

Peeking at quantum gravity with self-overlapping curves

TSVP Symposium: Frontiers in Nonlinear Differential Equations
and Stokes Phenomena

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1 Motivations

2 Immersions of the disk

3 Perspectives

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What is Quantum Gravity?

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From the point of view of quantum field theory, we would like to write and solve:

$$Z = \int_{\mathcal{M}} \mathcal{D}g_{\mu\nu} \mathcal{D}\Phi \exp(-S[g_{\mu\nu}, \Phi])$$

$g_{\mu\nu}$: metric structure on \mathcal{M} ; Φ : matter content.

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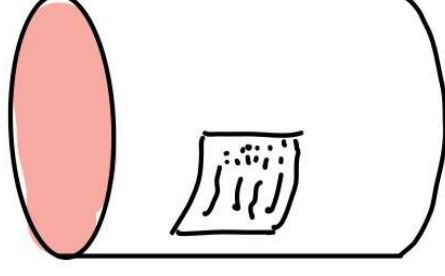
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What are \mathcal{D} , $S[\cdot]$, Z ?

One approach: Holography

Postulate ([’t Hooft 1993, Susskind 1995]):

Quantum Gravity ($D + 1$ dimensions)
= Quantum Theory (D dimensions)



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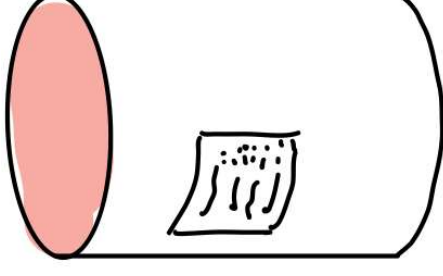
Quantum Gravity ($D + 1$ dimensions)
= **Quantum Theory (D dimensions)**

Hints for the emergence of gravity:

- Black hole entropy [Bekenstein 1972]:

$$S_{BH} = \frac{c^3}{4G\hbar} A_{\text{horizon}}$$

- Laws of black hole thermodynamics [Bardeen, Bekenstein, Carter, Hawking 1973]

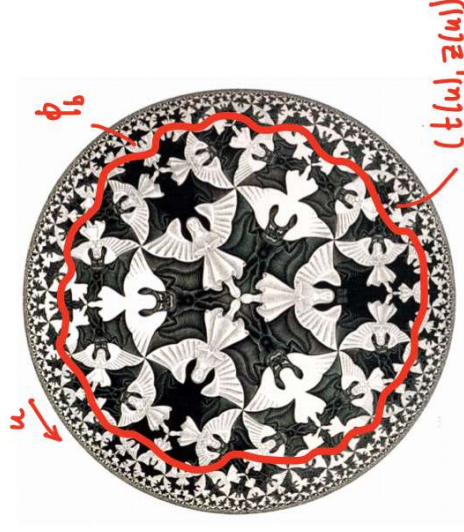


A simple model of holography

Bulk: Near-horizon limit of (near-extremal) black holes

→ **Jackiw-Teitelboim theory** ($D = 2$):

$$Z = \int \mathcal{D}g_{\mu\nu} \mathcal{D}\phi \exp \left(\frac{1}{16\pi G_N} \int d^2x \sqrt{g} \phi (R + 2) + \frac{\phi_b}{8\pi G_N} \int \phi \, dsk \right) .$$



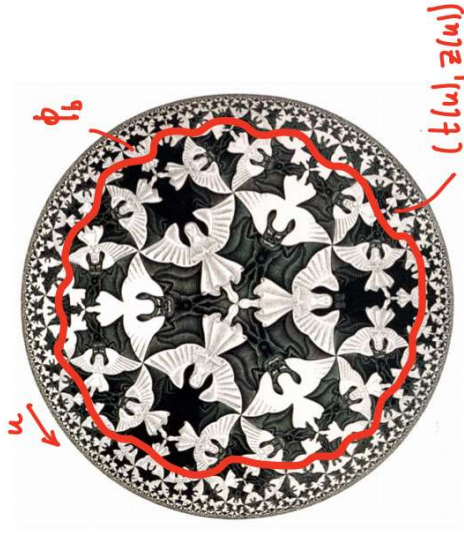
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Boundary: **Sachdev-Ye-Kitaev model** ($D = 1$)

$$H = \sum_{1 \leq i < j < k < l \leq N} J_{ijkl} \psi_i \psi_j \psi_k \psi_l, \quad \left\langle J_{ijkl}^2 \right\rangle \propto \frac{J^2}{N^3} .$$

A simple model of holography

Same effective action on the boundary (a comprehensive review is [\[Mertens 2022\]](#)):

$$\frac{\phi_b}{8\pi G_N} \int_{S^1} du Sch[t, u] ,$$

with $t : S^1 \rightarrow S^1$, $t' > 0$ (reparametrization) and $Sch[t, u] = \left(\frac{t''}{t'} \right)' - \frac{1}{2} \left(\frac{t''}{t'} \right)^2$.

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Also equivalent to a random matrix theory [\[Saad, Shenker, Stanford 2019\]](#):

$$Z = \int dH \exp(-\text{Tr } V(H)) ,$$
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But... no Hilbert space interpretation!

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We are interested in metrics on the disk.

Conformal gauge:

$$ds^2 = e^{2\Sigma} |dz|^2, \quad z = x + iy.$$

Metric of constant curvature:

$$4\partial_z\partial_{\bar{z}}\Sigma = -\kappa e^{2\Sigma}, \quad \kappa = \pm 1, 0.$$

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Theorem

- a) Let $\Sigma_b : S^1 \rightarrow \mathbb{R}$ be a continuous function defined on the boundary of the disk. Then there exists a unique solution $\Sigma \in C^\infty(D)$ of the Liouville equation such that $\Sigma = \Sigma_b$ on the boundary.
- b) The most generic solution (up to disk automorphisms, $PSL(2, \mathbb{R})$) takes the form:

$$e^\Sigma = \frac{2|F'(z)|}{1 + \kappa|F(z)|^2}.$$

We parametrize metrics on the disk \mathcal{D} :

$$ds^2 = \frac{4|F'(z)|^2}{(1 + \kappa|F(z)|^2)^2} |dz|^2, \quad \begin{cases} F : \mathcal{D} \rightarrow H^2, \mathbb{R}^2, S^2 \text{ holomorphic,} \\ F'(z) \neq 0 \quad \forall z \in \mathcal{D}. \end{cases}$$

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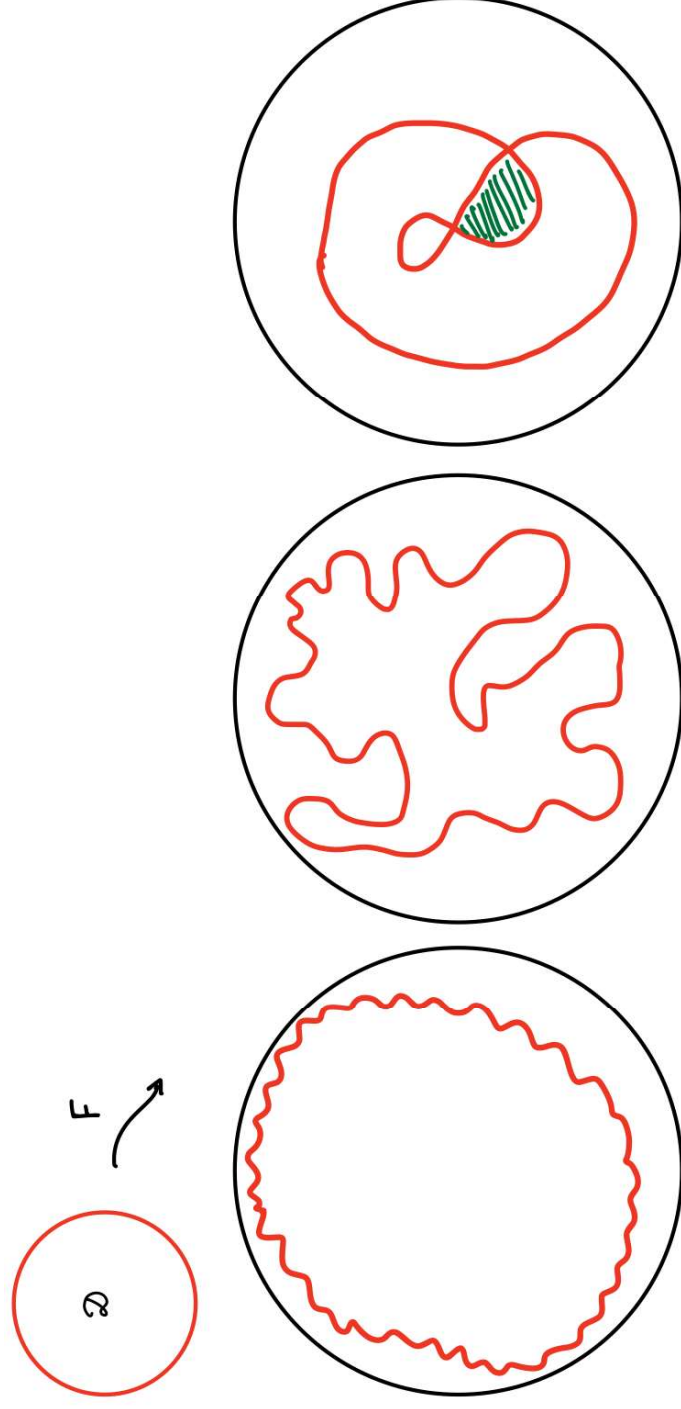


