Exact Asymptotics TSVP

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Black Hole Entropy and the Asymptotic Growth of Cohomology

Hardy-Ramanujan Reloaded

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- **Physics**

$$CFT_D \longleftrightarrow Gravity_{D+1}$$

Thermal Ensemble　　　Black Hole

Old story: D=2

Recent progress: D>2 and D=1

- **Maths**

$$\begin{array}{c} \text{Geometry/Topology} \\ \Big\downarrow \;\text{Localisation} \\ \text{Combinatorics} \\ \Big\downarrow \;\text{Modularity} \\ \text{Asymptotics} \end{array}$$

Outline

- Review Hardy-Ramanujan formula for the asymptotic growth of,

$$p(K) \quad := \qquad \text{Number of partitions of } K \in \mathbb{N}$$

- New results for growth of cohomology of K instanton moduli space

Multi-dimensional generalisation. Complex saddle points.

ASYMPTOTIC FORMULÆ IN COMBINATORY ANALYSIS

By G. H. Hardy and S. Ramanujan.*

[Preliminary communication December 14th, 1916.—Read January 18th, 1917.—
Received February 28th, 1917.]

1.

Introduction and summary of results.

1 . 1. The present paper is the outcome of an attempt to apply to the principal problems of the theory of partitions the methods, depending upon the theory of analytic functions, which have proved so fruitful in the theory of the distribution of primes and allied branches of the analytic theory of numbers.

The most interesting functions of the theory of partitions appear as the coefficients in the power-series which represent certain elliptic modular functions. Thus $p(n)$, the number of unrestricted partitions of n, is the coefficient of x^n in the expansion of the function

$$f(x) = 1 + \sum_1^\infty p(n) x^n = \frac{1}{(1-x)(1-x^2)(1-x^3)\ldots}. \dagger \tag{1 . 11}$$

TABLE IV*: $p(n)$.

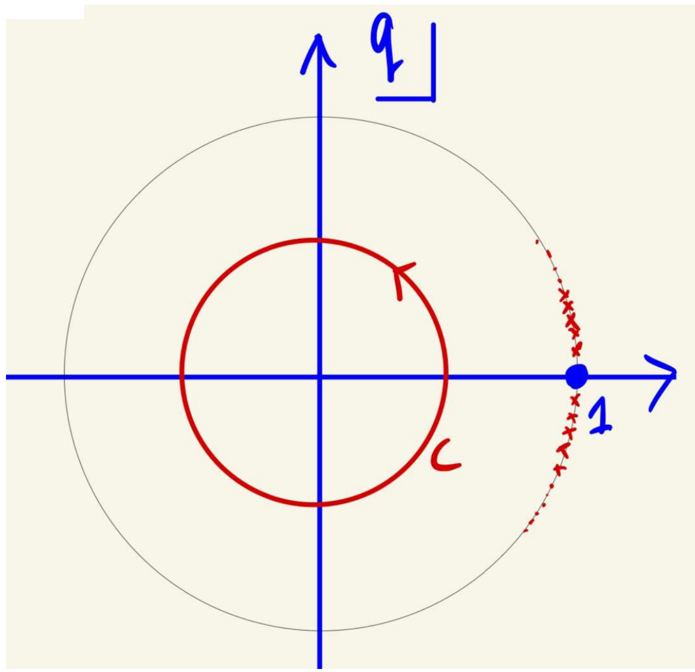
1 ...	1	39 ...	31185	77 ...	10619863	115 ...	1064144451
2 ...	2	40 ...	37338	78 ...	12132164	116 ...	1188908248
3 ...	3	41 ...	44583	79 ...	13848650	117 ...	1327710076
4 ...	5	42 ...	53174	80 ...	15796476	118 ...	1482074143
5 ...	7	43 ...	63261	81 ...	18004327	119 ...	1653668665
6 ...	11	44 ...	75175	82 ...	20506255	120 ...	1844349560
7 ...	15	45 ...	89134	83 ...	23338469	121 ...	2056148051
8 ...	22	46 ...	105558	84 ...	26543660	122 ...	2291320912
9 ...	30	47 ...	124754	85 ...	30167357	123 ...	2552338241
10 ...	42	48 ...	147273	86 ...	34262962	124 ...	2841940500
11 ...	56	49 ...	173525	87 ...	38887673	125 ...	3163127352
12 ...	77	50 ...	204226	88 ...	44108109	126 ...	3519222692
13 ...	101	51 ...	239943	89 ...	49995925	127 ...	3913864295
14 ...	135	52 ...	281589	90 ...	56634173	128 ...	4351078600
15 ...	176	53 ...	329931	91 ...	64112359	129 ...	4835271870
16 ...	231	54 ...	386155	92 ...	72533807	130 ...	5371315400
17 ...	297	55 ...	451276	93 ...	82010177	131 ...	5964539504
18 ...	385	56 ...	526823	94 ...	92669720	132 ...	6620830889
19 ...	490	57 ...	614154	95 ...	104651419	133 ...	7346629512
20 ...	627	58 ...	715220	96 ...	118114304	134 ...	8149040695
21 ...	792	59 ...	831820	97 ...	133230930	135 ...	9035836076
22 ...	1002	60 ...	966467	98 ...	150198136	136 ...	10015581680
23 ...	1255	61 ...	1121505	99 ...	169229875	137 ...	11097645016
24 ...	1575	62 ...	1300156	100 ...	190569292	138 ...	12292341831
25 ...	1958	63 ...	1505499	101 ...	214481126	139 ...	13610949895
26 ...	2436	64 ...	1741630	102 ...	241265379	140 ...	15065878135
27 ...	3010	65 ...	2012558	103 ...	271248950	141 ...	16670689208
28 ...	3718	66 ...	2323520	104 ...	304801365	142 ...	18440293320
29 ...	4565	67 ...	2679689	105 ...	342325709	143 ...	20390982757
30 ...	5604	68 ...	3087735	106 ...	384276336	144 ...	22540654445
31 ...	6842	69 ...	3554345	107 ...	431149389	145 ...	24908858009
32 ...	8349	70 ...	4087968	108 ...	483502844	146 ...	27517052599
33 ..	10143	71 ...	4697205	109 ...	541946240	147 ...	30388671978
34 ...	12310	72 ...	5392783	110 ...	607163746	148 ...	33549419497
35 ...	14883	73 ...	6185689	111 ...	679903203	149 ...	37027355200
36 ...	17977	74 ...	7089500	112 ...	761002156	150 ...	40853235313
37 ...	21637	75 ...	8118264	113 ...	851376628	151 ...	45060624582
38 ...	26015	76 ...	9289091	114 ...	952050665	152 ...	49686288421

