Strange identities, the Habiro ring and resurgence

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Main goals

- O. Costin, S. Garoufalidis, "Resurgence of the Kontsevich-Zagier series", Ann. Inst. Fourier (Grenoble) **61** (2011), no. 3, 1225–1258.
- K. Habiro, "Cyclotomic completions of polynomial rings", Publ. Res. Inst. Math. Sci. **40** (2004), no. 4, 1127–1146.
- S. Crew, A. Goswami, –, "Resurgence of Habiro elements", available at `https://arxiv.org/abs/2304.07001`

Fishburn numbers

- Consider the usual q-Pochhammer symbol

$$(a)_n = (a;q)_n := \prod_{k=1}^{n}(1 - aq^{k-1}),$$

valid for $n \in \mathbb{N}_0 \cup \{\infty\}$ and the generating function

$$\sum_{n\geq 0}(1-q;1-q)_n = \sum_{n\geq 0}\xi(n)\,q^n = 1 + q + 2q^2 + 5q^3 + 15q^4 + 53q^5 + \cdots .$$

- A *Fishburn matrix* is an upper-triangular matrix with non-negative integer entries and without zero rows or columns such that the sum of all entries is n. So, $\xi(3) = 5$ since

$$(3)\,, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

- For other interesting combinatorial interpretations of $\xi(n)$, see A022493.

Arithmetic and asymptotic properties of $\xi(n)$

- Andrews and Sellers (2016), Guerzhoy, Kent and Rolen (2014), Garvan (2015), Ahlgren and Kim (2015), Straub (2015) studied prime power congruences for $\xi(n)$.

- For example, we have for all $r, m \in \mathbb{N}$

$$\xi(5^r m - 1) \equiv \xi(5^r m - 2) \equiv 0 \pmod{5^r},$$

$$\xi(7^r m - 1) \equiv 0 \pmod{7^r}$$

and

$$\xi(11^r m - 1) \equiv \xi(11^r m - 2) \equiv \xi(11^r m - 3) \equiv 0 \pmod{11^r}.$$

- In 2001, Zagier proved that as $n \to \infty$

$$\xi(n) \sim \left(\frac{6}{\pi^2}\right)^n n! \sqrt{n} \frac{12\sqrt{3}}{\pi^{\frac{5}{2}}} e^{\frac{\pi^2}{12}}.$$

Strange series

- On October 14, 1997, Kontsevich introduced the expression

$$F(q) := \sum_{n\geq 0} (q)_n,$$

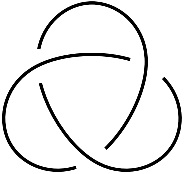
which does not converge on any open subset of $\mathbb{C}$, but is well-defined when q is a root of unity and q is replaced by $1-q$.

- $F(q)$ is a "quantum modular form" (Zagier, 2010).

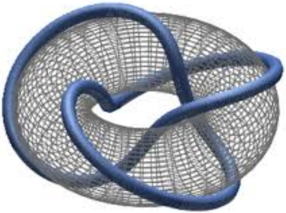
- There are other examples of such strange series arising in knot theory ...

Knots

- Let K be a knot in $\mathbb{R}^3$. For example, the right-handed trefoil knot is given by

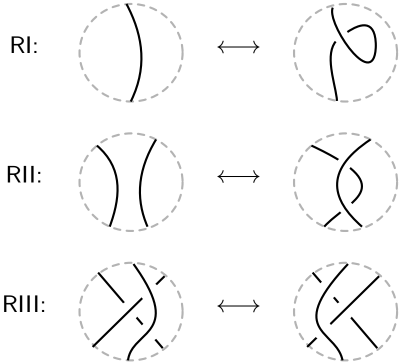


- We will consider the family of *torus knots* $T(p, q)$. For example, $T(3, 2)$ is given by



Knots

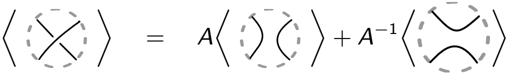
- (Reidemeister, 1927) Let K and K' be two knots with diagrams D and D'. Then K is isotopic to K' in $\mathbb{R}^3$ if and only if D is related to D' by a sequence of isotopies of $\mathbb{R}^2$ and the moves RI, RII and $RIII$ given by the following:



The Kauffman bracket

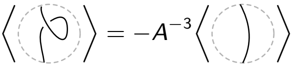
- The *Kauffman bracket* $\langle D \rangle$ of D is defined by

$$\left\langle D \sqcup \bigcirc \right\rangle = (-A^2 - A^{-2})\langle D \rangle$$



$$\left\langle \text{empty diagram} \right\rangle = 1.$$

- $\langle D \rangle$ is invariant under RII and RIII, but not RI as



The Jones polynomial

- The Jones polynomial $V(K) = V(K;q)$ is given by

$$V(K) = \frac{1}{(-A^2 - A^{-2})}(-A)^{-3w(D)}\langle D\rangle\Big|_{A^2=q^{-1/2}}$$

where

$$w(D) = \quad \#$$

$$- \quad \#$$


is the "writhe" of D.