Figure: Reparametrization embedding – General embedding – Immersion

- How does considering immersions change the previous results?
- What are the properties of those immersions?
Number of self-overlaps, fractals,...?
- Of their boundaries?
Do they characterise the whole immersion?
Minimal combinatorial properties that lead to an immersion?

Self-overlapping curves = curves that are boundary of an immersed disk.

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Evans & Wenk [2020] (interior boundary, minimum homotopy area).

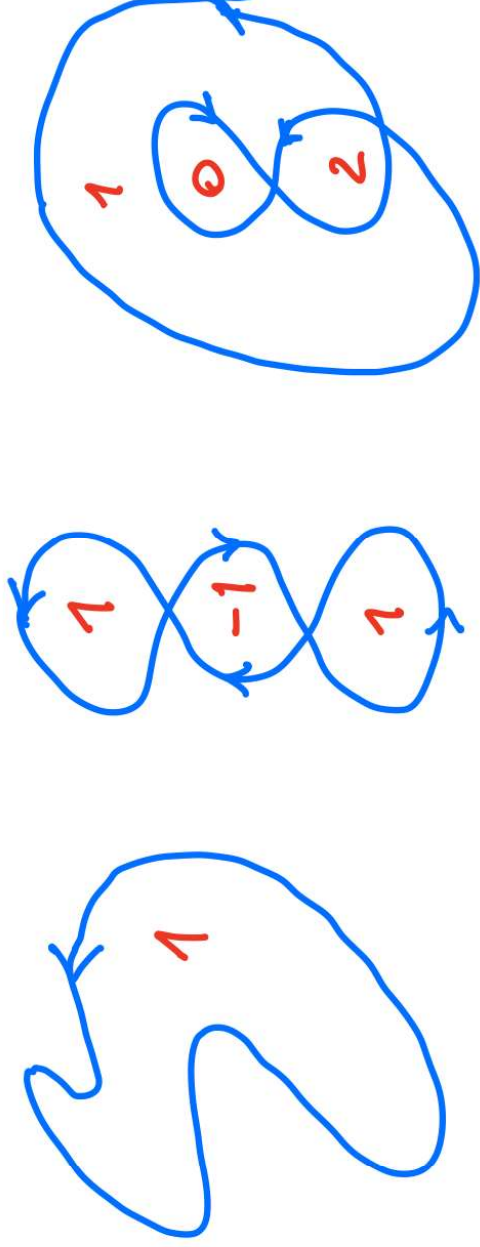
Self-overlapping curves: Numbers

- Turning number

$$\text{turn}(\gamma) = \frac{1}{2\pi} \oint_{\gamma} k ds = 1, \quad k = \frac{x'y'' - y'x''}{(x'^2 + y'^2)^{3/2}}.$$

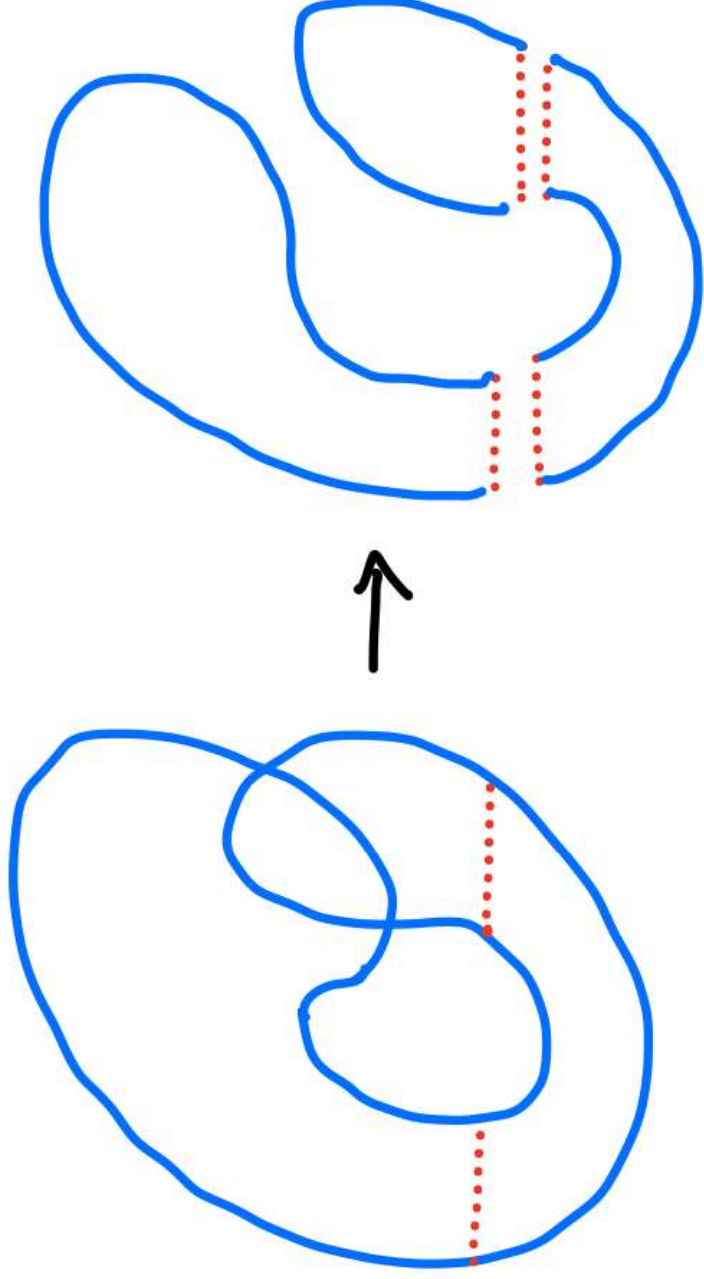
- Winding number (number of overlaps)

$$\text{wind}_{\gamma}(z_0) = \frac{1}{2\pi i} \oint_{\gamma} \frac{\gamma'(t)}{\gamma(t) - z_0} dt \geq 0.$$



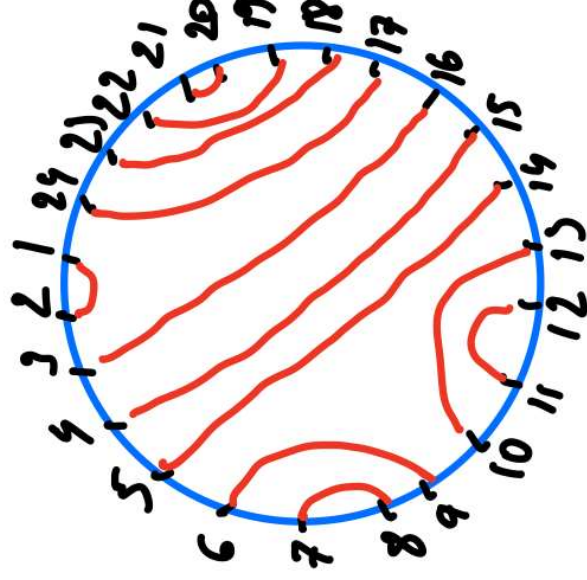
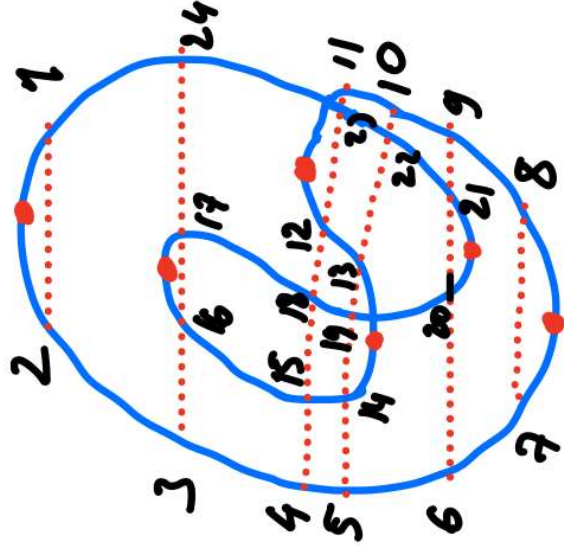
Self-overlapping curves: Cuts

Curves that can be decomposed into simple curves through well-chosen cuts.