Generating Function

$$Z(q) \;=\; \sum_{K=0}^{\infty} p(K)\, q^K \;=\; \prod_{l=1}^{\infty} \frac{1}{(1-q^l)}$$

Extract coefficient of q^K,

$$p(K) \;=\; \frac{1}{2\pi i} \oint_C \frac{dq}{q^{K+1}}\, Z(q)$$



Plethystic Exponential

$$\mathrm{Pexp}\left[f(x_1, x_2, \ldots, x_M)\right] \quad := \quad \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} f(x_1^n, x_2^n, \ldots, x_M^n)\right)$$

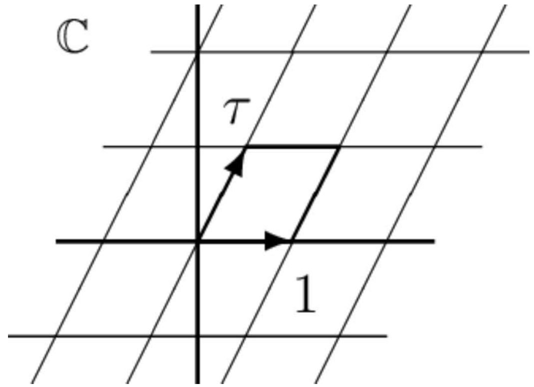
$$\mathrm{Pexp}\,[\mathrm{x}] \;=\; 1/(1-x) \qquad\qquad \mathrm{Pexp}\,[f+g] \;=\; \mathrm{Pexp}\,[f]\cdot \mathrm{Pexp}\,[g]$$

- Generating function,

$$Z(q) = \prod_{l=1}^{\infty} \frac{1}{(1-q^l)} = \mathrm{Pexp}\left[q + q^2 + q^3 + \ldots\right] = \mathrm{Pexp}\left[\frac{q}{1-q}\right]$$

Modular Invariance

$$Z(q) = \prod_{l=1}^{\infty} \frac{1}{(1-q^l)} = \frac{q^{\frac{1}{24}}}{\eta(\tau)}$$