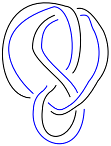
- $V(K)$ is invariant under RI, RII and RIII.

The colored Jones polynomial

- The colored Jones polynomial $J_N(K;q)$ is a linear combination of cablings of D using Chebyshev polynomials: $S_1(x) = 1$, $S_2(x) = x$, $S_N(x) = xS_{N-1}(x) - S_{N-2}(x)$.

- For example, $S_3(x) = x^2 - 1$. So, we have

$$J_3(4_1;q) \quad = \quad \star \left\langle \qquad \right\rangle - 1$$



- The $N = 2$ case recovers the Jones polynomial.

Strange series and knot theory

- Habiro (2000), T. Lê (2003) proved the expansion

$$J_N(\text{trefoil}; q) = q^{1-N} \sum_{n \geq 0} q^{-nN} (q^{1-N})_n.$$

- Observe that

$$F(\zeta_N) = \zeta_N^{-1} J_N(\text{trefoil}; \zeta_N)$$

and so

$$F(q) := \sum_{n \geq 0} (q)_n$$

is "extracted" from the colored Jones polynomial of the trefoil.

The Habiro ring

- Habiro (2004): Consider the projective system

$$\left(\left(\mathbb{Z}[q] / \langle (q)_i \rangle \right)_{i \in \mathbb{N}}, \ (f_{ij})_{i \le j \in \mathbb{N}} \right)$$

  where

$$f_{ij} : \mathbb{Z}[q] / \langle (q)_j \rangle \longrightarrow \mathbb{Z}[q] / \langle (q)_i \rangle$$

$$m(q) = \sum_{k=0}^{j-1} \underbrace{m_k(q)}_{\deg \le k} (q)_k \mapsto m(q) \pmod{(q)_i}.$$

  Then, $\widehat{\mathbb{Z}[q]} := \varprojlim_n \mathbb{Z}[q] / \langle (q)_n \rangle$.

- Every $m \in \widehat{\mathbb{Z}[q]}$ can be expressed as

$$m(q) = \sum_{n \ge 0} m_n(q) \, (q)_n$$

  where $m_n(q) \in \mathbb{Z}[q]$, $\deg(m_n) \le n$ for all $n \in \mathbb{N}$. Note that $F(q) \in \widehat{\mathbb{Z}[q]}$.

Strange identities

- The key to the arithmetic and asymptotic properties for the Fishburn numbers $\xi(n)$ and the quantum modularity of $F(q)$ is the "strange" identity

$$F(q) \text{``}=\text{''} -\frac{1}{2}\sum_{n\geq 1} n\chi(n) q^{\frac{n^2-1}{24}}$$

  where

$$\chi(n) := \begin{cases} -1 & \text{if } n \equiv 1, 11 \pmod{12}, \\ 1 & \text{if } n \equiv 5, 7 \pmod{12}, \\ 0 & \text{otherwise.} \end{cases}$$

- Behind every strange identity is *actual* q-series identity. For example, we have

$$2\sum_{n\geq 0}\left[(q)_n - (q)_\infty\right] + (q)_\infty\left(-1 + 2\sum_{n\geq 1}\frac{q^n}{1-q^n}\right) = -\sum_{n\geq 1} n\chi(n) q^{\frac{n^2-1}{24}}$$

  using Bailey pairs (Lovejoy, 2022).

Other strange identities

- Hikami (2006): For $m \in \mathbb{N}$ and $0 \leq \ell \leq m-1$, consider

$$X_m^{(\ell)}(q) := \sum_{k_1,k_2,\ldots,k_m=0}^{\infty} (q)_{k_m} q^{k_1^2+\cdots+k_{m-1}^2+k_{\ell+1}+\cdots+k_{m-1}} \prod_{i=1}^{m-1} \underbrace{\begin{bmatrix} k_{i+1}+\delta_{i,\ell} \\ k_i \end{bmatrix}}_{q\text{-binomial coefficient}} \in \widehat{\mathbb{Z}[q]}.$$

- We have that

$$X_m^{(0)}(\zeta_N) = \zeta_N^{-m} J_N(T(2,2m+1);\zeta_N)$$

  and $X_1^{(0)}(q) = F(q)$.

- $X_m^{(\ell)}(q)$ satisfies a strange identity.