Self-overlapping curves: Maximally Planar Matchings

[Bonsma, Breuer 2012] Mapping the curve, together with good Blank cuts*, to a chordal graph, the problem of counting inequivalent disks corresponds to counting **Maximum Independent Sets** in the circle graph (for n vertices of the circle graph, $\mathcal{O}(n^2)$).



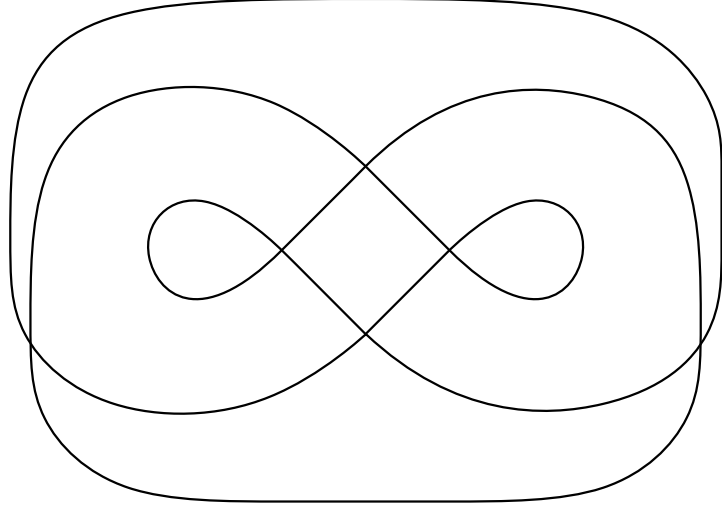
*such that it can define the interior of a disk

Self-overlapping curves: Inequivalent disks

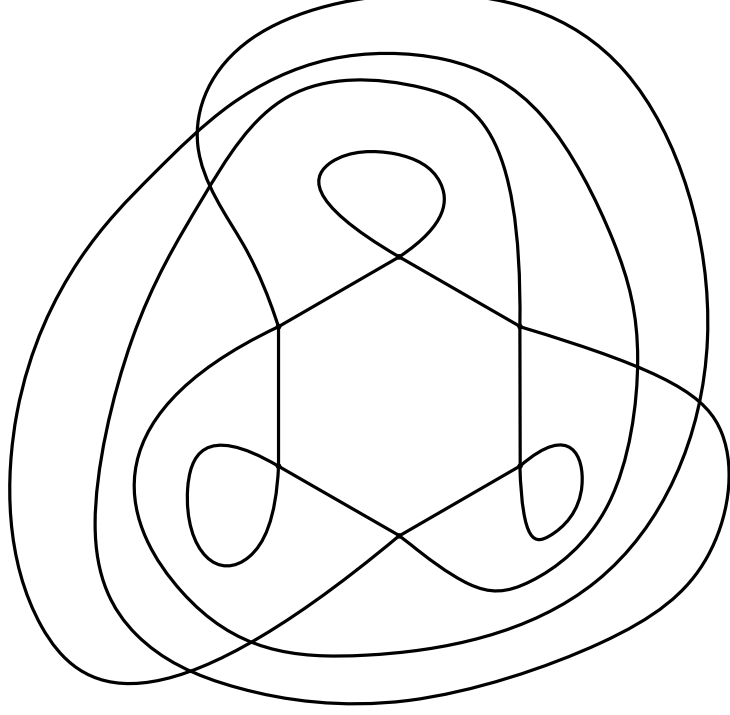
Examples of boundary curves that don't have a unique holomorphic extension:

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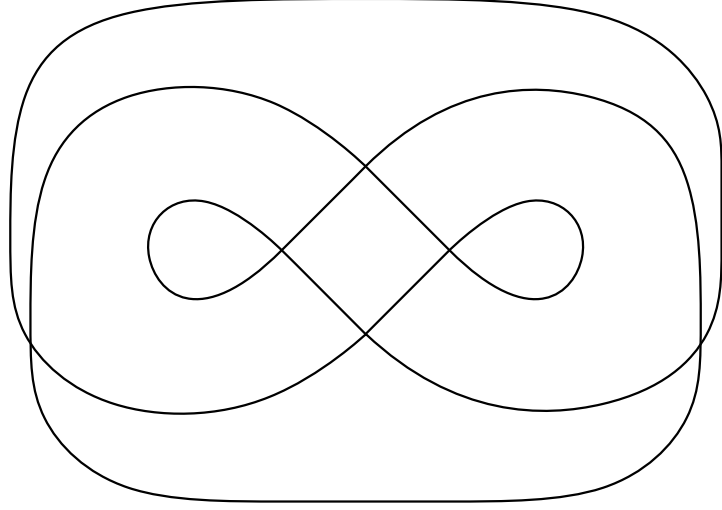
(a) Milnor (2 disks)



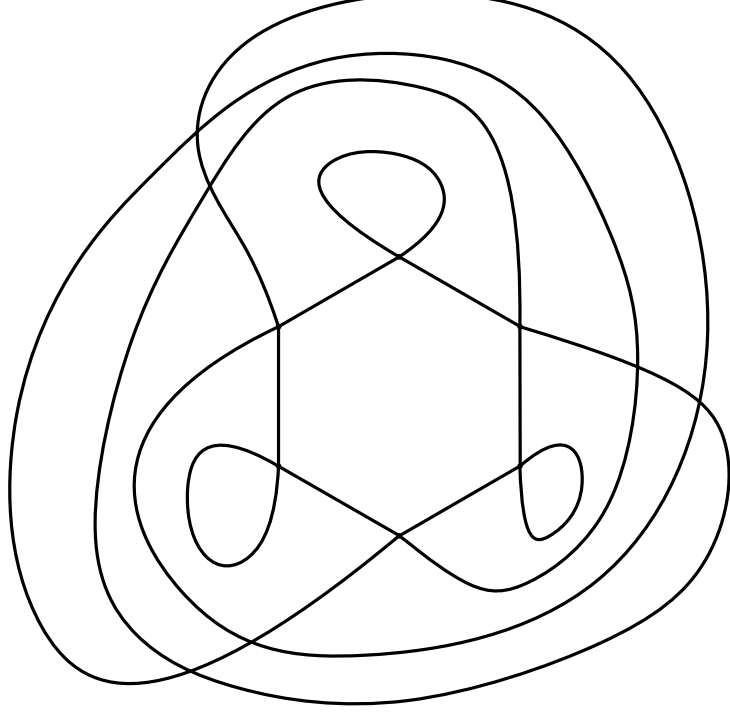
(b) Bennequin (5 disks)

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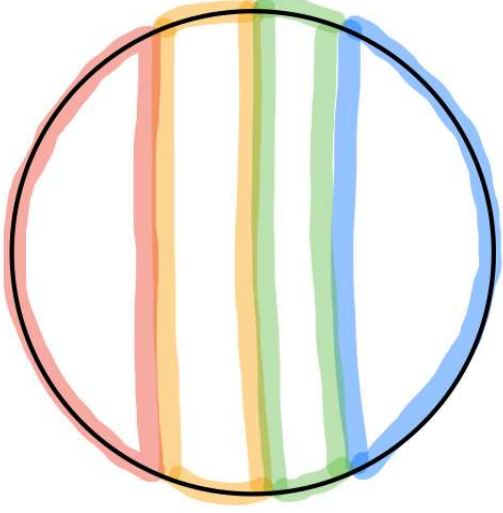
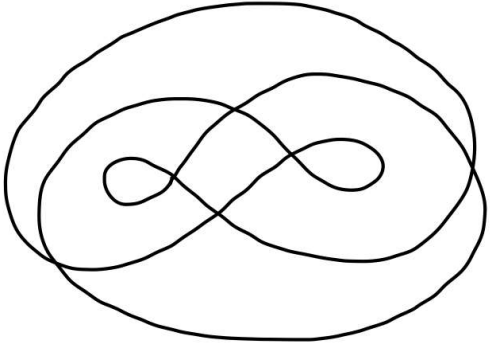


