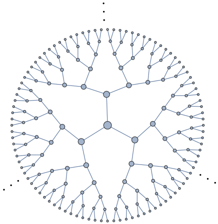
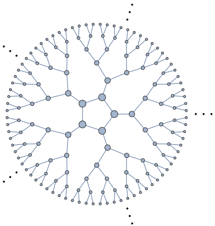
OIST Seminar

An Atlas of p-adic AdS/CFT





Christian Baadsgaard Jepsen

KIAS KOREA INSTITUTE FOR ADVANCED STUDY

OIST

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Atlas of an atlas

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Part I: Fundamentals

Norms

Let x and y be elements of some field $\mathbb{F}$. A norm $|\cdot|$ is a map from $\mathbb{F}$ to the non-negative real numbers possessing these three properties:

1. $|x| = 0 \Leftrightarrow x = 0\,,$
2. $|x \cdot y| = |x| \cdot |y|\,,$
3. $|x + y| \leq |x| + |y|\,.$

Two norms $|\cdot|_a$ and $|\cdot|_b$ are equivalent if $|\cdot|_a = (|\cdot|_b)^s$ for some positive s.

Norms on the rationals

Each rational number $x \in \mathbb{Q}$ has a unique prime decomposition:

$$x = \pm \prod_{\text{primes } p} p^{e_p}\,, \qquad\qquad e_p \in \mathbb{Z}\,.$$

Ostrowski's theorem: the only inequivalent norms on the rationals are

the Archimedean norm : $\quad |x|_\infty = \prod_{\text{primes } p} p^{e_p}\,,$

the p-adic norm : $\quad |x|_p = p^{-e_p}\,,$

the trivial norm : $\quad |x|_{\text{trivial}} = 1\,.$

We immediately see that $\quad |x|_\infty \prod_{\text{primes } p} |x|_p = 1\,.$

The p-adic numbers

$$\mathbb{R} = \text{ completion of } \mathbb{Q} \text{ wrt. } |\cdot|_\infty$$

$$\mathbb{Q}_p = \text{ completion of } \mathbb{Q} \text{ wrt. } |\cdot|_p$$

Each p-adic number $x \in \mathbb{Q}_p$ admits of a decomposition:

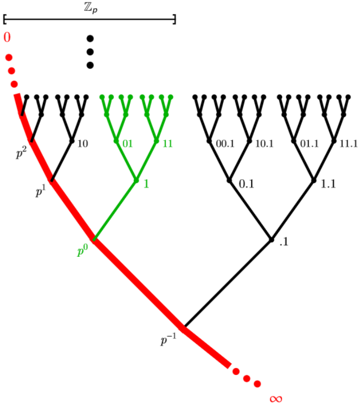
$$x = p^v \sum_{n=0}^{\infty} a_n \, p^n \,, \qquad v \in \mathbb{Z}\,, \qquad a_n \in \{0, 1, ..., p-1\}\,, \quad a_0 \neq 0\,.$$

The expansion is unique, unlike the decimal expansion of the reals:

$$0.999999... = 1.00000...$$

The Bruhat-Tits Tree T_p

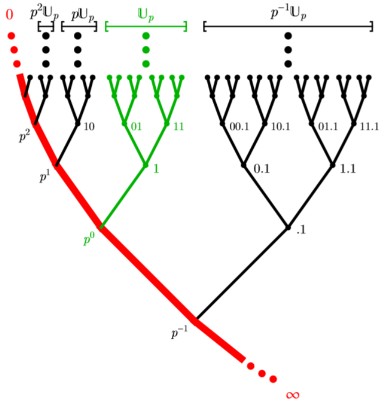
Bruhat-Tits Tree = Bethe lattice with coordination number $p + 1$



$$x = p^v \sum_{n=0}^{\infty} a_n \, p^n$$

Boundary of the Bruhat-Tits tree $= \mathbb{Q}_p \cup \{\infty\} = \mathbb{P}^1(\mathbb{Q}_p)$

Translational invariant integration measure



the p-adic integers:

$$\mathbb{Z}_p = \{z \in \mathbb{Q}_p \mid |z|_p \leq 1\}$$

$$\int_{\mathbb{Z}_p} dx = 1$$

the p-adic units:

$$\mathbb{U}_p = \{z \in \mathbb{Q}_p \mid |z|_p = 1\}$$

$$\int_{\mathbb{U}_p} dx = 1 - \frac{1}{p}$$

$$\int_{\mathbb{Q}_p} dx\, f(|x|_p) = \sum_{m=-\infty}^{\infty} f(p^m) \int_{p^{-m}\mathbb{U}_p} dx = \sum_{m=-\infty}^{\infty} f(p^m)\, p^m\, \frac{p-1}{p}$$

This kind of integration over $\mathbb{Q}_p$ produces a complex-valued answer.

p-adic String Theory

Two conceptions of p-adic string theory

There exists two notions of p-adic strings, both formulated in 1987.

	spacetime and momenta	scattering amplitudes
[Volovich 87'; Grossman '87]	p-adic	Morita gamma functions
[Freund, Olson '87]	real	Gelfand-Graev gamma functions

The most important, and by far the most studied, is the formulation due to Freund and Olson.

The Freund-Olson amplitudes

The Veniano amplitude is given (in mostly-positive signature) by

$$A_4^{(\infty)} = \int_{\mathbb{R}} dx \, |x|^{2\alpha' k_1 \cdot k_2} |1-x|^{2\alpha' k_1 \cdot k_3} \, ,$$

where the momenta are tachyonic: $k_i^2 = \frac{1}{\alpha'}$.

Freund and Olson observed that changing the integration domain from $\mathbb{R}$ to $\mathbb{Q}_p$,

$$A_4^{(p)} = \int_{\mathbb{Q}_p} dx \, |x|_p^{2\alpha' k_1 \cdot k_2} |1-x|_p^{2\alpha' k_1 \cdot k_3}$$

produces a tentative amplitude A_p that, along with its higher-point cousins, is

- meromorphic,
- (p-adic) Möbius invariant,
- factorizing, and
- free of pairs of incompatible poles.

However, the amplitude A_p is periodic in the imaginary direction and only has the tachyon pole.

Adelic amplitudes

[Freund, Witten '87] observed that when written in terms of the special functions

$$\zeta_p(s) \equiv \frac{1}{1-p^{-s}}\,, \qquad \zeta_\infty(s) \equiv \pi^{-\frac{s}{2}}\,\Gamma(\frac{s}{2})\,,$$

and using the shorthands

$$a = 2\alpha' k_1 \cdot k_2 + 1\,, \qquad b = 2\alpha' k_1 \cdot k_3 + 1\,.$$

the four-point amplitudes take on similar forms:

$$A_4^{(p)} = \int_{\mathbb{Q}_p} dx\,|x|_p^{a-1}|1-x|_p^{b-1} = \frac{\zeta_p(a)\,\zeta_p(b)\,\zeta_p(1-a-b)}{\zeta_p(1-a)\,\zeta_p(1-b)\,\zeta_p(a+b)}\,,$$

$$A_4^{(\infty)} = \int_{\mathbb{R}} dx\,|x|^{a-1}|1-x|^{b-1} = \frac{\zeta_\infty(a)\,\zeta_\infty(b)\,\zeta_\infty(1-a-b)}{\zeta_\infty(1-a)\,\zeta_\infty(1-b)\,\zeta_\infty(a+b)}\,,$$

and from Euler's product formula and Riemann's functional equation:

$$\prod_{p\in\mathbb{P}} \zeta_p(s) = \zeta(s)\,, \qquad \zeta_\infty(s)\zeta(s) = \zeta_\infty(1-s)\zeta(1-s)\,,$$

it appears to follow that

$$A_4^{(\infty)} \prod_{p\in\mathbb{P}} A_4^{(p)} = 1\,.$$

Intricacies of infinite products

Actually the product $\prod_p A_4^{(p)}(k)$ does not converge for any values of momenta k_i. The regularization procedure consists in the identification

$$\prod_p \frac{f_1(p) f_2(p) \ldots}{g_1(p) g_2(p) \ldots} \qquad \leftrightarrow \qquad \frac{\Big(\prod_p f_1(p)\Big)\Big(\prod_p f_1(p)\Big) \ldots}{\Big(\prod_p g_1(p)\Big)\Big(\prod_p g_2(p)\Big) \ldots},$$

where the subproducts are analytically continued separately.

But [Arefeva, Dragovic, Volovich '88] observed that $\prod_p \big(-A_4^{(p)}(k)\big)$ does sometimes converge, but not to $\pm 1/A_4^{(\infty)}(k)$.

It follows then that

$$A_4^{(\infty)}(k)\, A_4^{(\infty)}(k') \prod_p A_4^{(p)}(k)\, A_4^{(p)}(k') \neq 1\,,$$

which seems to contradict the four-point formula.

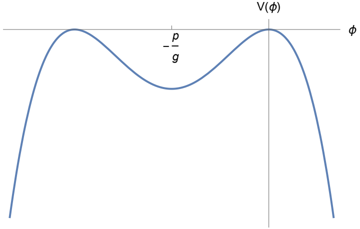
p-adic string field theory

[Brekke, Freund, Olson, Witten '88] determined an effective Lagrangian of a real spacetime field $\phi(x)$ that reproduces the p-adic tree-amplitudes $g^n A_n^{(p)}$:

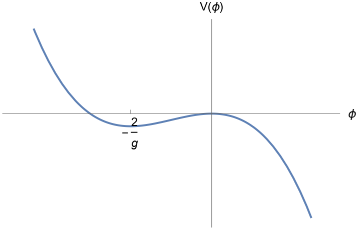
$$L^{(p)} = \frac{p}{p-1}\left(-\frac{1}{2p}\phi(x)\,p^{-\frac{1}{2}\nabla^2}\phi(x) + \frac{1}{g^2}\frac{p}{p+1}\left(1+\frac{g}{p}\phi(x)\right)^{p+1} - \frac{1}{g}\phi(x) - \frac{1}{g^2}\frac{p}{p+1}\right).$$

From this Lagrangian we can read off the tachyon potential:

$p > 2:$



$p = 2:$



[Ghoshal, Sen '00]: solitonic lump solutions to eom. $\leftrightarrow$ p-adic D-branes,

[Gerasimov, Shatashvili '00]: $p \to 1$ limit gives the real tachyon potential.

p-adic AdS/CFT

p-adic AdS/CFT before AdS/CFT

In p-adic string theory, $\mathbb{Q}_p$ furnishes the boundary of the worldsheet. What is the worldsheet itself then? The Bruhat-Tits tree!