2d CFT: Free Boson

$$q = \exp 2\pi i \tau$$

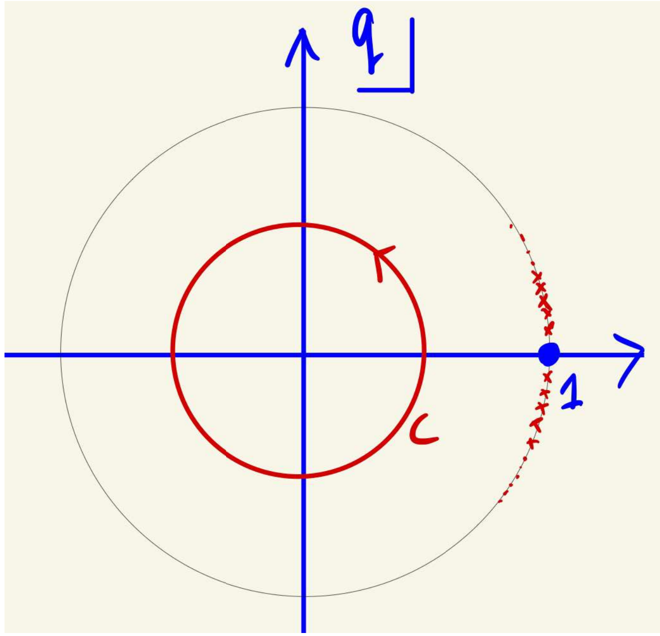
Modular transformation,

$$\eta\left(\frac{1}{\tau}\right) = \sqrt{-i\tau}\,\eta(\tau)$$

yields asymptotic,

$$Z(q) \overset{\tau \to 0}{\sim} \sqrt{-i\tau}\,\exp\left(\frac{i\pi}{12}\frac{1}{\tau}\right)$$

Saddle-Point



- Using asymptotic form near q=1

$$p(K) = \frac{1}{2\pi i} \oint_C \frac{dq}{q^{K+1}} Z(q) \overset{K\to\infty}{\sim} \int d\tau \exp S(\tau)$$

with

$$S(\tau) = 2\pi i \left[\frac{1}{24\tau} - K\tau \right]$$

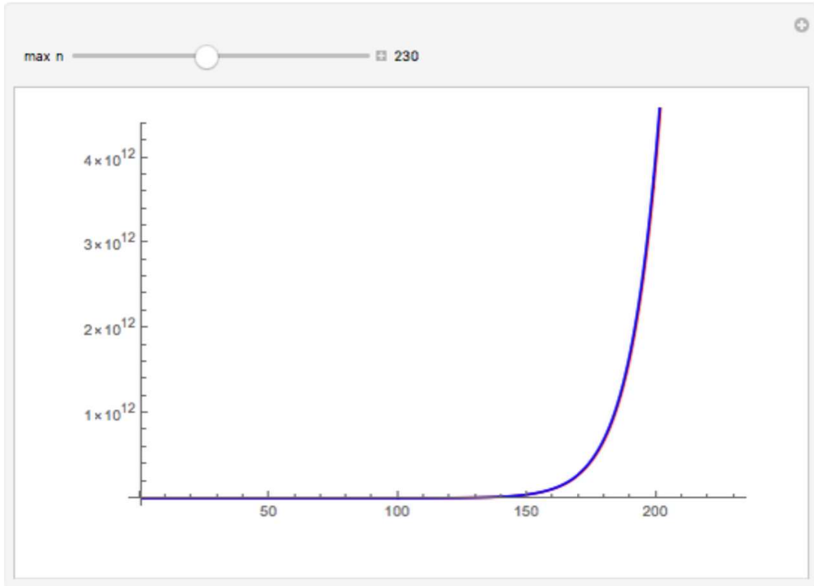
- Saddle-points at $S = S(\tau^*)$

$$\left.\frac{\partial S}{d\tau}\right|_{\tau=\tau^*} = 0 \quad \Rightarrow \quad \left(\frac{S}{2\pi}\right)^2 = \frac{K}{6}$$

Asymptotics

Hardy-Ramanujan formula,

$$p(K) \overset{K\to\infty}{\sim} \sqrt{\frac{\pi}{S''(\tau^*)}}\,\exp S(\tau^*) = \frac{1}{4K\sqrt{3}}\exp\left(\pi\sqrt{\frac{2K}{3}}\right)$$



Wolfram Demonstrations Project

2d CFT: Cardy formula

Subleading corrections: convergent Rademacher expansion

Instanton Moduli Space

- ADHM moduli space of K Yang-Mills instantons, $G = U(N)$

$$\mathcal{M}_{K,N} \quad = \quad \frac{\left\{ \, [X, \tilde{X}] + \sum_{i=1}^{N} Q_i \tilde{Q}_i = 0 \, \right\}}{GL(K, \mathbb{C})}$$