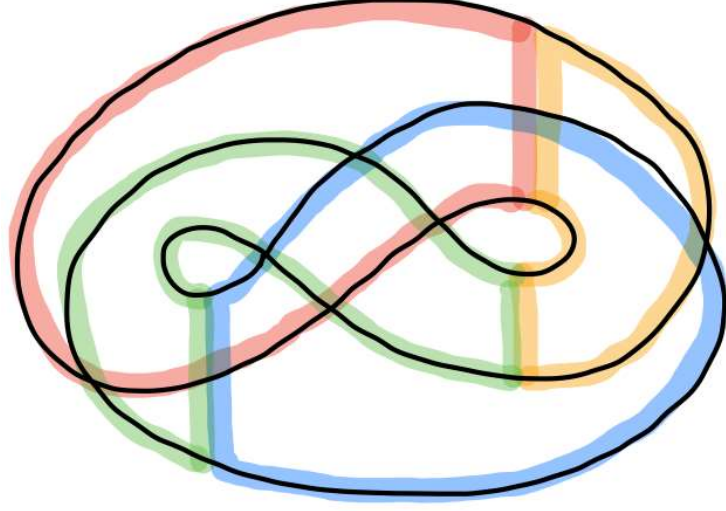
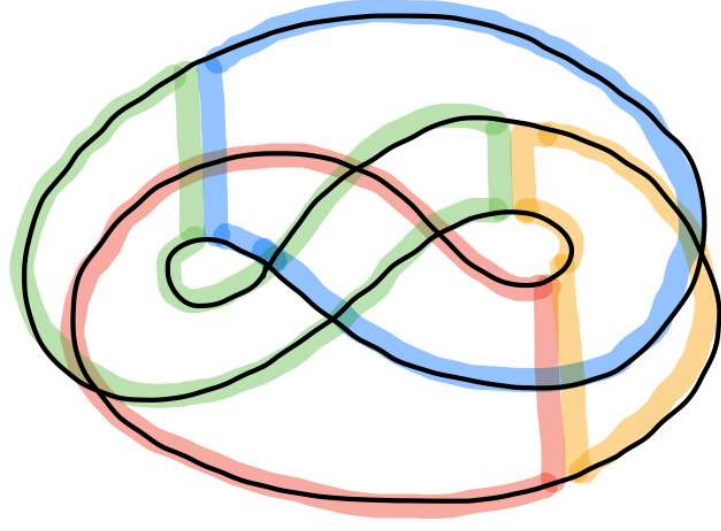
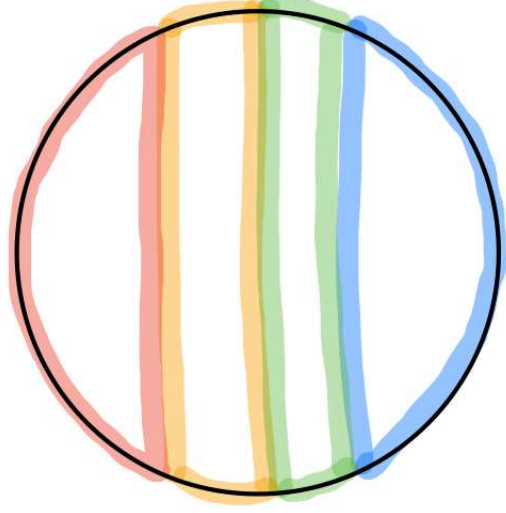
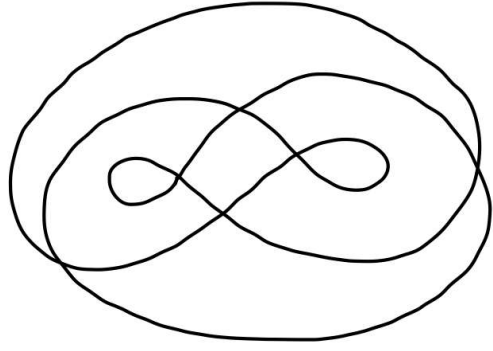
(a) Milnor (2 disks)



(b) Bennequin (5 disks)

NB: They can also be glued together.





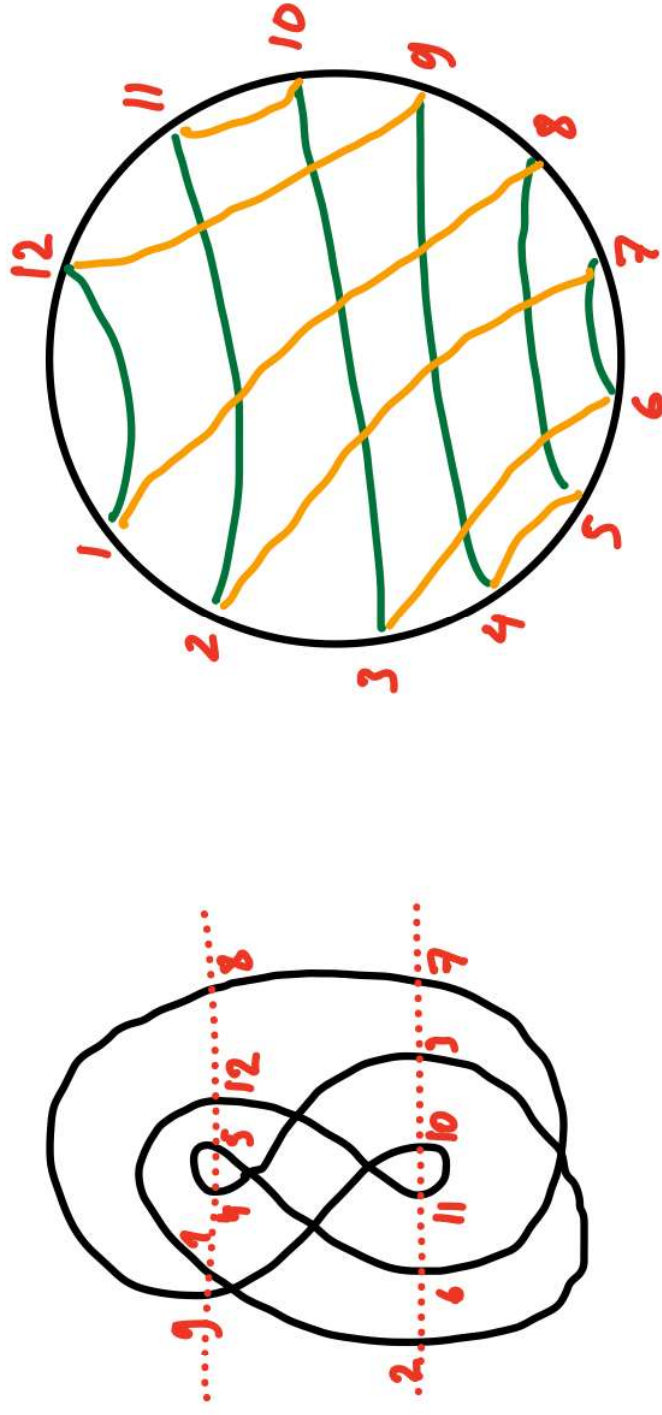


Figure: Minimal number of cuts and the associated “good” pairings.

Theorem (Graver, Cargo 2011)

An oriented normal curve γ , with $0 \leq \text{wind}_\gamma(f) \leq 2$, admits a unique extension if:

- (i) the number of faces with $\text{wind}_\gamma(f) = 2$ equals the number of faces with $\text{wind}_\gamma(f) = 0$,
- (ii) all faces with $\text{wind}_\gamma(f) = 2$ have boundaries of even length.