[Zabrodin '89] provided the worldsheet action of the theory

$$S_{\text{bulk}} = \sum_{z \in T_p} \phi(z) \Box \phi(z) = \sum_{z \in T_p} \phi(z) \sum_{z' \text{ neighbours } z} \Big(\phi(z') - \phi(z) \Big) ,$$

and explicitly showed that this worldsheet action is dual to a boundary action

$$S_{\text{boundary}} = \int_{\mathbb{Q}_p \times \mathbb{Q}_p} dx\, dy \frac{\Big(\phi_{\text{boundary}}(x) - \phi_{\text{boundary}}(y)\Big)^2}{|x-y|_p^2} .$$

From a subsequent AdS/CFT perspective, his derivation establishes the free theory instance of the key holographic identity:

$$\frac{Z_{\text{bulk}}[\phi_{\text{boundary}}]}{Z_{\text{bulk}}[0]} = \left\langle \exp \left[\int_{\mathbb{Q}_p} dx\, \phi_{\text{boundary}}(x) \mathcal{O}(x) \right] \right\rangle_{\text{CFT}} .$$

Under this reinterpretation, the Bruhat-Tits tree furnishes the bulk AdS space, while the p-adic numbers parametrize the spacetime of the boundary CFT.

CFT two-point function

From Zabrodin's boundary action we read off the CFT two-point function:

$$\langle \mathcal{O}(x)\mathcal{O}(y)\rangle = \frac{1}{|x-y|_p^2}\,.$$

An analog of higher dimensions d can be obtained in the p-adic setting by performing an algebraic extension of $\mathbb{Q}_p$ to $\mathbb{Q}_{p^d}$. In effect, this changes the coordination number of T_p from $p+1$ to p^d+1. Numbers are now given as:

$$x = p^v \sum_{n=0}^{\infty} a_n\, p^n\,, \qquad v\in\mathbb{Z}\,, \qquad a_n \in \{0,1,...,p^d-1\}\,, \qquad a_0 \neq 0\,.$$

In the extended case the two point-function is given by

$$\langle \mathcal{O}(x)\mathcal{O}(y)\rangle = \frac{1}{|x-y|_p^{2d}}\,.$$

By further turning on masses and interactions, one can get more general scaling dimensions Δ:

$$\langle \mathcal{O}(x)\mathcal{O}(y)\rangle = \frac{1}{|x-y|_p^{2\Delta}}\,.$$

The inceptions of p-adic AdS/CFT

The notion of p-adic AdS/CFT rose independently amongst two research groups in 2016. Both wrote down the two-point function.

The Princeton group [Gubser, Knaute, Parikh, Samberg, Witaszczyk '16] also determined the three-point function.

Their work was inspired by bubbling cosmology paper [Harlow, Shenker, Stanford, Susskind '11], which discussed p-adic string theory and dS space.

The Caltech group [Heydeman, Marcolli, Saberi, Stoica '16] also studied the analog of the BTZ black hole.

Their motivation was the observation that the BTZ black hole has a construction as the quotient of an isometry group, which generalizes to the Bruhat-Tits tree.

Further motivation for p-adic AdS/CFT derives from its relation to critical phenomena for a special class of statistical systems.

The Dyson hierarchical model

[Dyson '68] formulated a hierarchical statistical mechanical model. The Hamiltonian is given by

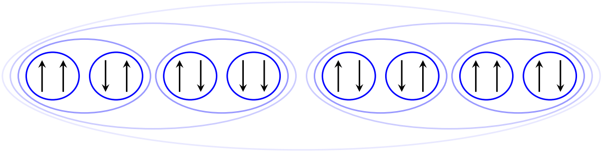
$$H = H_1 + H_2 + H_3 + H_4$$

where the first three terms are given by

$$H_1 = g_1 \sum_{\text{spin-pairs } \circ} \sum_{s_i, s_j \in \circ} s_i s_j \,,$$

$$H_2 = g_2 \sum_{\text{pairs of spin-pairs } \bigcirc} \sum_{s_i, s_j \in \bigcirc} s_i s_j \,,$$

$$H_3 = g_3 \sum_{\text{pairs of pairs of spin-pairs } \bigcirc} \sum_{s_i, s_j \in \bigcirc} s_i s_j \,.$$



g_1 g_2 g_3 g_4

Owing to the high degree of symmetry of this model, Dyson was able to rigorously establish the existence of a phase transition.

p-adic QFTs

The study of p-adic QFTs is motivated by the claim of [Lerner, Missarov '89] that they describe the continuum limit of hierarchical lattice models.

Like real QFTs, p-adic QFTs exhibit renormalization group flows. Critical behaviour is captured by the RG fixed point.

Two key properties of the RG flow:

- RG transformations are discrete even in the continuum limit: values of $|k|_p$ are integrated out shell by shell.
- The quantum fields $\phi(x)$ take values over the p-adics, $x \in \mathbb{Q}_{p^d}$, but are themselves real-valued, $\phi \in \mathbb{R}$.

  $\Rightarrow$ no derivative operator

  $\Rightarrow$ non-local kinetic terms

  $\Rightarrow$ no wavefunction renormalization

The enhancement of scale invariance to (p-adic) conformal invariance is a lot less understood than in the real case.

p-adic $O(N)$ model

[Missarov, Stepanov 2006], [Missarov 2012], [Gubser, Parikh, Jepsen, Trundy 2017]: Quartic interaction of p-adic ϕ^4 theory or more generally $O(N)$ theory:

$$S_{\text{int}} = \frac{\lambda}{4!} \int_{\mathbb{Q}_{p^d}} dx \left(\phi^i(x) \phi^i(x) \right)^2 .$$

Bilocal kinetic term given in terms of a real parameter s:

$$S_{\text{kin}} = \frac{1}{4} \frac{\zeta_p(s+d)}{\zeta_p(-s)} \int_{\mathbb{Q}_{p^d} \times \mathbb{Q}_{p^d}} dx\, dy \, \frac{\left(\phi(x) - \phi(y) \right)^2}{|x-y|_p^{d+s}} = \frac{1}{2} \int_{\mathbb{Q}_{p^d}} dk \, \widetilde{\phi}(-k) \, |k|_p^s \, \widetilde{\phi}(k) .$$

A perturbative expansion in $\epsilon \equiv 2s - d$ gives the RG relation

$$\overline{\lambda}_{\text{soft}} = p^{\epsilon} \, \overline{\lambda} \left[1 - \frac{N+8}{6\zeta_p(d)} \overline{\lambda} \right] + \mathcal{O}(\overline{\lambda}^3) .$$

where $\overline{\lambda}$: renormalized coupling for UV cutoff Λ,

$\overline{\lambda}_{\text{soft}}$ renormalized coupling for UV cutoff Λ/p.

At the RG fixed point: $$\Delta_{\phi^4} = s - \frac{6}{N+8} \epsilon + \mathcal{O}(\epsilon^3) .$$

Bulk-boundary matching

In order to match the two- and four-point functions of a free boundary theory for general scaling dimension Δ, [Gubser, Parikh '17] had to introduce cubic and quartic bulk interactions:

$$S = \frac{1}{2} \sum_{\text{neighbours } a,b \in T_{p^d}} (\phi_a - \phi_b)^2 + \sum_{a \in T_{p^d}} \left(\frac{1}{2} m^2 \phi_a^2 + \frac{g_3}{3!} \phi_a^3 + \frac{g_4}{4!} \phi_a^4 \right) + \frac{\widetilde{g}_4}{4!} \sum_{\text{neighbours } a,b \in T_{p^d}} \phi_a^2 \phi_b^2 \, .$$

To match higher-point correlators, presumable more interaction terms will be needed.

For the special case $\Delta = d$, all interactions go away.

The utility and inutility of p-adic AdS/CFT

Owing to the absence of derivatives, locality does not serve as a constraining principle in the space of theories.

Moreover, because of the topology of the p-adic numbers, it is not possible to perform the OPE in distinct channels. $\Rightarrow$ No conformal bootstrap.

The upshot is that we do not have islands of special p-adic CFTs and so it is unclear if we can construct meaning p-adic parallel theories to real theories.

Where p-adic AdS/CFT has provided a useful toy model is in the computation of generic conformal data, like conformal blocks and generic Witten diagrams.

No derivatives implies no descendants. Hence, complicated infinite sums at the real place reduce to a finite number of terms.

Part II: Core Results

Propagators and correlators

Bulk-to-bulk propagator

From the free massive bulk action

$$S = \sum_{a \in T_{p^n}} \left(\frac{1}{2} \sum_{a \sim b} (\phi_a - \phi_b)^2 + \frac{1}{2} m_p^2 \, \phi_a^2 \right),$$

we obtain the equation of motion:

$$0 = (p^n + 1 + m_p^2)\phi_a - \sum_{a \sim b} \phi_b \, .$$

It is a simple exercise to check that, letting $n(a,b) =$ tree distance between vertices a and b, the Green's function is given by

$$G_\Delta(a,b) = p^{-\Delta n(a,b)} \, ,$$

provided Δ and m_p^2 are related by the mass relation mass relation:

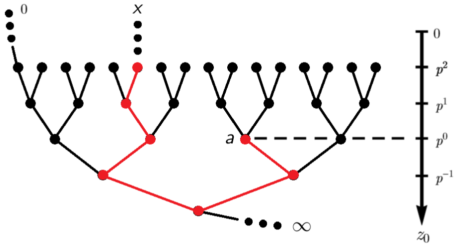
$$m_p^2 = -\frac{1}{\zeta_p(\Delta - n)\zeta_p(-\Delta)} \, , \qquad \left(\text{recall that } m_\infty^2 = \Delta(\Delta - n) \right).$$

Bulk-to-boundary propagator

To obtain the bulk-to-boundary propagator, take the limit of $G_\Delta(a,b)$ as b tends to a boundary point x while performing a suitable rescaling:

$$K_\Delta(a;x) = \frac{|z_0|_p^\Delta}{|z_0, z-x|_s^{2\Delta}}\,,$$

where z_0 is the depth coordinate of the bulk point a on the tree and z is any boundary point in the direction of a away from ∞.