← Non-compact hyper-kähler manifold of complex dimension 2KN

- N=1 case,

$$\mathcal{M}_K \;:=\; \mathcal{M}_{K,1} \;=\; \mathrm{Hilb}_K \left[\mathbb{C}^2\right]$$

ADHM Data:

	$U(K)$	$U(N)$
$X, \tilde{X}$	$\underline{K}^2 \oplus \underline{K}^2$	$\underline{1}$
$Q, \tilde{Q}$	$\underline{K} \oplus \bar{\underline{K}}$	$\underline{N} \oplus \bar{\underline{N}}$

Counting Problem

Differential: Twisted Dolbeault cohomology

Algebraic: Sheaf cohomology

- Compute cohomology of $\mathbb{Q} = \bar{\partial} - \bar{\partial}\mathcal{K}\wedge$

  graded by $T = \mathbb{C}_1^* \times \mathbb{C}_2^*$ action

$$\left(X, \tilde{X}, \ldots\right) \to \left(q_1 X, q_2 \tilde{X}, \ldots\right)$$

- Subspace decomposition, $$H_{p,q}\left(\mathcal{M}_K\right) = \bigoplus_{K_1, K_2 \in \mathbb{Z}} H_{p,q}^{(K_1,K_2)}\left(\mathcal{M}_K\right)$$

- Define, $$d(K, K_1, K_2, p, q) := \dim_{\mathbb{C}}\left[H_{p,q}^{(K_1,K_2)}\left(\mathcal{M}_K\right)\right] \in \mathbb{N}$$

Index

$$\chi_y(\mathcal{M}_K) := \sum_{p,q=0}^{2K} (-1)^{p+q} y^{p-K}\, \mathrm{Tr}_{H_{p,q}(\mathcal{M}_K)} \left[q_1^{\mathbb{T}_1} q_2^{\mathbb{T}_2} \right]$$

Equivariant χ_y-genus

- Generating function = Nekrasov partition function

$$Z[q, q_1, q_2, y] := \sum_{K=0}^{\infty} q^K \chi_y(\mathcal{M}_K)$$

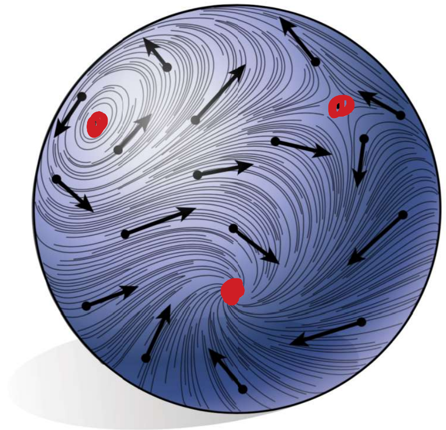
- Extract "index degeneracy",

$$\mathcal{A}(K, K_1, K_2, p) := \sum_{q=0}^{2K} (-1)^q\, d(K, K_1, K_2, p, q) \quad \in \mathbb{Z}$$

$$= \frac{1}{(2\pi i)^4} \oint_C \frac{dq}{q^{K+1}} \oint_C \frac{dq_1}{q_1^{K_1+1}} \oint_C \frac{dq_2}{q_2^{K_2+1}} \oint_C \frac{dy}{y^{p+1}} Z[q, q_1, q_2, y]$$

Index Theorem

Grothendieck–Riemann–Roch



• = fixed point

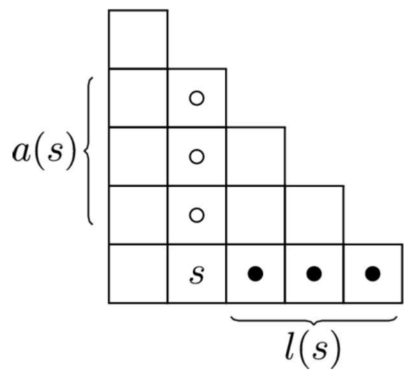
Manifold X with torus action T, fixed point set X^T

$$\chi_y(X) = \sum_{x\in X^T} y^{d_{\mathbb{C}}/2}\,\mathrm{Pexp}\left[(1-y)\,\mathrm{char}_T\,(T_x^* X)\right]$$