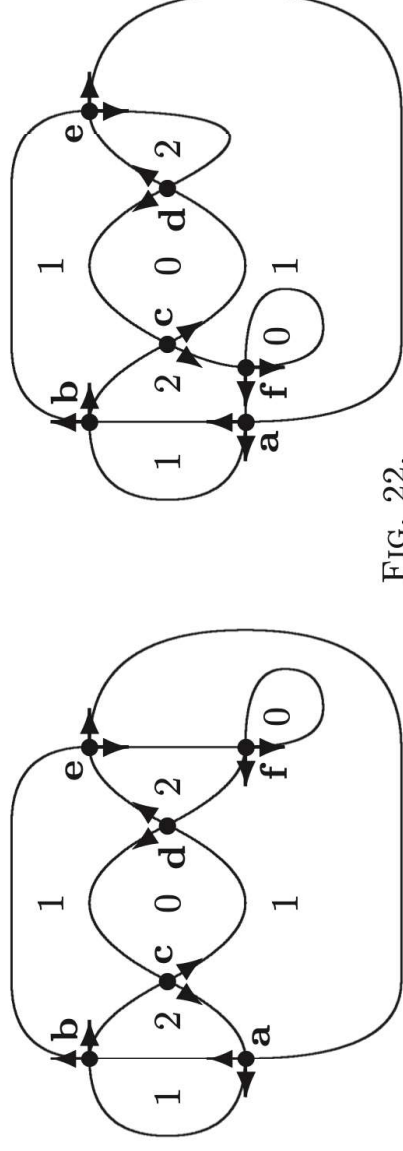


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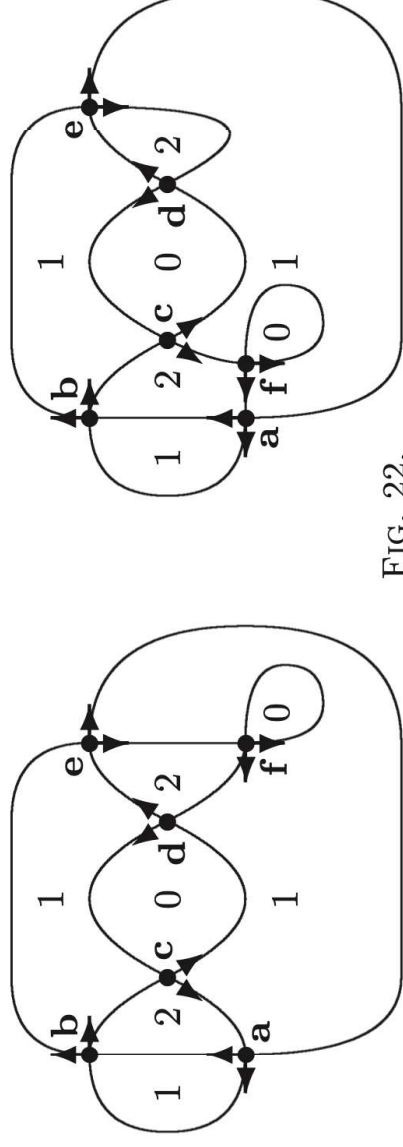


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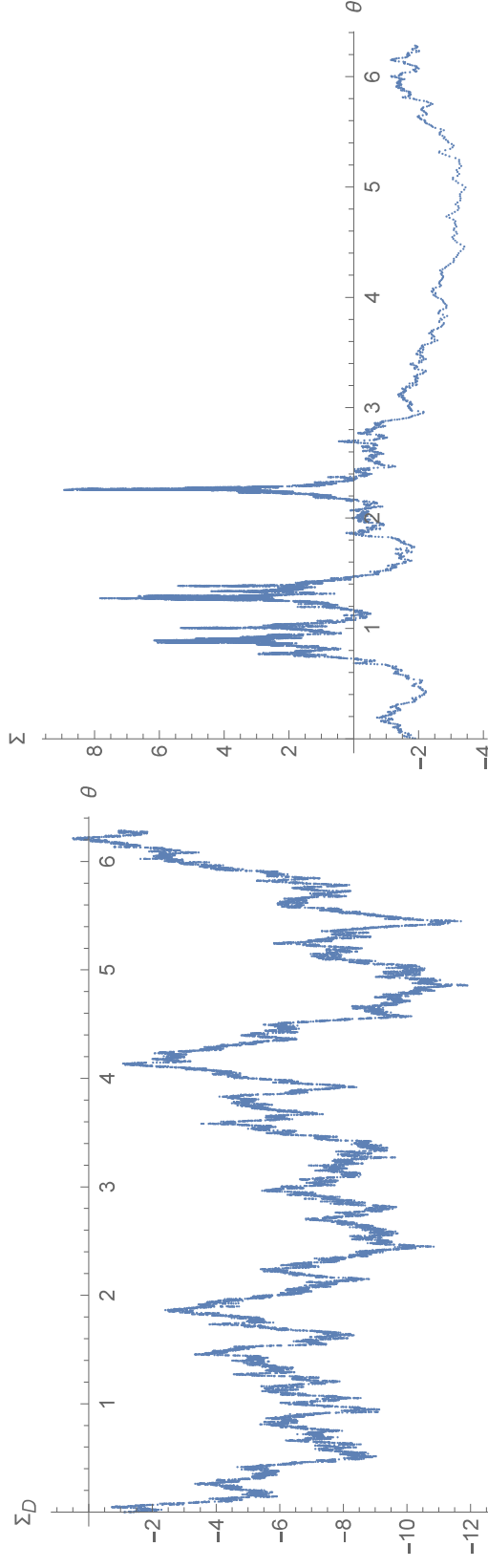
Theorem (Shor, Van Wyk 1992)

The number of incompatible decompositions is equal to the number of combinatorially inequivalent constrained Delaunay triangulations.

Random samples of immersed disks in \mathbb{R}^2 (i.e. random flat metrics on the disk)

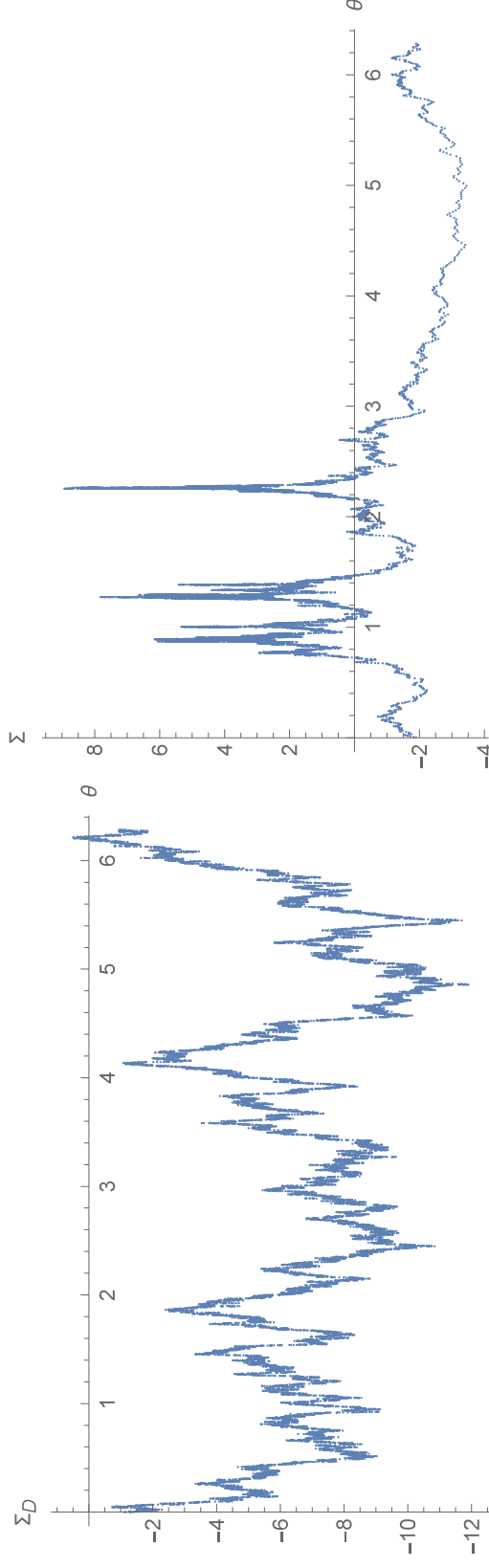
Random samples of immersed disks in \mathbb{R}^2 (i.e. random flat metrics on the disk)

- 1) Generate random Gaussian field $\Sigma_D(\theta)$: 2 parameters: $\{N, \sigma\}$
(2π -uniform: $d\theta = \frac{2\pi}{N}$)
- 2) $\ell = \int d\theta e^{\Sigma_D}$
(arclength-uniform: $d\vartheta = \frac{2\pi}{\ell} e^{\Sigma_D} d\theta$)
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Σ has an action invariant under $PSL(2, \mathbb{R})$.