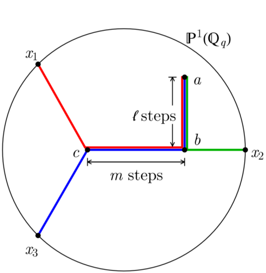
Three-point correlator

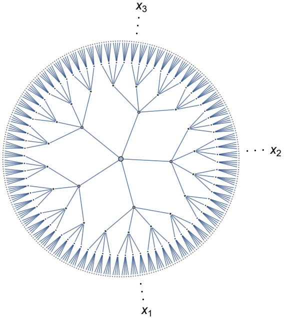
Tree-level three-point Witten diagram:

$$\langle \mathcal{O}_\Delta(x_1)\mathcal{O}_\Delta(x_2)\mathcal{O}_\Delta(x_3)\rangle = -g \sum_{a\in T_p} K(a,x_1)K(a,x_2)K(a,x_3)\,.$$

Three boundary points define a unique bulk point c where the paths between them meet. It turns out that

$$K(c,x_1)K(c,x_2)K(c,x_3) = \frac{1}{|x_1-x_2|_p^\Delta |x_2-x_3|_p^\Delta |x_1-x_3|_p^\Delta}\,.$$





We sum over all bulk points, backtracing wrt. c.

$$1+(p^d-2)p^{-3\Delta}\sum_{l=0}^{\infty}p^{(d-3\Delta)l}+3\sum_{m=1}^{\infty}p^{-\Delta m}\left[1+(p^d-1)p^{-3\Delta}\sum_{l=0}^{\infty}p^{(d-3\Delta)l}\right]=\frac{\zeta_p(\Delta)^3\zeta_p(3\Delta-d)}{\zeta_p(2\Delta)^3}$$

Real and p-adic three-point diagrams

Let's introduce the shorthands

$$x_{i,j} \equiv x_i - x_j\,, \qquad \Delta_{123} \equiv \Delta_1 + \Delta_2 + \Delta_3\,, \qquad \Delta_{ij,k} \equiv \Delta_i + \Delta_j - \Delta_k\,.$$

For the p-adics, generalizing to thee distinct scalars [Gubser, Knaute, Parikh, Samberg, Witaszczyk '16; Gubser, Parikh '17]:

$$\langle \mathcal{O}_{\Delta_1}(x_1)\mathcal{O}_{\Delta_2}(x_2)\mathcal{O}_{\Delta_3}(x_3)\rangle = -g\frac{C(\Delta_1,\Delta_2,\Delta_3)}{|x_{12}|_p^{\Delta_{12,3}}|x_{23}|_p^{\Delta_{23,1}}|x_{13}|_p^{\Delta_{13,2}}}\,.$$

In the real case, letting $L =$ the AdS radius
[Mueck, Viswanathan '98; Freedman, Mathur, Matusis, Rastelli '99]:

$$\langle \mathcal{O}_{\Delta_1}(\vec{x}_1)\mathcal{O}_{\Delta_2}(\vec{x}_2)\mathcal{O}_{\Delta_3}(\vec{x}_3)\rangle = -\frac{g}{2}\frac{C(\Delta_1,\Delta_2,\Delta_3)}{|\vec{x}_{12}|_\infty^{\Delta_{12,3}}\;|\vec{x}_{23}|_\infty^{\Delta_{23,1}}\;|\vec{x}_{13}|_\infty^{\Delta_{13,2}}}L^{n-1}\,.$$

The coefficient is given in each case by

$$C(\Delta_1,\Delta_2,\Delta_3) = \zeta_v(\Delta_{123}-d)\frac{\zeta_v(\Delta_{12,3})\,\zeta_v(\Delta_{13,2})\,\zeta_v(\Delta_{23,1})}{\zeta_v(2\Delta_1)\,\zeta_v(2\Delta_2)\,\zeta_v(2\Delta_3)}\,,$$

where $v=\infty$ or $v=p$ and we recall that

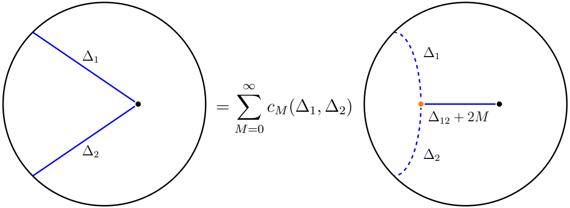
$$\zeta_p(s) \equiv \frac{1}{1-p^{-s}}\,, \qquad \zeta_\infty(s) \equiv \pi^{-s/2}\,\Gamma(\frac{s}{2})\,.$$

Propagator identities and conformal block decompositions

KK identity

[Hijano, Kraus, Perlmutter, Snively '16]:

$$K_{\Delta_1}(x_1,z)K_{\Delta_2}(x_2,z) = \sum_{M=0}^{\infty} c_M(\Delta_1,\Delta_2)\int_{w\in\gamma_{12}} K_{\Delta_1}(x_1,w)K_{\Delta_2}(x_2,w)\, G_{\Delta_{12}+2M}(w,z)$$



$$c_M(\Delta_1,\Delta_2) = \frac{2\zeta_\infty(2\Delta_{12}+4M)}{\zeta_\infty(2\Delta_1)\,\zeta_\infty(2\Delta_1)}\,\frac{(-\pi^2)^M}{M!(\Delta_1+\Delta_2+M-h)_M}$$

[Gubser, Parikh '17]:

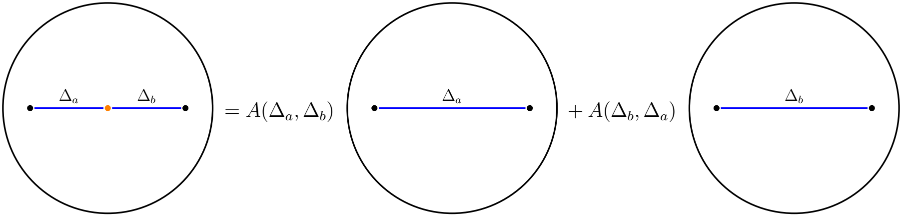
$$K_{\Delta_1}(x_1,z)K_{\Delta_2}(x_2,z) = c(\Delta_1,\Delta_2)\sum_{w\in\gamma_{12}} K_{\Delta_1}(x_1,w)\,K_{\Delta_2}(x_2,w)\,G_{\Delta_{12}}(w,z)$$

$$c(\Delta_1,\Delta_2) = \frac{\zeta_p(2\Delta_{12})}{\zeta_p(2\Delta_1)\,\zeta_p(2\Delta_1)}$$

GG identity

[Hijano, Kraus, Perlmutter, Snively '16]:

$$\int_{z\in \mathrm{AdS}_{d+1}} G_{\Delta_a}(w_a,z)\,G_{\Delta_b}(w_b,z) = A_{\Delta_a\Delta_b} G_{\Delta_a}(w_a,w_b) + A_{\Delta_b\Delta_a} G_{\Delta_b}(w_a,w_b)$$



$$A_{\Delta_a\Delta_b} = \frac{(2\Delta_b - d)\zeta_\infty(2\Delta_b - d)}{(\Delta_{ab} - d)\,\Delta_{b,a}\,\zeta_\infty(2\Delta_b)}$$

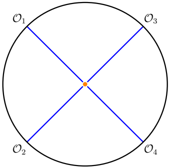
[Gubser, Parikh '17]:

$$\int_{z\in T_{p^d}} G_{\Delta_a}(w_a,z)\,G_{\Delta_b}(w_b,z) = A_{\Delta_a\Delta_b}\,G_{\Delta_a}(a,b) + A_{\Delta_b\Delta_a}\,G_{\Delta_b}(w_a,w_b)$$

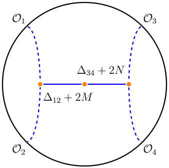
$$A_{\Delta_a\Delta_b} = \frac{\zeta_p(\Delta_{ab} - d)\zeta_p(\Delta_{b,a})}{\zeta_p(2\Delta_b)}$$

Four-point conformal block decomposition

[Hijano, Kraus, Perlmutter, Snively '16]:

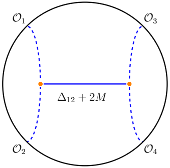


$$= \sum_{M,N=0}^{\infty} c_M(\Delta_1, \Delta_2)\, c_N(\Delta_3, \Delta_4)$$

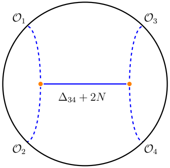


$$= \sum_{M,N=0}^{\infty} c_M(\Delta_1, \Delta_2)\, c_N(\Delta_3, \Delta_4) \cdot \Big[$$

$A(\Delta_{12}+2M, \Delta_{34}+2N)$



$+A(\Delta_{34}+2N, \Delta_{12}+2M)$



$\Big]$

These manipulations in effect project the contact diagram onto conformal families associated to primaries $\mathcal{O}_1 \partial^M \mathcal{O}_2$ and $\mathcal{O}_3 \partial^N \mathcal{O}_4$.

The geodesic diagrams match the conformal blocks determined by [Ferrara, Gatto, Grillo, Parisi '71, '72, '75] and [Dolan, Osborn '04, '11].

The same manipulations work for p-adic AdS/CFT, but now without the infinite sums.