T character of tangent space at fixed point $x \in X^T$

Fixed points Nakajima

Young Diagram $Y(\lambda)$



- Labelled by partitions of K,
  $\lambda : \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{\ell(\lambda)} > \lambda_{\ell(\lambda)+1} = 0$

$$|\lambda| = \sum_{i\geq 1} \lambda_i = K$$

$$\chi_y(\mathcal{M}_K) = \sum_{\lambda\in\mathcal{P}:|\lambda|=K} \prod_{s\in Y(\lambda)} \frac{(1 - y q_1^{a(s)+1} q_2^{l(s)})(1 - y^{-1} q_1^{a(s)} q_2^{l(s)+1})}{(1 - q_1^{a(s)+1} q_2^{l(s)})(1 - q_1^{a(s)} q_2^{l(s)+1})}$$

Numerical Evaluation of $\mathscr{A}(K, K_1, K_2 = K_1, p = K)$

	$K_1 = K_2 =$	50	52	54
	50	4773158006473089778	13722136087430823474	10728665632616173124
	51	6943033937905529622	14870285533157146362	-1493086283031736666
	52	9190525712121239144	13822179413239343452	-2163073510148185436 6
	53	11174425419671147488	9447220680249748082	-5074469518684269411 4
K	54	12412734295210394496	5431158129565572 90	-8893822944634145587 0
	57	4868162317987965318	-6300908921769692954 6	-23389739952962751693 2
	58	-4069732338637176522	-9726567340028439510 8	-27244877884370581861 4
	59	-17508654288324952938	-13621685813308401908 8	-28899000730892369695 4
	60	-35960421557696977470	-17674458392244946650 8	-26882184022072047695 8

Problem: determine $K, K_1, K_2 \to \infty$ asymptotics....

Plethystic Resummation

Conjecture: Awata, Kanno...

Proof: Rains, Ole Warnaar

$$
\begin{aligned}
Z\left[q, q_1, q_2, y\right] &= \sum_{\lambda \in \mathcal{P}} q^{|\lambda|} \prod_{s \in Y(\lambda)} \frac{(1 - y q_1^{a(s)+1} q_2^{l(s)})(1 - y^{-1} q_1^{a(s)} q_2^{l(s)+1})}{(1 - q_1^{a(s)+1} q_2^{l(s)})(1 - q_1^{a(s)} q_2^{l(s)+1})} \\
&= \prod_{i,j,k=1}^{\infty} \frac{(1 - y q^i q_1^j q_2^{k-1})(1 - y^{-1} q^i q_1^{j-1} q_2^k)}{(1 - q^i q_1^{j-1} q_2^{k-1})(1 - q^i q_1^j q_2^k)} \\
&= \mathrm{Pexp}\left[\frac{q(1 - y q_1)(1 - y^{-1} q_2)}{(1 - q)(1 - q_1)(1 - q_2)}\right]
\end{aligned}
$$

Physics: 6d CFT on $T^2 \times \mathbb{R}^4$

Second Elliptic Gamma Function

Nishizawa

$$
\begin{aligned}
G_2(m;\tau,\tau_1,\tau_2) \quad &:= \quad \mathrm{Pexp}\left[\frac{y+\frac{qq_1q_2}{y}}{(1-q)(1-q_1)(1-q_2)}\right] \\
&= \quad \prod_{i,j,k=1}^{\infty}\left(1-yq^iq_1^jq_2^k\right)\left(1-y^{-1}q^{i+1}q_1^{j+1}q_2^{k+1}\right)
\end{aligned}
$$

$q = \exp 2\pi i\tau,\quad q_1 = \exp 2\pi i\tau_1,\quad q_2 = \exp 2\pi i\tau_2,\quad y = \exp 2\pi i m$

- Partition function, Vafa, Lockhart

$$
Z\left[q,q_1,q_2,y\right] \quad = \quad \frac{1}{\eta(\tau)\eta(\tau_1)\eta(\tau_2)}\,\frac{G_2'(0;\tau,\tau_1,\tau_2)}{G_2(m;\tau,\tau_1,\tau_2)}
$$

Related to counting “partitions” of a vector in $\mathbb{Z}^3$

Modular Transformation

Narukawa

4th Bernoulli Polynomial

$$G_2(m;\tau,\tau_1,\tau_2) = \exp\left(\frac{\pi i}{12}B_{44}(m;\tau,\tau_1,\tau_2,1)\right)\times$$
$$G_2\left(\frac{m}{\tau};-\frac{1}{\tau},\frac{\tau_1}{\tau},\frac{\tau_2}{\tau}\right)G_2\left(\frac{m}{\tau_1};\frac{\tau}{\tau_1},-\frac{1}{\tau_1},\frac{\tau_2}{\tau_1}\right)G_2\left(\frac{m}{\tau_2};\frac{\tau}{\tau_2},\frac{\tau_1}{\tau_2},-\frac{1}{\tau_2}\right)$$