Other strange identities

- Consider the family of torus knots $T(3,2^t)$, $t \geq 1$. In 2016, Konan proved

$$J_N(T(3,2^t);q) = (-1)^{h''(t)} q^{2^t-1-h'(t)-N} \sum_{n\geq 0} (q^{1-N})_n q^{-Nnm(t)}$$
$$\times \sum_{3\sum_{\ell=1}^{m(t)-1} j_\ell \ell \equiv 1 \pmod{m(t)}} (-q^{-N})^{\sum_{\ell=1}^{m(t)-1} j_\ell} q^{\frac{-a(t)+\sum_{\ell=1}^{m(t)-1} j_\ell \ell}{m(t)} + \sum_{\ell=1}^{m(t)-1} \binom{j_\ell}{2}}$$
$$\times \sum_{k=0}^{m(t)-1} q^{-kN} \prod_{\ell=1}^{m(t)-1} \begin{bmatrix} n+I(\ell \leq k) \\ j_\ell \end{bmatrix}.$$

- Let $\mathcal{F}_t(q) := (-1)^{h''(t)} q^{-h'(t)} \sum_{n\geq 0} (q)_n \sum_{j_\ell}{}' (-1)^{\sum_{\ell=1}^{m(t)-1} j_\ell} q^{v} \sum_{k=0}^{m(t)-1} \prod_{\ell=1}^{m(t)-1} \begin{bmatrix} n+I(\ell \leq k) \\ j_\ell \end{bmatrix}$.

  Then
$$\mathcal{F}_t(\zeta_N) = \zeta_N^{1-2^t} J_N(T(3,2^t);\zeta_N),$$
  $\mathcal{F}_1(q) = F(q)$ and $\mathcal{F}_t(q) \in \widehat{\mathbb{Z}[q]}$ satisfies a strange identity.

Setup

- Consider a Gevrey-1 formal power series

$$\sum_{n=0}^{\infty} a_n x^{-n} \in \mathbb{C}[[1/x]]$$

and its Borel transform

$$\mathcal{B}\left(\sum_{n=0}^{\infty} a_n x^{-n}\right) = \sum_{n=0}^{\infty} a_{n+1} \frac{p^n}{n!} =: G(p).$$

- For $x \in \mathbb{C}$, define the median resummation as

$$S^{\mathbf{med}}(x) = \frac{1}{2}\Big(S^{\mathbf{L}}(x) + S^{\mathbf{R}}(x)\Big)$$

where

$$S^{\mathbf{L}}(x) = \int_{\gamma_l} e^{-px} G(p)\, dp$$

and

$$S^{\mathbf{R}}(x) = \int_{\gamma_r} e^{-px} G(p)\, dp.$$

The contours γ_l and γ_r

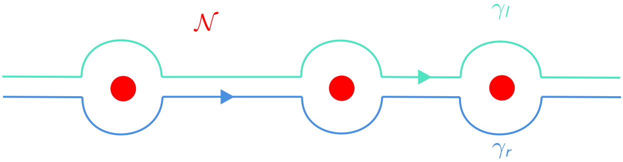


Figure: A singularity set $\mathcal{N} \subset \mathbb{C}$ and the contours γ_l and γ_r.

The CG conjecture

- (Huyuh, Lê, 2007) Given a knot K, there exists $\Phi_K(q) \in \widehat{\mathbb{Z}[q]}$ which is extracted from $J_N(K;q)$.

- Let $F_K(x) := \Phi_K(e^{-1/x})$ and write $G_K(p)$ for $G(p)$ and $S_K^{\mathbf{med}}(x)$ for $S^{\mathbf{med}}(x)$.

Conjecture (CG, 2011)

For every knot K,

(1) $G_K(p)$ *is resurgent,*

(2) $S_K^{\mathbf{med}}(x)$ *is an analytic function defined on* $\Re(x) > 0$*,*

(3) *For* $0 \neq \alpha \in \mathbb{Q}$*, we have* $S_K^{\mathbf{med}}\left(-\frac{1}{2\pi i\alpha}\right) = \Phi_K(e^{2\pi i\alpha})$.

- CG prove (1) and (2) (and claim a proof of (3)) for the trefoil knot.

Main result

- Let $f : \mathbb{Z} \to \mathbb{C}$ be a function of period $M \geq 2$. For integers $a \geq 0$ and $b > 0$, consider the partial theta series

$$\theta^{(\nu)}_{a,b,f}(q) := \sum_{n\geq 0} n^{\nu} f(n)\, q^{\frac{n^2-a}{b}}$$

where $q = e^{2\pi i z}$, $z \in \mathbb{H}$ and $\nu \in \{0, 1\}$.