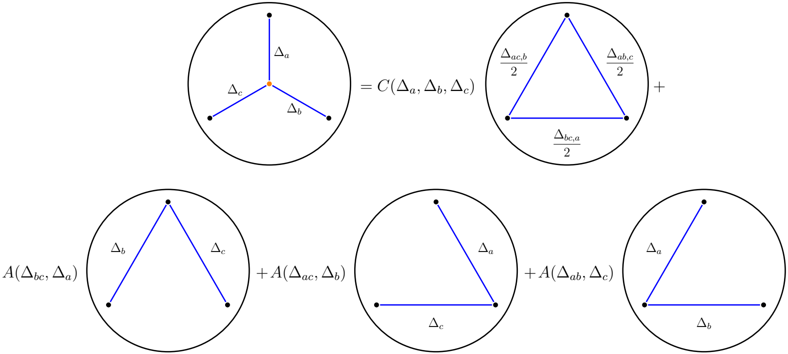
Beyond the imitation game: p-adic GGG identity

[Gubser, Parikh '17] derived a p-adic identity for which no real version was known:

$$\sum_{z \in T_{p^n}} G_{\Delta_a}(w_a, z) G_{\Delta_b}(w_b, z) G_{\Delta_c}(w_c, z)$$

$$= C(\Delta_a, \Delta_b, \Delta_c)\, G_{\frac{\Delta_{ac,b}}{2}}(w_a, w_c)\, G_{\frac{\Delta_{ab,c}}{2}}(w_a, w_b)\, G_{\frac{\Delta_{bc,a}}{2}}(w_b, w_c)$$

$$+ \Big(A(\Delta_{bc}, \Delta_a)\, G_{\Delta_b}(w_b, w_a)\, G_{\Delta_c}(w_c, w_a) + (a \leftrightarrow b) + (a \leftrightarrow c) \Big)$$



Reverse imitation: real GGG identity

Subsequently [Jepsen, Parikh '19] found the real GGG identity:

$$\int_{z\in \mathrm{AdS}_{n+1}} G_{\Delta_a}(w_a,z)\,G_{\Delta_b}(w_b,z)\,G_{\Delta_c}(w_c,z) =$$

$$C(\Delta_a,\Delta_b,\Delta_c)\sum_{k_a,k_b,k_c=0}^{\infty} c^{\Delta_a;\Delta_b;\Delta_c}_{k_a;k_b;k_c}\left(\frac{\xi_{a,c}}{2}\right)^{\frac{\Delta_{ac,b}}{2}+k_{ac,b}}\left(\frac{\xi_{a,b}}{2}\right)^{\frac{\Delta_{ab,c}}{2}+k_{ab,c}}\left(\frac{\xi_{b,c}}{2}\right)^{\frac{\Delta_{bc,a}}{2}+k_{bc,a}}$$

$$+\Bigg(A(\Delta_{bc},\Delta_a)\sum_{k_a,k_b,k_c=0}^{\infty} d^{\Delta_a;\Delta_b;\Delta_c}_{k_a;k_b;k_c}\left(\frac{\xi_{a,c}}{2}\right)^{\Delta_c+2k_c+k_a}\left(\frac{\xi_{a,b}}{2}\right)^{\Delta_b+2k_b+k_a}\left(\frac{\xi_{b,c}}{2}\right)^{-k_a}$$

$$+(a\leftrightarrow b)+(a\leftrightarrow c)\Bigg),$$

where $\xi_{a,b}$ is the chordal distance between bulk points w_a and w_b and

$$d^{\Delta_a;\Delta_b;\Delta_c}_{k_a;k_b;k_c} = \frac{1}{k_a!k_b!k_c!}\,\frac{(\Delta_b)_{2k_b+k_a}(\Delta_c)_{2k_c+k_a}}{(\frac{\Delta_{bc,a}}{2}+1)_{k_{abc}}}\,\frac{\Gamma(\frac{\Delta_{a,bc}}{2}+1)\Gamma(\frac{\Delta_{abc}-n}{2}+1)}{\Gamma(\Delta_a-\frac{n}{2}+1)}$$

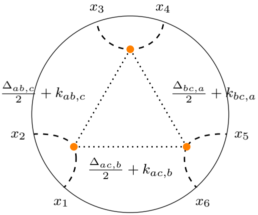
$$\sum_{\ell_a\ell_b\ell_c=0}^{\infty}\left(\frac{\Delta_{abc}-n}{2}\right)_{\ell_{abc}}\frac{(\frac{\Delta_{a,bc}}{2}-k_{abc})_{\ell_a}(-k_b)_{\ell_b}(-k_c)_{\ell_c}}{\ell_a!\ell_b!\ell_c!(\Delta_a-\frac{n}{2}+1)_{\ell_a}(\Delta_b-\frac{n}{2}+1)_{\ell_b}(\Delta_c-\frac{n}{2}+1)_{\ell_c}}\,.$$

Six-point conformal block

Through the use of propagator identities old and new, it was possible to perform higher-point conformal block decompositions, in the process identifying six-point conformal blocks:

$$W^{\Delta_1,\ldots,\Delta_6}_{\Delta_a,\Delta_b,\Delta_c}(x_1,\ldots,x_6) =$$

$$\sum_{k_a,k_b,k_c=0}^{\infty} \frac{8\, c^{\Delta_a;\Delta_b;\Delta_c}_{k_a;k_b;k_c}}{B(\frac{\Delta_{a1,2}}{2},\frac{\Delta_{a2,1}}{2})\,B(\frac{\Delta_{b3,4}}{2},\frac{\Delta_{b4,3}}{2})\,B(\frac{\Delta_{c5,6}}{2},\frac{\Delta_{c6,5}}{2})}$$



where the coefficient is given by

$$c^{\Delta_a;\Delta_b;\Delta_c}_{k_a;k_b;k_c} = \frac{(-1)^{k_{abc}}}{k_a!k_b!k_c!}\left(\frac{\Delta_{ac,b}}{2}\right)_{k_{ac,b}}\left(\frac{\Delta_{ab,c}}{2}\right)_{k_{ab,c}}\left(\frac{\Delta_{bc,a}}{2}\right)_{k_{bc,a}}$$

$$F^{(3)}_{A}\Big[\frac{\Delta_{abc}-n}{2};\{-k_a,-k_b,-k_c\};\{\Delta_a-\frac{n}{2}+1,\Delta_b-\frac{n}{2}+1,\Delta_c-\frac{n}{2}+1\};1,1,1\Big].$$

Easy and hard identity

The fact that each term in the conformal block decomposition has the same functional form is non-trivial statement.

In the p-adic case, the identification hinges on the identity

$$\frac{\zeta_p(\Delta_1+\Delta_2)}{\zeta_p(\Delta_1)\zeta_p(\Delta_2)}A(\Delta_{bc},\Delta_{12})=\frac{\zeta_p(\Delta_b+\Delta_c)}{\zeta_p(\frac{\Delta_{bc1,2}}{2})\zeta_p(\frac{\Delta_{bc2,1}}{2})}C(\Delta_{bc},\Delta_1,\Delta_2)\,.$$

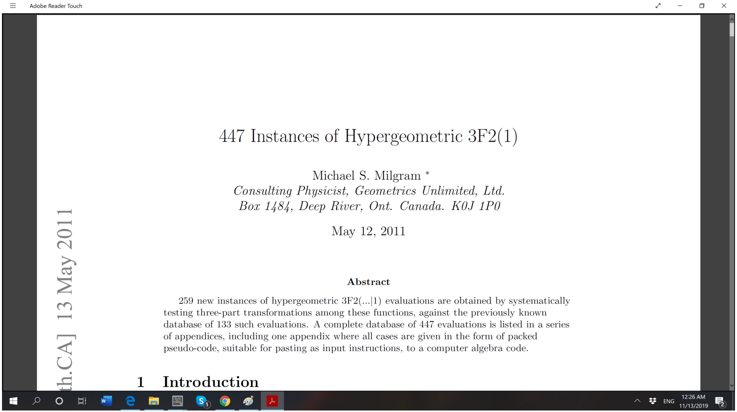
The uplift of this equation to the reals is given by

$$\sum_{M=0}^{\infty}\frac{\alpha_M^{\Delta_1\Delta_2}A(\Delta_{bc},\Delta_{12}+2M)}{B(\Delta_1+M,\Delta_2+M)}d_{k_a;k_b;k_c}^{\Delta_{12}+2M;\Delta_b;\Delta_c}=\sum_{M=0}^{\infty}\frac{\alpha_M^{\Delta_b\Delta_c}C(\Delta_{bc}+2M,\Delta_1,\Delta_2)}{B(\frac{\Delta_{bc1,2}}{2}+M,\frac{\Delta_{bc2,1}}{2}+M)}c_{k_{abc}-M;k_c;k_c}^{\Delta_{bc}+2M;\Delta_b;\Delta_c}$$

where $\qquad \alpha_M^{\Delta_1\Delta_2}\equiv\dfrac{(-1)^M(\Delta_1)_M(\Delta_2)_M}{M!(\Delta_1+\Delta_2+M-h)_M}\,.$

Establishing this latter identity required a pretty deep dive into the mathematics literature.