Random samples of immersed disks in \mathbb{R}^2 (i.e. random flat metrics on the disk)

4) Analytic continuation:

$$H(z)|_{z=e^{i\theta}} = \Sigma(\theta) + i\Gamma(\theta), \quad \Gamma(\theta) = \frac{1}{2\pi} \mathbf{P} \int d\theta' \frac{\Sigma(\theta')}{\tan \frac{\theta' - \theta}{2}},$$

5) Integrate the exponential of its analytic continuation:

$$F(z)|_{z=e^{i\theta}} = i \int_0^\theta d\theta' e^{i\theta'} \exp \left[H(e^{i\theta'}) \right], \quad (\text{gauge: } F(1) = 0).$$

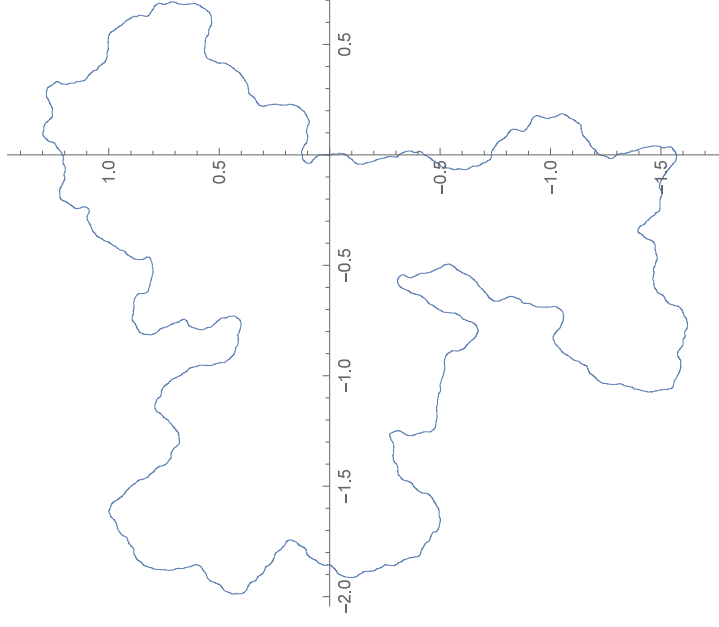
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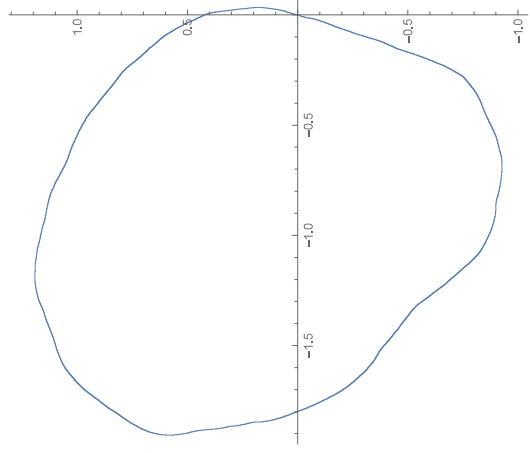
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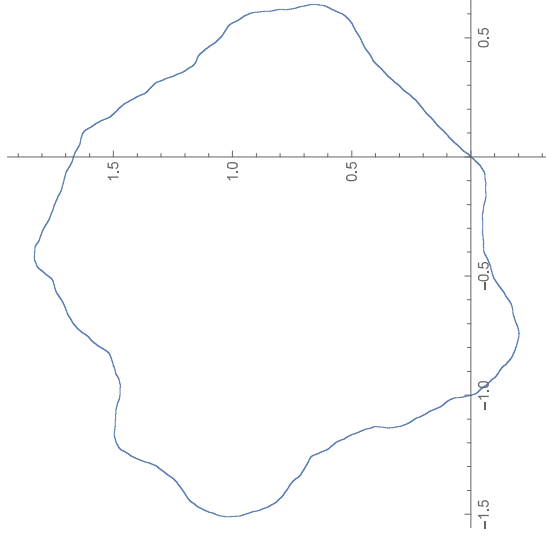
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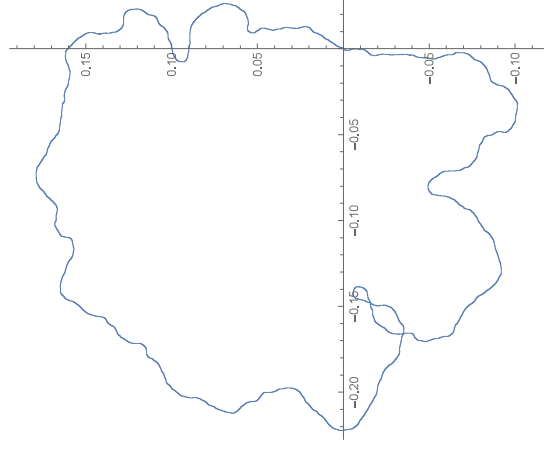
Monte Carlo: Samples