- Yields asymptotic behaviour,

$$Z[q,q_1,q_2,y] \overset{\tau,\tau_1,\tau_2\to 0}{\sim} \exp\left(-\frac{\pi i}{12}\frac{m^2(1-m)^2}{\tau\tau_1\tau_2}\right)$$

$q=\exp 2\pi i\tau,\quad q_1=\exp 2\pi i\tau_1,\quad q_2=\exp 2\pi i\tau_2,\quad y=\exp 2\pi i m$

Saddle-Point

$K, K_1, K_2 \to \infty$

$$\mathcal{A}(K, K_1, K_2, p) = \frac{1}{(2\pi i)^4} \oint_C \frac{dq}{q^{K+1}} \oint_C \frac{dq_1}{q_1^{K_1+1}} \oint_C \frac{dq_2}{q_2^{K_2+1}} \oint_C \frac{dy}{y^{p+1}} Z\left[q, q_1, q_2, y\right]$$

$$\sim \int d^4\vec{\tau} \, \exp S\left(\vec{\tau}\right)$$

$\vec{\tau} := (\tau, \tau_1, \tau_2, m)$

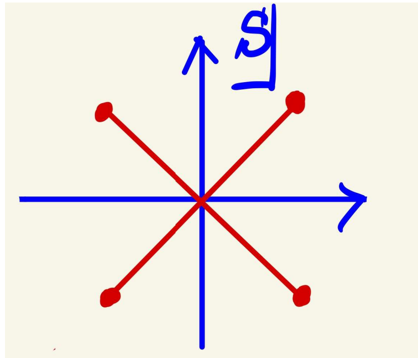
$$S\left(\vec{\tau}\right) := 2\pi i \left(-\frac{1}{24} \frac{m^2(1-m)^2}{\tau \tau_1 \tau_2} + K\tau + K_1\tau_1 + K_2\tau_2 + pm \right)$$

- Saddle points at $S = S\left(\vec{\tau}^*\right)$

$$\left.\frac{\partial S}{d\vec{\tau}}\right|_{\vec{\tau}=\vec{\tau}^*} = 0 \quad \Rightarrow \quad \left(\frac{S}{2\pi}\right)^4 = \frac{K K_1 K_2}{24}$$

- Four roots,

$$\left(\frac{S}{2\pi}\right)^4 = \frac{KK_1K_2}{24} \quad \Longrightarrow \quad S = \pm\chi \pm i\chi$$



$$\chi = 2\pi\left(\frac{KK_1K_2}{24}\right)^{\frac{1}{4}} > 0$$

- Leading contribution from two complex conjugate saddles,

$$\mathcal{A}(K, K_1, K_2, p = K) \sim \frac{3\sqrt{2}}{16\pi}\,\frac{\chi}{KK_1K_2}\,\exp(+\chi)\cos\left(\chi + \frac{\pi}{4}\right)$$

Theorem

ND + Zhao

Let $\mathscr{M}_K$ be the Hilbert Scheme of K points on $\mathbb{C}^2$ and,

$$\mathscr{C}[K,L] \;:=\; \sum_{q=1}^{2K} (-1)^q \dim_{\mathbb{C}} \left[H^{(L,L)}_{p=K,q} \left(\mathscr{M}_K \right) \right]$$

Then as, $K, L \to \infty$ with K/L held fixed,

$$\mathscr{C}[K,L] = \quad \exp(2\sqrt{2}\pi 24^{-1/4}\sqrt{L}K^{1/4} + P)(\cos(2\sqrt{2}\pi 24^{-1/4}\sqrt{L}K^{1/4} + Q) + o(1)$$

where $P < c_P\sqrt{K}$ and $Q < c_Q\sqrt{K}$ for some constants c_P and c_Q which depend on K/L.

Conclusion

- Matches numerical evaluation of $\mathscr{C}$ up to calculated errors

- $N > > 1$ version reproduces entropy of “ultra-spinning” BPS black hole in AdS_7

Outlook

- Similar results for other space: (affine) ADE quiver varieties, moduli spaces of vortices and instantons on compact manifolds

- Subleading corrections: Rademacher-type expansion? Convergence? Resurgence?