The power of AdS/CFT



Adobe Reader Touch

447 Instances of Hypergeometric 3F2(1)

Michael S. Milgram *
Consulting Physicist, Geometrics Unlimited, Ltd.
Box 1484, Deep River, Ont. Canada. K0J 1P0

May 12, 2011

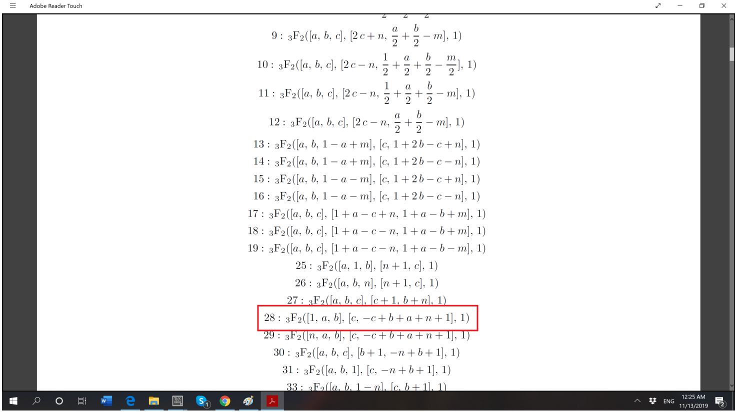
th.CA] 13 May 2011

Abstract

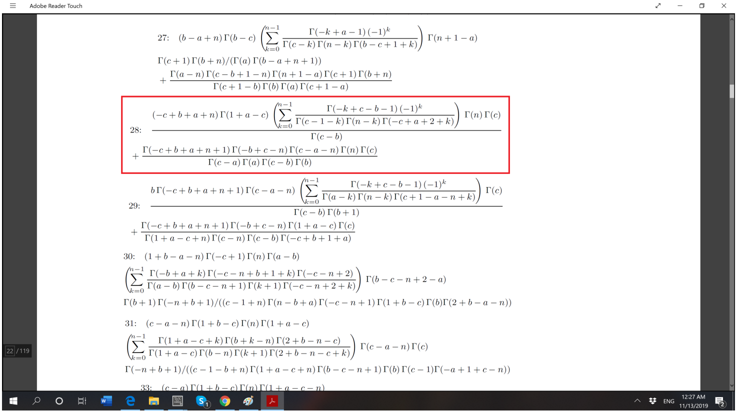
259 new instances of hypergeometric 3F2(...|1) evaluations are obtained by systematically testing three-part transformations among these functions, against the previously known database of 133 such evaluations. A complete database of 447 evaluations is listed in a series of appendices, including one appendix where all cases are given in the form of packed pseudo-code, suitable for pasting as input instructions, to a computer algebra code.

1 Introduction

The power of AdS/CFT



The power of AdS/CFT



$$27:\quad (b-a+n)\,\Gamma(b-c)\left(\sum_{k=0}^{n-1}\frac{\Gamma(-k+a-1)\,(-1)^k}{\Gamma(c-k)\,\Gamma(n-k)\,\Gamma(b-c+1+k)}\right)\Gamma(n+1-a)$$

$$\Gamma(c+1)\,\Gamma(b+n)/(\Gamma(a)\,\Gamma(b-a+n+1))$$

$$+\frac{\Gamma(a-n)\,\Gamma(c-b+1-n)\,\Gamma(n+1-a)\,\Gamma(c+1)\,\Gamma(b+n)}{\Gamma(c+1-b)\,\Gamma(b)\,\Gamma(a)\,\Gamma(c+1-a)}$$

$$28:\quad \frac{(-c+b+a+n)\,\Gamma(1+a-c)\left(\sum_{k=0}^{n-1}\frac{\Gamma(-k+c-b-1)\,(-1)^k}{\Gamma(c-1-k)\,\Gamma(n-k)\,\Gamma(-c+a+2+k)}\right)\Gamma(n)\,\Gamma(c)}{\Gamma(c-b)}$$

$$+\frac{\Gamma(-c+b+a+n+1)\,\Gamma(-b+c-n)\,\Gamma(c-a-n)\,\Gamma(n)\,\Gamma(c)}{\Gamma(c-a)\,\Gamma(a)\,\Gamma(c-b)\,\Gamma(b)}$$

$$29:\quad \frac{b\,\Gamma(-c+b+a+n+1)\,\Gamma(c-a-n)\left(\sum_{k=0}^{n-1}\frac{\Gamma(-k+c-b-1)\,(-1)^k}{\Gamma(a-k)\,\Gamma(n-k)\,\Gamma(c+1-a-n+k)}\right)\Gamma(c)}{\Gamma(c-b)\,\Gamma(b+1)}$$

$$+\frac{\Gamma(-c+b+a+n+1)\,\Gamma(-b+c-n)\,\Gamma(1+a-c)\,\Gamma(c)}{\Gamma(1+a-c+n)\,\Gamma(c-n)\,\Gamma(c-b)\,\Gamma(-c+b+1+a)}$$

$$30:\quad (1+b-a-n)\,\Gamma(-c+1)\,\Gamma(n)\,\Gamma(a-b)$$

$$\left(\sum_{k=0}^{n-1}\frac{\Gamma(-b+a+k)\,\Gamma(-c-n+b+1+k)\,\Gamma(-c-n+2)}{\Gamma(a-b)\,\Gamma(b-c-n+1)\,\Gamma(k+1)\,\Gamma(-c-n+2+k)}\right)\Gamma(b-c-n+2-a)$$

$$\Gamma(b+1)\,\Gamma(-n+b+1)/((c-1+n)\,\Gamma(n-b+a)\,\Gamma(-c-n+1)\,\Gamma(1+b-c)\,\Gamma(b)\Gamma(2+b-a-n))$$

$$31:\quad (c-a-n)\,\Gamma(1+b-c)\,\Gamma(n)\,\Gamma(1+a-c)$$

$$\left(\sum_{k=0}^{n-1}\frac{\Gamma(1+a-c+k)\,\Gamma(b+k-n)\,\Gamma(2+b-n-c)}{\Gamma(1+a-c)\,\Gamma(b-n)\,\Gamma(k+1)\,\Gamma(2+b-n-c+k)}\right)\Gamma(c-a-n)\,\Gamma(c)$$

$$\Gamma(-n+b+1)/((c-1-b+n)\,\Gamma(1+a-c+n)\,\Gamma(b-c-n+1)\,\Gamma(b)\,\Gamma(c-1)\Gamma(-a+1+c-n))$$

$$33:\quad (c-a)\,\Gamma(1+b-c)\,\Gamma(n)\,\Gamma(1+a-c-n)$$

Mellin space

Mellin space primer

The Mellin transform $F(s)$ of a function $f(x)$ is given by

$$F(s) = \int_0^\infty \frac{dx}{x}\, x^s f(x)\,.$$

The inverse transformation, where c is any complex number whose real part satisfies a mild lower bound,

$$f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} ds\, x^{-s}\, F(s)\,.$$

Mellin space provides the natural space for studying scattering amplitudes in AdS, analogous to momentum space for flat space scattering amplitudes.

Momenta are the conjugate variables to translations while Mellin variables are conjugate to dilatations.

Mellin amplitudes [Mack '09] are meromorphic functions of their arguments and exhibit a simple analytic structure related to the OPE.

In Mellin space one can formulate the AdS/CFT equivalent of tree-level Feynman rules. [Fitzpatrick, Kaplan, Penedones, Raju, van Rees '11; Paulos '11]

The formalism extends to p-adic AdS/CFT [Jepsen, Parikh '19].

Wikipedia excerpt

Barnes lemmas [edit]

The first Barnes lemma (Barnes 1908) states

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\Gamma(a+s)\Gamma(b+s)\Gamma(c-s)\Gamma(d-s)ds=\frac{\Gamma(a+c)\Gamma(a+d)\Gamma(b+c)\Gamma(b+d)}{\Gamma(a+b+c+d)}.$$

This is an analogue of Gauss's ${}_2F_1$ summation formula, and also an extension of Euler's beta integral. The integral in it is sometimes called **Barnes's beta integral**.

The second Barnes lemma (Barnes 1910) states

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{\Gamma(a+s)\Gamma(b+s)\Gamma(c+s)\Gamma(1-d-s)\Gamma(-s)}{\Gamma(e+s)}ds$$
$$=\frac{\Gamma(a)\Gamma(b)\Gamma(c)\Gamma(1-d+a)\Gamma(1-d+b)\Gamma(1-d+c)}{\Gamma(e-a)\Gamma(e-b)\Gamma(e-c)}$$

where $e = a + b + c - d + 1$. This is an analogue of Saalschütz's summation formula.

Rewriting Barnes lemmas

Suppose we re-write the Barnes lemmas in terms of the local zeta function $\zeta_\infty(s) \equiv \pi^{-s/2}\,\Gamma(\frac{s}{2})$:

$$\int_{-i\infty}^{i\infty} \frac{dz}{2\pi i}\,\zeta_\infty(a+z)\zeta_\infty(b+z)\zeta_\infty(c-z)\zeta_\infty(d-z)$$

$$= 2\,\frac{\zeta_\infty(a+c)\zeta_\infty(a+d)\zeta_\infty(b+c)\zeta_\infty(b+d)}{\zeta_\infty(a+b+c+d)}$$

$$\int_{-i\infty}^{i\infty} \frac{dz}{2\pi i}\,\frac{\zeta_\infty(a+z)\zeta_\infty(b+z)\zeta_\infty(c+z)\zeta_\infty(d-z)\zeta_\infty(-z)}{\zeta_\infty(a+b+c+d+z)}$$

$$= 2\,\frac{\zeta_\infty(a)\zeta_\infty(b)\zeta_\infty(c)\zeta_\infty(a+d)\zeta_\infty(b+d)\zeta_\infty(c+d)}{\zeta_\infty(b+c+d)\zeta_\infty(a+c+d)\zeta_\infty(a+b+d)}$$

We can then ask, do similar identities exist for $\zeta_p(s) \equiv \frac{1}{1-p^{-s}}$?

p-adic Barnes lemmas

The answer is yes, very close analogs exist for the p-adics:

$$\int_{-\frac{i\pi}{\log p}}^{\frac{i\pi}{\log p}} \frac{dz}{2\pi i}\, \zeta_p(a+z)\zeta_p(b+z)\zeta_p(c-z)\zeta_p(d-z)$$
$$= \frac{1}{\log p}\, \frac{\zeta_p(a+c)\zeta_p(a+d)\zeta_p(b+c)\zeta_p(b+d)}{\zeta_p(a+b+c+d)}\,,$$

$$\int_{-\frac{i\pi}{\log p}}^{\frac{i\pi}{\log p}} \frac{dz}{2\pi i}\, \frac{\zeta_p(a+z)\zeta_p(b+z)\zeta_p(c+z)\zeta_p(d-z)\zeta_p(-z)}{\zeta_p(a+b+c+d+z)}$$
$$= \frac{1}{\log p}\, \frac{\zeta_p(a)\zeta_p(b)\zeta_p(c)\zeta_p(a+d)\zeta_p(b+d)\zeta_p(c+d)}{\zeta_p(b+c+d)\zeta_p(a+c+d)\zeta_p(a+b+d)}\,.$$

The integration range is finite because the function $\zeta_p(s) = (1-p^{-s})^{-1}$ is periodic in the imaginary direction.