(a) $\sigma = 0.5$

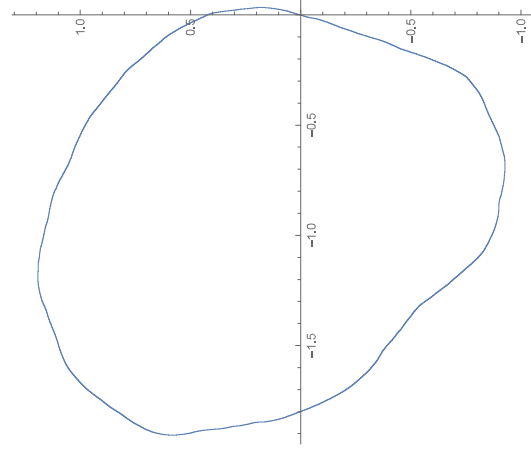


(b) $\sigma = 1.5$

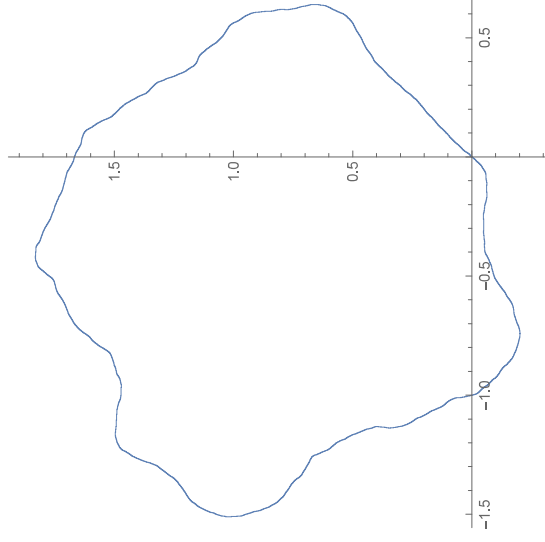


(c) $\sigma = 3$

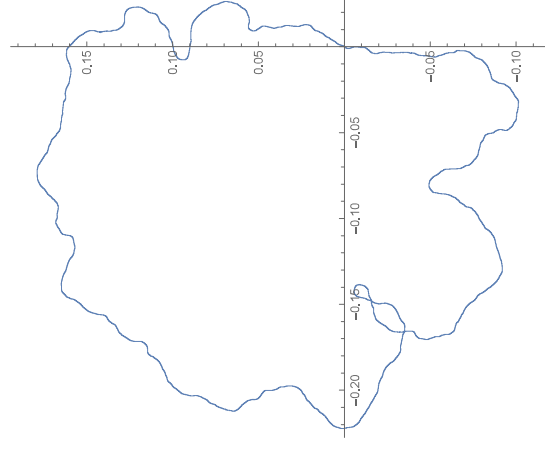
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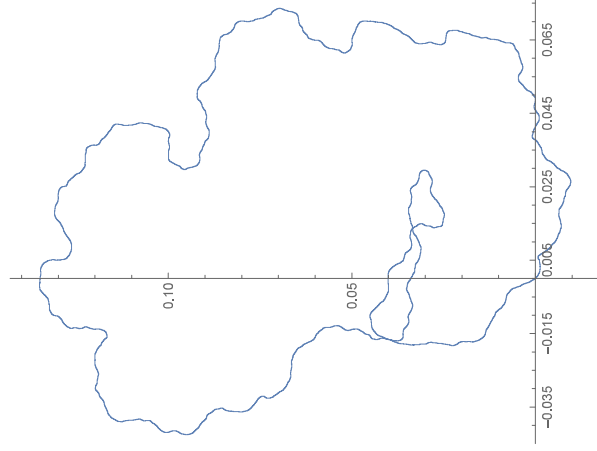
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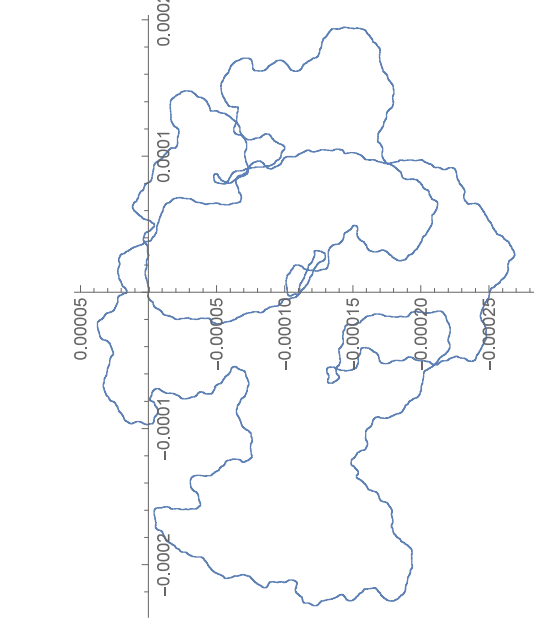
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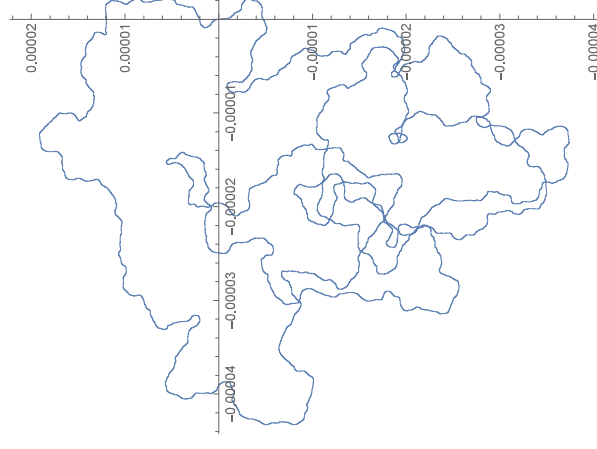
(c) $\sigma = 3$



(d) $\sigma = 4$



(e) $\sigma = 7$



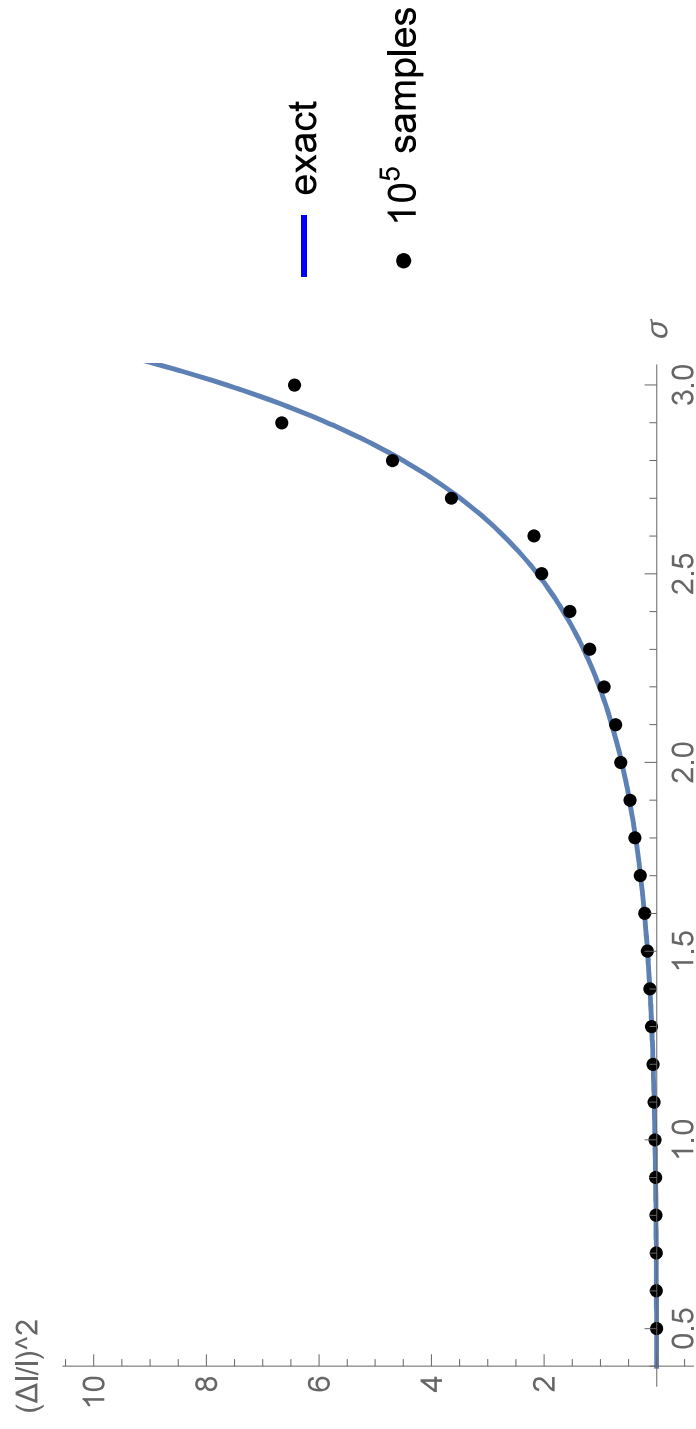
(f) $\sigma = 8$

$$\ell = \int d\theta \mathbf{e}^{\Sigma(\theta)}, \quad \langle \ell \rangle = 2\pi,$$

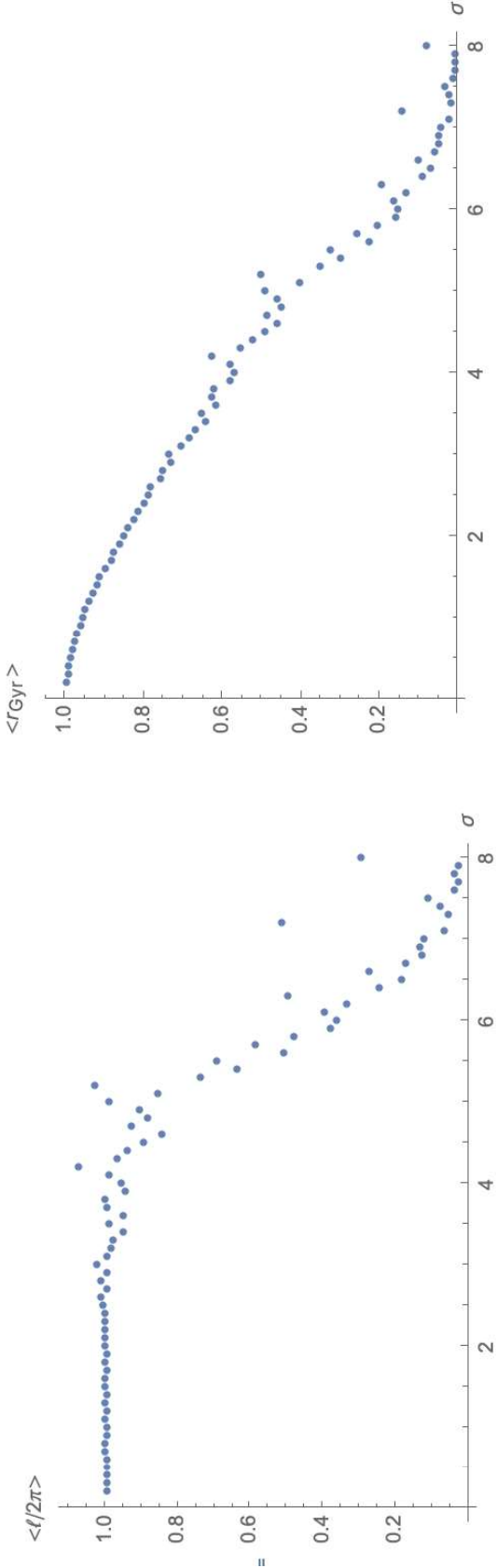
$$\langle \ell^2 \rangle = \frac{4\pi^2}{\sigma} \exp\left(-\frac{\pi\sigma^2}{12}\right) \operatorname{Erfi}\left(\frac{\sqrt{\pi}\sigma}{2}\right),$$