- Suppose there exists

$$\Phi_f(q) := \sum_{n=0}^{\infty} A_{n,f}(q)\,(q)_n \in \widehat{\mathbb{Z}[q]}$$

which satisfies

$$\Phi_f(q) \text{``}=\text{''} \theta^{(\nu)}_{a,b,f}(q). \qquad (\star)$$

Main result

- Let $k_1, k_2 \in \mathbb{Z}$ with $k_1 < k_2$, $0 \neq c \in \mathbb{R}$ and g be the function

$$g(n) := \begin{cases} c & \text{if } n \equiv k_1, M - k_1 \pmod{M}, \\ -c & \text{if } n \equiv k_2, M - k_2 \pmod{M}, \\ 0 & \text{otherwise.} \end{cases}$$

- Let $\mathcal{F}_g(x) := e^{-\frac{a}{bx}} \Phi_g(e^{-\frac{1}{x}})$ and $G_g(p)$ denote the Borel transform of $\mathcal{F}_g(x)$.

- We write $S_g^{\mathbf{med}}(x)$ for $S^{\mathbf{med}}(x)$ and $\theta_{a,b,g}^{(\nu)}(z)$ for $\theta_{a,b,g}^{(\nu)}(q)$.

Theorem (Crew, Goswami, –)

Assume $(\star)$ *is true. Then*

(1) $G_g(p)$ *is resurgent,*

(2) $S_g^{\mathbf{med}}(x)$ *is an analytic function defined on* $\Re(x) > 0$,

(3) *For* $0 \neq \alpha \in \mathbb{Q}$, *we have*

$$S_g^{\mathbf{med}}\left(-\frac{1}{2\pi i\alpha}\right) = -\frac{ibc}{M\pi\alpha^{\frac{3}{2}}}\int_0^{i\infty}\frac{\theta^{(0)}_{0,4M^2,\tilde{g}}(bp)}{\left(p+\frac{1}{\alpha}\right)^{\frac{3}{2}}}\,dp - C_M + \tilde{\delta}_{b,c,g}\left(-\frac{1}{2\pi i\alpha}\right)$$

where $\tilde{g}$ *is a periodic function associated to* g, C_M *is a constant depending on* M *and*

$$\tilde{\delta}_{b,c,g}\left(-\frac{1}{2\pi i\alpha}\right) := \left(\frac{b}{\alpha}\right)^{\frac{3}{2}}\frac{\sqrt{2}c\,e^{\frac{\pi i}{4}}}{M^2}\sum_{\ell\in\mathbb{N}}{}''(-1)^{\ell}\ell\,\mathcal{S}(k_1,k_2,\ell,M)\,e^{-i\frac{b\ell^2\pi}{2M^2\alpha}}.$$

Here, $\mathcal{S}(k_1,k_2,\ell,M) := \sin\left(\frac{(k_2-k_1)\ell\pi}{M}\right)\sin\left(\frac{(M-k_1-k_2)\ell\pi}{M}\right)$.

- As a consequence, the CG conjecture (1) and (2) are true for the torus knots $T(2,2m+1)$ and $T(3,2^t)$, but (3) is false for the trefoil.

Future work

- Verify the CG conjecture (1) and (2) for all torus knots $T(r,s)$. For $(\star)$, we know the partial theta series, but *not* the element in $\widehat{\mathbb{Z}[q]}$.
- (Hikami, –) Strange identities for WRT invariants. Consider the twist knots $K_{p>0} \rightsquigarrow M = \Sigma(2,3,6p+1) \rightsquigarrow \tau_M(q) \in \widehat{\mathbb{Z}[q]}$. We observed

$$\sum_{s_p \geq \cdots \geq s_2 \geq s_1 \geq 0} q^{-s_p(s_p+2)} (q^{s_p+1})_{s_p+1} \prod_{i=1}^{p-1} q^{s_i(s_i+1)} \begin{bmatrix} s_{i+1} \\ s_i \end{bmatrix} \text{ “} = \text{” } -\frac{1}{2} \underbrace{\Phi_{(2,3,6p+1)}(q)}_{\text{partial theta series}}.$$

Study asymptotic, arithmetic and resurgence properties.

Thank you!

Modularity

- A *modular form* of weight k is a holomorphic function $f : \mathbb{H} \to \mathbb{C}$ satisfying

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$$

for all $z \in \mathbb{H}$ and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$.

- (Zagier, 2010) A *quantum modular form* of weight k is a function $g : \mathbb{Q} \to \mathbb{C}$ such that for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$,

$$r_\gamma(\alpha) := g(\alpha) - (c\alpha + d)^{-k} g\left(\tfrac{a\alpha+b}{c\alpha+d}\right)$$

has "nice" properties (e.g., continuity or analyticity).

- (Zagier, 2010) The function $\phi(\alpha) := e^{\frac{\pi i \alpha}{12}} F(e^{2\pi i \alpha})$ is a quantum modular form of weight $3/2$ with respect to $\mathrm{SL}_2(\mathbb{Z})$.