Mellin amplitudes

Relation between position and Mellin space amplitudes ($v = \infty, p$):

$$A_v(x_1, ..., x_N) = \int [d\gamma]_v \, M_v(\{\gamma\}) \prod_{1\le i<j\le N} \frac{\zeta_v(2\gamma_{ij})}{|x_{ij}|_v^{2\gamma_{ij}}}$$

The Mellin integration measure:

$$[d\gamma]_v \equiv \left(\frac{c_v}{2\pi i}\right)^{\frac{N(N-3)}{2}} \left[\prod_{1\le i<j\le N} d\gamma_{ij}\right] \left[\prod_{i=1}^{N} \delta(\sum_{j=1}^{N} \gamma_{ij})\right]$$

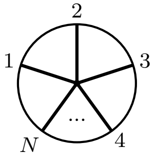
$$\gamma_{ij} = \gamma_{ji}, \qquad \gamma_{ii} = -\Delta_i, \qquad c_\infty = 1, \qquad c_p = 2\log p.$$

We can think of the Mellin variables as Mandelstam invariants associated with auxiliary moment k_i with masses squared Δ_i:

$$\gamma_{ij} = k_i \cdot k_j \,, \qquad \sum_{i=1}^{N} k_i = 0\,, \qquad k_i^2 = -\Delta_i \,.$$

The number of independent Mellin variables equals the number of independent conformal cross ratios.

N-point contact amplitude

$\mathcal{M}^{\text{contact}} =$ 

For contact Witten diagrams, the Mellin amplitude is a constant wrt. the Mellin variables:

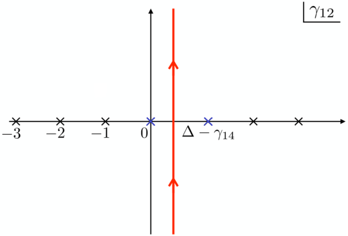
$$\mathbb{R}^d) \qquad \mathcal{M}^{\text{contact}} = \frac{1}{2}\zeta_\infty\Big(\sum_{i=1}^{N}\Delta_i - n\Big)\,,$$

$$\mathbb{Q}_{p^d}) \qquad \mathcal{M}^{\text{contact}} = \zeta_p\Big(\sum_{i=1}^{N}\Delta_i - n\Big)\,.$$

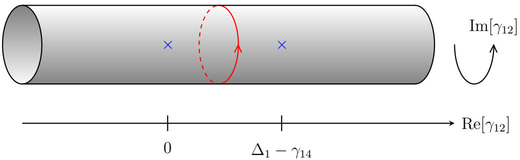
Contours for the 4-point contact amplitude

$$\int \frac{d\gamma_{12} d\gamma_{14}}{(2\pi i)^2} \zeta_v(2\gamma_{12})^2 \zeta_v(2\gamma_{14})^2 \zeta_v(2\Delta - 2\gamma_{12} - 2\gamma_{14})^2 u^{-\gamma_{12}} v^{-\gamma_{14}}$$

$u, v =$ conformal cross-ratios



$$\zeta_\infty(s) \equiv \pi^{-s/2}\,\Gamma(\frac{s}{2})$$

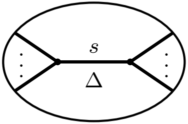


$$\zeta_p(s) \equiv \frac{1}{1 - p^{-s}}$$

The compact integration range is ultimately a consequence of the discreteness of the p-adic norm: dilatations are discrete ⇒ compact Mellin space integral.

The exchange amplitude

$\mathcal{M}^{\text{exchange}} = \quad i_L$



i_R

Once we introduce internal legs to Witten diagrams, the Mellin amplitudes begin to grow more complicated.

But all the dependency on Mandelstam variables enter through a single quantity analogous to a Mandelstam invariant:

$$s = \sum \Delta_{i_L} - 2 \sum \gamma_{i_L j_L} = \sum \Delta_{i_R} - 2 \sum \gamma_{i_R j_R} \,.$$

Real and p-adic exchange amplitudes:

$$\mathbb{R}^d) \qquad \mathcal{M}^{\text{exc}} = \frac{\zeta_\infty(\sum \Delta_{i_L} + \Delta - \frac{d}{2})\, \zeta_\infty(\sum \Delta_{i_R} + \Delta - \frac{d}{2})}{(s - \Delta)\,(2\Delta - d)\,\zeta_\infty(2\Delta - n)} \times {}_3F_2\left(\frac{2 - \sum \Delta_{i_L} + \Delta}{2}, \frac{2 - \sum \Delta_{i_R} + \Delta}{2}, \frac{2 + \Delta - s}{2}; 1 + \Delta - \frac{d}{4}; 1\right) \,,$$

$$\mathbb{Q}_{p^d}) \quad \mathcal{M}^{\text{exc}} = -\zeta_p(\Delta + \sum \Delta_{i_L} - n)\zeta_p(\Delta + \sum \Delta_{i_R} - d) \times \left(\zeta_p(s - \Delta) - \zeta_p(\sum \Delta_i - d)\right) \,.$$

Mellin pre-amplitudes

The real and p-adic exchange amplitudes looked rather different and the real answered involved a ${}_3F_2$ hypergeometric function.

But there exists a trick for dealing with the infinite sums: re-express them as contour integrals with gamma functions.

To this end, [Yuan '17] introduced the concept of a pre-amplitude $\widetilde{M}$:

$$\mathcal{M} = \left[\prod_{\text{internal legs } I} \int \frac{dc_I}{2\pi i} f_{\Delta_I}(c_I) \right] \widetilde{\mathcal{M}} .$$

The pre-amplitude formalism can also be applied in the p-adic case. In each case the weigh function is given by

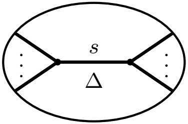
$\mathbb{Q}_{p^d})$ $$f_{\Delta_I} = \frac{\log p}{2} \frac{\zeta_p\left(\Delta_I - \frac{d}{2} + c_I\right) \zeta_p\left(\Delta_I - \frac{d}{2} - c_I\right)}{\zeta_p(2c_I)\, \zeta_p(-2c_I)} ,$$

$\mathbb{R}^n)$ $$f_{\Delta_I} = \frac{(2\Delta_I - d)\, \zeta_\infty(2\Delta_I - d)}{(\Delta_I - \frac{d}{2} + c_I)\, (\Delta_I - \frac{d}{2} - c_I)\, \zeta_\infty(2c_I) \zeta_\infty(-2c_I)} .$$

The exchange pre-amplitudes

Let us revisit the exchange diagram.

$\mathcal{M}^{\text{exchange}} = \quad i_L$



i_R

The exchange pre-amplitudes take on similar forms in the two formalisms:

$$\mathbb{Q}_{p^d}) \qquad \widetilde{\mathcal{M}}^{\text{exchange}} = \zeta_p\Big(\sum \Delta_{i_L} + c - \frac{d}{2}\Big)\zeta_p\Big(\sum \Delta_{i_R} - c - \frac{d}{2}\Big) \times \beta_p\left(\frac{d}{2} + c - s, \sum \Delta_{i_L} - \frac{d}{2} - c\right)\beta_p\left(\frac{d}{2} - c - s, \sum \Delta_{i_R} - \frac{d}{2} + c\right),$$

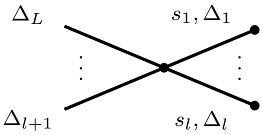
$$\mathbb{R}^d) \qquad \widetilde{\mathcal{M}}^{\text{exchange}} = \frac{1}{4}\zeta_\infty\Big(\sum \Delta_{i_L} + c - \frac{d}{2}\Big)\zeta_\infty\Big(\sum \Delta_{i_R} - c - \frac{d}{2}\Big) \times \beta_\infty\left(\frac{d}{2} + c - s, \sum \Delta_{i_L} - \frac{d}{2} - c\right)\beta_\infty\left(\frac{d}{2} - c - s, \sum \Delta_{i_R} - \frac{d}{2} + c\right),$$

where $\quad \beta_p(s,t) \equiv \dfrac{\zeta_p(s)\zeta_p(t)}{\zeta_p(s+t)}, \qquad \beta_\infty(s,t) \equiv \dfrac{\zeta_\infty(s)\zeta_\infty(t)}{\zeta_\infty(s+t)}.$

Pre-amplitude Feynman rules

internal legs $i \in \{1, ..., l\}$: $\widetilde{\Delta}_i = h \pm c_I$ external legs $i \in \{l+1, ..., L\}$: $\widetilde{\Delta}_i = \Delta_i$

Associate to each vertex



a factor of:

$$\widetilde{V}_p = \zeta_p\Big(\sum_{i=1}^{L} \widetilde{\Delta}_i - n\Big)\left[\prod_{i=1}^{l} \frac{2\,\zeta_p(1)}{|2|_p} \int_{\mathbb{Q}_p^2} \frac{dx_i}{|x_i|_p}\, |x_i|_p^{\frac{\widetilde{\Delta}_i - s_i}{2}} |1, x_i|_s^{\frac{\widetilde{\Delta}_i + s_i - n}{2}}\right] |1, x_1, \ldots, x_l|_s^{\frac{n - \sum_{i=1}^{L} \widetilde{\Delta}_i}{2}}$$

$$\widetilde{V}_\infty = \zeta_\infty\Big(\sum_{i=1}^{L} \widetilde{\Delta}_i - n\Big)\left[\prod_{i=1}^{l} \int_{\mathbb{R}^2} \frac{dx_i}{|x_i|} |x_i|^{\frac{\widetilde{\Delta}_i - s_i}{2}} |1 + x_i|^{\frac{\widetilde{\Delta}_i + s_i - n}{2}}\right] \Big|1 + \sum_{i=1}^{l} x_i\Big|^{\frac{n - \sum_{i=1}^{L} \widetilde{\Delta}_i}{2}}$$

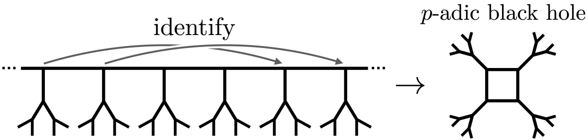
Multiply the vertex factors to get the pre-amplitude : $\widetilde{\mathcal{M}} = \prod_{\text{vertices}} \widetilde{V}\,.$

Finite temperature CFTs

The arithmetic BTZ black hole

[Bañados, C. Teitelboim '92] found an arithmetic construction of the BTZ black hole as quotient of the AdS isometry group by a discrete subgroup.

