectured, arguing from the lattice, that the 3d Ising model should be exactly solvable by a fermionic string theory. This conjecture has never been made precise and it is stuck in limbo ever since. There is no clear connection to the recent bootstrap results.

10.4 Other topics worth looking into

10.4.1 Low-dimension operator results - severe truncation (Gliozzi et al)

The linear/SDP results are nice since completely rigorous (errors on the exclusion region boundaries due to truncation, discretization etc can be easily made negligibly small). However rather heavy-weight if precision is needed. Current state-of-the-art computations require clusters with every point taking a day or so of computer time. (Still orders of magnitude faster than lattice Monte Carlo simulations of comparable precision). Another disadvantage is that linear/SDP operates in the "black box" mode. For every point it tells you if it's allowed or disallowed after a long sequence of minimization steps, but it does not necessarily provide intuition what's going on.

Question: can we sacrifice some rigour for speed/intuition?

The only proposal currently on the market is due to Gliozzi and collaborators 1307.3111,1403.6003,1502.072

The idea is to keep a few low-dim operators, and drop the “high” part altogether. With a finite number of operators we can only hope to satisfy the bootstrap equations approximately. For example, we can select a subset of the bootstrap eqs corresponding to setting a few derivatives around $z = 1/2$ to zero (and forgetting about the rest). Sometimes this works pretty well, sometimes not. It’s not systematic, and there is no control on the error of the method. It’s more of an art. It would be nice to upgrade this method to a more systematic procedure.

For a reviw see lectures by Qualls 1511.04074 and by Sheer El-Showk
<http://home.kias.re.kr/MKG/upload/9thasianwinterschool/BootstrapLectures.pdf>.

10.4.2 Light cone bootstrap (low-twist, high dimension operators)

One can get analytic results about the spectrum of low-twist, high dimension operators by imposing bootstrap consistency in the region near the light cone. This means that one goes to Minkowski space and considers $0 < z, \bar{z} < 1$ real and independent. The relevant region is then $z \to 0, \bar{z} \to 1$. The typical result is as follows. Consider the OPE $\phi \times \phi$. Then at large spin it will contain an infinite series of operators whose dimension approaches

$$(l + d - 2) + 2\gamma_\phi$$

where $\gamma_\phi = \Delta_\phi - (d/2 - 1)$. The asymptotic approach rate is also known. See 1212.3616,1212.4103.

It would be nice to put together the numerical methods for low dimensions with analytic results for high dimensions. For some steps in this direction see 1506.04659,1510.08091.