$$\Delta\ell = \sqrt{\langle \ell^2 \rangle - \langle \ell \rangle^2},$$

$$\operatorname{Erfi}(z) = \frac{-2i}{\sqrt{\pi}} \int_0^z dt e^{t^2}.$$

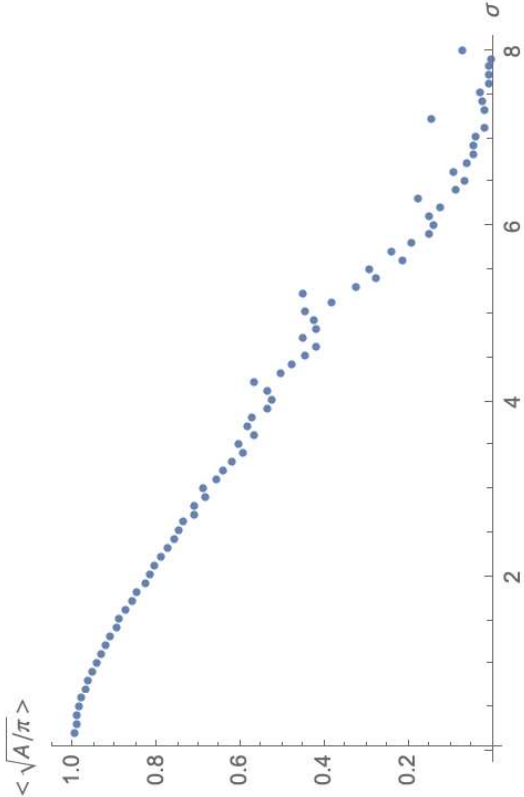


Monte Carlo: Various Statistics (10⁴ curves)



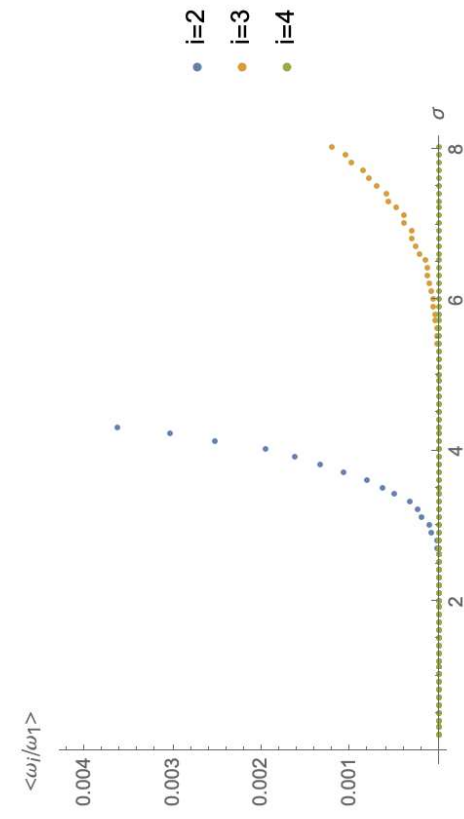
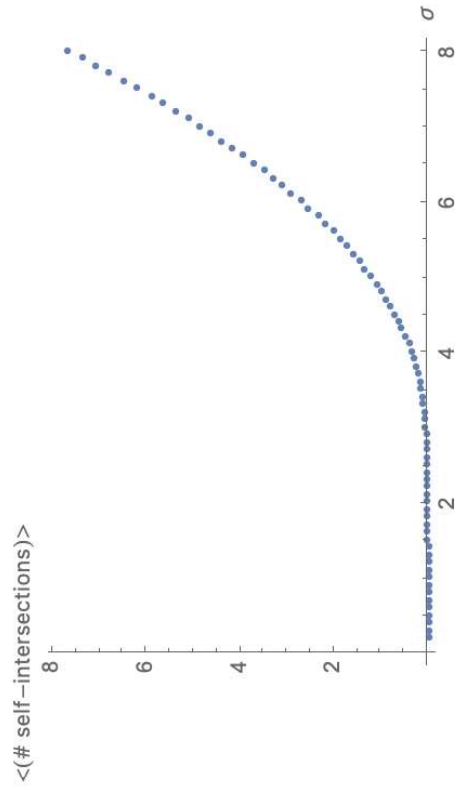
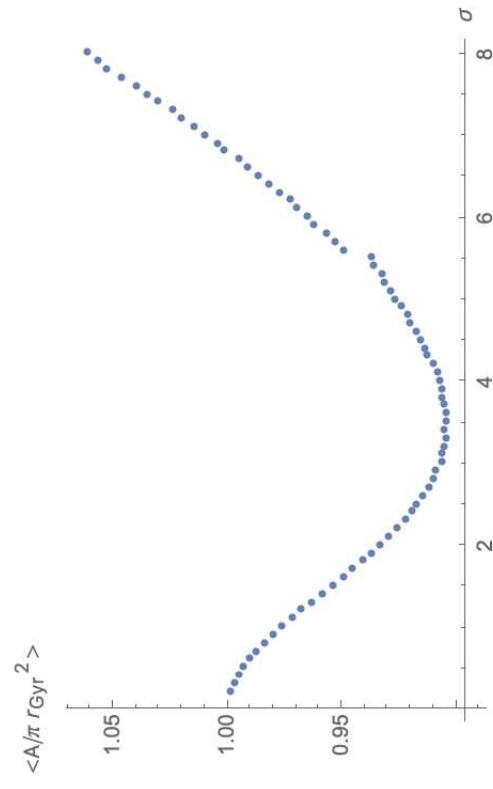
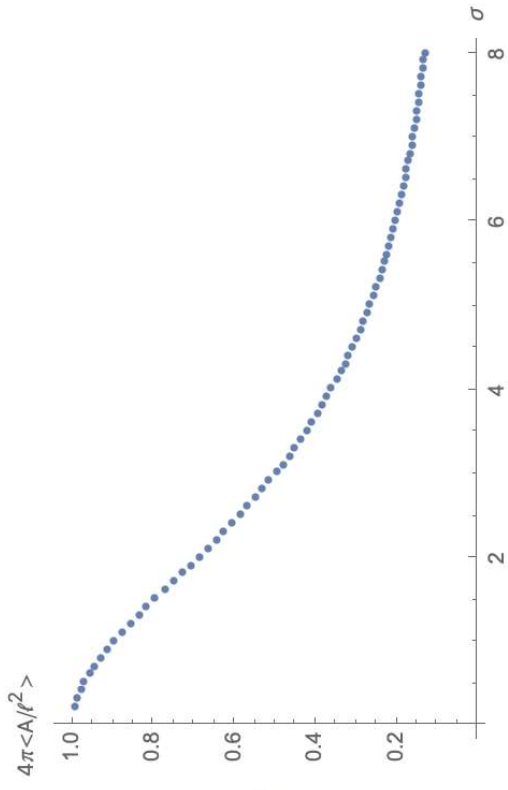
(a) $\ell = \int d\theta e^{\Sigma}$

(b) $r_G^2 = \int d\theta d\theta' |r(\theta) - r(\theta')|^2$



(c) $A = \int d\theta d\rho \rho e^{2\Sigma}$

Monte Carlo: Various Statistics (10^4 curves)



1 Motivations

2 Immersions of the disk

3 Perspectives

- What is this kind of new random object?
(fixed length, extrinsic curvature, order parameter...)
- Implement counting and identification of distinct immersions.
(faster algorithms using minimal number of cuts?)
- Partition function.
- Hyperbolic case, other topologies...

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Thank you!