[Heydeman, Marcolli, Saberi, Stoica '16] and [Manin, Marcolli '02] proposed a p-adic BTZ black hole based on a discrete quotient of the Bruhat-Tits tree isometry group.



The same geometry was studied by [Chekhov, Mironov, Zabrodin '89] in the context of higher-genus p-adic string amplitudes.

The holographic dual to the quotient bulk geometry is a CFT at finite temperature.

Thermal CFT basics

Foundational literature on finite temperature CFTs: [El-Showk, Papadodimas '11], [Katz, Sachdev, Sørensen, Witczak-Kremp '14], [Witczak-Krempa, '15], [Iliesiu, Koloğlu, Mahajan, Perlmutter, Simons-Duffin '18].

Thermal CFTs are CFTS that live on $S^1 \times \mathbb{R}^{d-1}$. Locally, the geometry remains flat, and so the OPE remains valid

$$\mathcal{O}_i(x_1)\mathcal{O}_j(x_2) = \sum_k c_{ij}^k(x_{12}, \partial_2)\, \mathcal{O}_k(x_2)\,,$$

provided the boundary points are contained within a sphere of radius β.

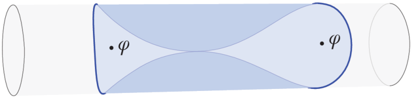


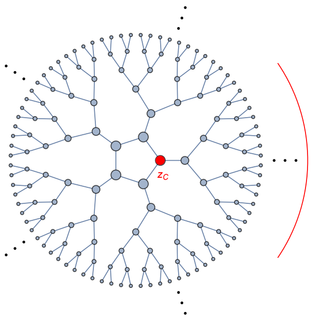
Figure 1: The OPE on $S^1_\beta \times \mathbb{R}^{d-1}$ is valid if the two operators lie inside a sphere. The largest possible sphere has diameter β, wrapping entirely around the S^1 such that it is tangent to itself. Here, we illustrate such a sphere (blue) in $d = 2$.

But because the thermal circle introduces a scale, traceless-symmetric operators can pick up non-zero one-point functions:

$$\langle \mathcal{O}^{\mu_1\cdots\mu_J}(x)\rangle_\beta = \frac{b_{\mathcal{O}}}{\beta^{\Delta_{\mathcal{O}}}}\left(e^{\mu_1}...e^{\mu_J} - \text{traces}\right).$$

Finite temperature at finite places

The OPE convergence condition in the p-adic case is the condition that two bulk points emanate form the same vertex z_c on the thermal cycle.



A scale $|\beta|$ is introduced through the volume of the boundary points sitting on the same branch of the thermal cycle.

$$\int_{z_C \to x} dx = \frac{p^d - 1}{p^d} \, |\beta|_p^d \, .$$

Let us call the number of links on the cycle w.

Thermal mean field theory

[Iliesiu, Koloğlu, Mahajan, Perlmutter, Simons-Duffin '18]: the thermal two-point conformal blocks are given in terms of Gegenbauer polynomials $C_\ell^{(\nu)}(\eta)$ and the mean field two-point correlator has the expansion

$$\langle \mathcal{O}_\Delta(x_1)\mathcal{O}_\Delta(x_2)\rangle^{(0)} = \frac{1}{|x_1-x_2|^{2\Delta}} + \sum_{n\in\mathbb{N}_0}\sum_{\ell\in 2\mathbb{N}_0} a_{[\phi\phi]_{n,\ell}}\, C_\ell^{(\nu)}(\eta)\, \frac{|x_1-x_2|^{2n+\ell}}{|\beta|^{2\Delta+2n+\ell}}\,,$$

where $\quad a_{[\phi\phi]_{n,\ell}} = 2\zeta(2\Delta+2n+\ell)\dfrac{(\ell+\nu)(\Delta)_{\ell+n}(\Delta-\nu)_n}{n!(\nu)_{\ell+n+1}}\,, \quad \nu = \dfrac{d-2}{d}\,, \quad \eta = \dfrac{\tau_1-\tau_2}{|x_1-x_2|}\,.$

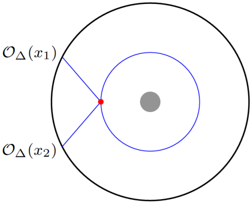
For comparison with the p-adics, let us omit derivative terms:

$$\langle \mathcal{O}_\Delta(x_1)\mathcal{O}_\Delta(x_2)\rangle^{(0)} = \frac{1}{|x_1-x_2|^{2\Delta}} + \frac{2\zeta(\Delta)}{|\beta|^{2\Delta}} + (\text{terms from ops. with derivatives})\,.$$

For p-adic mean field theory the answer is [Huang, Jepsen '24]:

$$\langle \mathcal{O}_\Delta(x_1)\mathcal{O}_\Delta(x_2)\rangle^{(0)} = \frac{1}{|x_1-x_2|_p^{2\Delta}} + \frac{2\zeta_p(w\Delta)p^{-w\Delta}}{|\beta|_p^{2\Delta}}\,.$$

One-loop computation



[Alday, Koloğlu, Zhiboedov '20]

$$\langle \mathcal{O}_\Delta(x_1)\mathcal{O}_\Delta(x_2)\rangle^{(1)} = -\frac{2^{-d-2}\pi^{-d/2}}{\Gamma(\Delta)\Gamma(\Delta-\frac{d-2}{2})}\int_{s_0-i\infty}^{s_0+i\infty}\frac{ds}{2\pi i}\Gamma(s)^2\Gamma(\Delta-s)^2\times$$

$$\frac{\zeta(2\Delta-2s)\Gamma(\Delta-\frac{d-1}{2}-s)\Gamma(2\Delta-\frac{d}{2}-s)}{\Gamma(\Delta+\frac{1}{2}-s)\Gamma(2\Delta-d+1-s)}\frac{\langle \mathcal{O}_s(x_1)\mathcal{O}_s(x_2)\rangle^{(0)}}{|\beta|^{2\Delta-2s}},$$

"suggestive of a thermal Mellin amplitude"

[Huang, Jepsen '24]

$$\langle \mathcal{O}_\Delta(x_1)\mathcal{O}_\Delta(x_2)\rangle^{(1)} = -p^{-w\Delta}\zeta_p(w\Delta)\frac{\zeta_p(4\Delta-d)}{\zeta_p(2\Delta)^2}\times$$

$$\frac{\log p}{2\pi i}\int_{s_0-\frac{i\pi}{\log p}}^{s_0+\frac{i\pi}{\log p}} ds\,\frac{\zeta_p(\Delta-s)^2\zeta_p(2s)^2}{\zeta_p(2\Delta+2s)}\frac{\langle \mathcal{O}_s(x_1)\mathcal{O}_s(x_2)\rangle^{(0)}}{|\beta|_p^{2\Delta-2s}}.$$

Fermions

Quadratic and higher extensions

$\mathbb{C}$ obtained by a field extension of $\mathbb{R}$ with respect to $\sqrt{-1}$. Extending $\mathbb{R}$ by the square root of any other negative number still gives $\mathbb{C}$.

Three different quadratic extensions of $\mathbb{Q}_p$ by $\sqrt{\tau}$ depending on the choice of non-square $\tau \in \mathbb{Q}_p$.

While $\mathbb{C}$ is algebraically closed, higher extensions are also possible for $\mathbb{Q}_p$.

Infinitely many extensions required to get an algebraically closed field from $\mathbb{Q}_p$. And the field so obtained is not complete.

But the completion of $\mathbb{C}_p$ this field is algebraically closed.

Sign functions

To have something like a fermion, you want a sign function in the two-point correlator:

$$\langle \psi(x)\psi(y)\rangle = \frac{\mathrm{sgn}_\tau(x-y)}{|x-y|_p^{2\Delta}}\,.$$

Every element in a quadratic extension $\mathbb{Q}_p(\sqrt{\tau})$ can be written as $x + \sqrt{\tau}\,y$ with $x, y \in \mathbb{Q}$.

Given a τ we can define a sign function by the condition that for any $z \in \mathbb{Q}_p$

$$\mathrm{sgn}_\tau(z) = \begin{cases} 1 & \text{if } z = (x+\sqrt{\tau}y)(x-\sqrt{\tau}y) \text{ for some } x, y \in \mathbb{Q}\,, \\ -1 & \text{otherwise.} \end{cases}$$

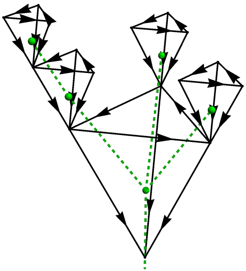
Such sign functions were used in the context of p-adic string theory in [Freund, Witten '87].

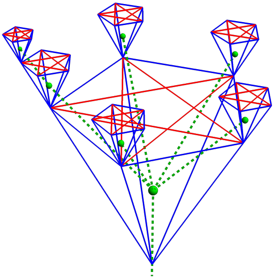
In the AdS/CFT context they were used to formulate a kind of SYK models in [Gubser, Heydeman, Jepsen, Parikh, Saberi, Stoica Trundy 2018], [Gubser, Jepsen, Ji, Trundy '18].

Since $\mathbb{Q}_p$ has three quadratic extensions, there are three sign functions. And some of them are even: $\mathrm{sgn}_\tau(-1) = 1$.

Bulk Fermions

Fermionic 2-point function obtained holographically from a bulk theory on the line graph of the Bruhat-Tits tree [Gubser, Jepsen, Trundy '18]:





Caveat: for the special case $p = 2$, there are seven sign functions, and a fermionic bulk dual has not been determined.

Recent progress on the Dirac operator on graphs: [Casiday, Contreras '22], [Delporte, Sen, Toriumi '24].

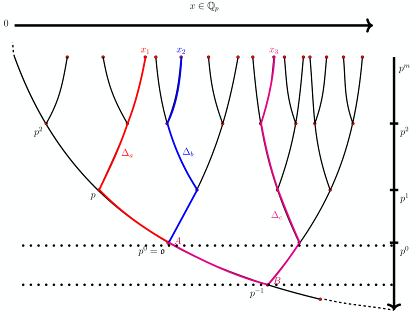
Tensor Networks

Tensor Network realization of p-adic CFTs

A set of axioms for p-adic CFT correlators were given in [Melzer '88]

[Bhattacharyya, Hung, Lei, Li, '18] pointed to the possibility of proving a bulk realization of these axioms via a tensor network construction.

This goal was achieved in [Hung, Li, Melby-Thompson 2019] for purely scalar theories on the Bruhat-Tits tree and the BTZ geometry.



$$\langle \mathcal{O}_a(x_1)\mathcal{O}_b(x_2)\mathcal{O}_c(x_3)\rangle = \sum_d C_{cd} C^d{}_{ab} |z_\epsilon|_p^{-\Delta_{abc}} \left(\frac{|z_\epsilon|_p}{|x_{12}|_p}\right)^{\Delta_{abc}} \left(\frac{|x_{12}|_p}{|x_{32}|_p}\right)^{\Delta_{cd}}$$

$$= \frac{C_{abc}}{|x_{12}|_p^{\Delta_a+\Delta_b-\Delta_c} |x_{23}|_p^{\Delta_b+\Delta_c-\Delta_a} |x_{13}|_p^{\Delta_a+\Delta_c-\Delta_b}} .$$

Tensor Networks and p-adic AdS/CFT

[Chen, Liu, Hung '21] extended the tensor network construction of p-adic CFTs to more deformed versions of the Bruhat-Tits tree and thereby obtained graph analogs of the Einstein equations.

In the process they recovered a notion of graph curvature previously known in the math literature.

Heydeman, Marcolli, Parikh, Saberi '18] studied quantum error-correcting codes on graphs motivated by p-adic AdS/CFT.

They establishined that a p-adic type of entropy obeys a Ryu-Takayangi type formula as well as strong subadditivity and monogamy of mutual information

Part III: Open Problems

Pushing the finite temperature frontier

In the context of p-adic AdS/CFT, it is comparatively easy to compute many finite temperature correlators that haven't been computed at the real place.

For example, [Huang, Jepsen '24] carried out the three-point thermal conformal block expansion:

$$\left\langle \mathcal{O}_{\Delta_1}(x_1)\mathcal{O}_{\Delta_2}(x_2)\mathcal{O}_{\Delta_3}(x_3)\right\rangle = \sum_{z\in T_{p^d}^{(w)}} K_{\Delta_1}(x_1,z)K_{\Delta_2}(x_2,z)K_{\Delta_3}(x_3,z)$$

$$= \quad C(\Delta_1,\Delta_2,\Delta_3)\times$$

$$\Bigg(\frac{1}{|x_{1,2}|_p^{\Delta_{12,3}}|x_{1,3}|_p^{\Delta_{13,2}}|x_{2,3}|_p^{\Delta_{23,1}}}+\frac{2\zeta_p(w\Delta_1)p^{-w\Delta_1}}{|x_{2,3}|_p^{\Delta_{23,1}}|\beta|_p^{2\Delta_1}}+\frac{2\zeta_p(w\Delta_2)p^{-w\Delta_2}}{|x_{1,3}|_p^{\Delta_{13,2}}|\beta|_p^{2\Delta_2}}+\frac{2\zeta_p(w\Delta_3)p^{-w\Delta_3}}{|x_{1,2}|_p^{\Delta_{12,3}}|\beta|_p^{2\Delta_3}}$$

$$+\frac{2\zeta_p(w\Delta_2)\zeta_p(w\Delta_3)p^{-w\Delta_{23}}}{|\beta|_p^{\Delta_{123}}}+\frac{2\zeta_p(w\Delta_1)\zeta_p(w\Delta_3)p^{-w\Delta_{13}}}{|\beta|_p^{\Delta_{123}}}+\frac{2\zeta_p(w\Delta_1)\zeta_p(w\Delta_2)p^{-w\Delta_{12}}}{|\beta|_p^{\Delta_{123}}}\Bigg).$$

The analogous real decomposition appears to involve multiple-zeta functions.

Perhaps the p-adic toy model can help better understand the real correlators, ultimately paving the way for finite temperature Feynman rules.

Arithmetic quantum gravity?

The p-adic BTZ black holes are topologically distinct geometries, labelled by the w of edges in the thermal cycle.

Perhaps only the $w = 1$ geometry should be thought of as the p-adic BTZ black hole, with $w > 1$ instead giving other black hole geometries.

On the real side, the BTZ black hole is part of a larger family of solutions to the Einstein equations known as the $SL(2, \mathbb{Z})$ black holes.

The sum over these geometries produces the gravitational path integral. [Witten '07; Maloney, Witten '07, Maloney, Song, Strominger '09], have brought to light the many connections between this object and wormholes, CFT ensembles, modular transformations, and the j-invariant.

Can a similar sum be performed over p-adic geometries?

$O(N)$/higher-spin analog?

An important instance of holography is the duality between the $O(N)$ model and higher spin theory [Klebanov, Polyakov '02], [Giombi, Yin '09], [Neiman '15], [Sleight, Tarona '16], [many many more].

[Aharony, Chester, Urbach '20] and [Aharony, Chester, Sheaffer, Urbach '22] presented mathematical steps for a derivation of this duality.

They used the earlier work of [de Mello Koch, Jevicki, Jin, Rodrigues '11; '13], [de Mello Koch, Jevicki, Rodrigues, Yoon '15], [de Mello Koch, Jevicki, Rodrigues, Yoon '15], and [de Mello Koch, Jevicki, Suzuki, Yoon '19], which expands the vector models in bi-local fields.

Some of the core identities used in their work have direct p-adic parallels:

$$\int dy \, \langle \mathcal{O}_\Delta(x_1)\mathcal{O}_\Delta(y)\rangle \, \langle \mathcal{O}_{\Delta_0}(x_2)\mathcal{O}_{\Delta_0}(x_3)\mathcal{O}_{d-\Delta}(y)\rangle = S_\Delta^{(p)} \, \langle \mathcal{O}_\Delta(x_1)\mathcal{O}_{\Delta_0}(x_2)\mathcal{O}_{\Delta_0}(x_3)\rangle \,,$$

$$\int dx_1 dx_2 \, \langle \mathcal{O}_{\Delta_0}(x_1)\mathcal{O}_{\Delta_0}(x_2)\mathcal{O}_\Delta(y)\rangle \, \langle \mathcal{O}_{d-\Delta_0}(x_1)\mathcal{O}_{d-\Delta_0}(x_2)\mathcal{O}_{d-\Delta'}(y')\rangle$$

$$= 2\pi i N_\Delta^{(p)} \left(\delta(y-y')\delta(\Delta-\Delta') + \frac{1}{S_\Delta^{(p)}} \delta(\Delta+\Delta'-d) \right),$$

where $$S_\Delta^{(p)} = \frac{\Gamma_p(\Delta)^2}{\Gamma_p(2\Delta)} \,, \qquad N_\Delta^{(p)} = -\frac{\Gamma_p(2\Delta-d)\Gamma_p(d-2\Delta)}{\zeta_p(d)\log p} \,.$$

Supersymmetric explorations

[Ruelle, Thiran, Verstegen, Weyers 1989] proposed a p-adic superstring amplitude. They did not have an action or integration formula, the proposal was based on adelic considerations:

The adelic formula of [Freund, Witten '87] hinges on the functional equation for the Riemann zeta function. Similar formulas can be conceived of for other Dirichlet L functions.

[Ubriaco 1990] described supersymmetry transformations acting on a p-adic quantum field theory.

If the later progress on p-adic fermions can be leveraged to determine the superstring N-point amplitudes, it may be possible to compute the effective spacetime action in p-adic superstring theory.

Exotic universality classes

Quantum system in d dimension = classical system in $d+1$ dimensions.

Hierarchical quantum system = classical system with p-adic and real directions.

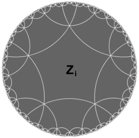
[Gubser, Jepsen, Ji, Trundy '18] studied the critical behaviour of such systems.

[Gubser, Jepsen, Ji, Trundy '19] presented simple lattice systems with continuum limits that interpolate between real and p-adic smoothness as a parameter is varied.

Proposed experimental realization: [Bentsen, Hashizume, Buyskikh, Davis, Daley, Gubser, Schleier-Smith '20].

Actual data from an optical cavity: [Periwal, Cooper, Kunkel, Wienand, Davis, Schleier-Smith' 21].

[Yan, Jepsen, Oz '23]: $\mathbb{Z}_2$ model on a hyperbolic tesselation with edge and vertex interactions described by decoupled Zabrodin Bruhat-Tits models in the weak-edge limit.



Is a more systematic understanding of exotic critical phenomena possible?

Conclusion

Roughly speaking, the four types of motivation given for studying p-adic AdS/CFT, and the status of those goals are:

∗ Adelic construction — studying p-adic theories because they secretly provide the building blocks of real physics: so far, this has not proven itself to be a useful approach.

∗ Using physics thinking to learn new math: this goal is as of yet unrealized.

∗ Using p-adic AdS/CFT as a toy model to tackle difficult computations: this goal has been realized.

∗ Using p-adic theories to understand unusual behaviour of statistical models and AMO sytems: incipient realization has taken place.

Thank You