An Introduction to Exponential Asymptotics

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Outline of the Lecture

- The modified Bessel function $K_0(z)$ and its asymptotic expansion.
- Berry–Howls analysis for $K_0(z)$.
- Error bounds.
- Exponentially improved asymptotic expansion.
- The general case.
- Late coefficients.
- Concluding remarks.

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Modified Bessel Function of Order Zero

The ***modified Bessel function of order zero***, denoted by $K_0(z)$, is a particular solution of the second order linear ordinary differential equation

$$\frac{\mathrm{d}^2 w(z)}{\mathrm{d}z^2} + \frac{1}{z}\frac{\mathrm{d}w(z)}{\mathrm{d}z} = w(z).$$

It is uniquely determined by the property that

$$K_0(z) \sim \sqrt{\frac{\pi}{2z}}\mathrm{e}^{-z},$$

as $z \to \infty$ in the sector $|\arg z| \le \frac{3\pi}{2} - \delta$ $(< \frac{3\pi}{2})$. It can be written in the form

$$K_0(z) = F(z)\log z + G(z),$$

where $F(z)$ and $G(z)$ are (even) entire functions of z. Thus, $K_0(z)$ is an analytic function on the Riemann surface $\widehat{\mathbb{C}}$ associated with the logarithm.

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Kummer's Asymptotic Expansion

It was shown by E. Kummer in 1837 that

$$K_0(z) \sim \sqrt{\frac{\pi}{2z}} e^{-z} \sum_{n=0}^{\infty} \frac{a_n}{z^n},$$

as $z \to \infty$ in the sector $|\arg z| \le \frac{3\pi}{2} - \delta$ ($< \frac{3\pi}{2}$), with

$$a_n \stackrel{\text{def}}{=} (-1)^n \frac{(2n)!^2}{32^n n!^3}.$$

The series is divergent for any finite z. It is interpreted as follows. If we define for any non-negative integer N the remainder $R_N(z)$ after N terms by

$$K_0(z) = \sqrt{\frac{\pi}{2z}} e^{-z} \left(\sum_{n=0}^{N-1} \frac{a_n}{z^n} + R_N(z) \right),$$

then $R_N(z) = \mathcal{O}_{N,\delta}(z^{-N})$ as $z \to \infty$ in the sector $|\arg z| \le \frac{3\pi}{2} - \delta$ ($< \frac{3\pi}{2}$). Empty sums are taken to be zero.

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Connection Formula

Since $K_0(z)$, $K_0(ze^{\pi i})$ and $K_0(ze^{-\pi i})$ are all solutions of the same second order linear ordinary differential equation, there exist constants c_1 and c_2 such that

$$K_0(ze^{-\pi i}) = c_1 K_0(z) + c_2 K_0(ze^{\pi i}).$$

By the leading order asymptotics, we have

$$i\sqrt{\frac{\pi}{2z}}e^{z} \sim c_1\sqrt{\frac{\pi}{2z}}e^{-z} - ic_2\sqrt{\frac{\pi}{2z}}e^{z}$$

as $z \to +\infty$, which implies that $c_2 = -1$. So $c_1 K_0(z) = K_0(ze^{\pi i}) + K_0(ze^{-\pi i})$. The exact form involving the logarithm then yields $c_1 = 2$, leading to the *connection formula*

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Stokes Phenomenon

Substituting the asymptotic expansion into this connection formula yields

$$K_0(z) \sim \sqrt{\frac{\pi}{2z}} \mathrm{e}^{-z} \sum_{n=0}^{\infty} \frac{a_n}{z^n} + 2\mathrm{i} \sqrt{\frac{\pi}{2z}} \mathrm{e}^{z} \sum_{\ell=0}^{\infty} (-1)^{\ell} \frac{a_\ell}{z^\ell},$$

as $z \to \infty$ in the sector $\frac{\pi}{2} + \delta \le \arg z \le \frac{5\pi}{2} - \delta$. Thus, in the sector $\frac{\pi}{2} + \delta \le \arg z \le \frac{3\pi}{2} - \delta$ both this and the original expansion

$$K_0(z) \sim \sqrt{\frac{\pi}{2z}} \mathrm{e}^{-z} \sum_{n=0}^{\infty} \frac{a_n}{z^n}$$

hold true. Note, however, that the difference

$$2\mathrm{i} \sqrt{\frac{\pi}{2z}} \mathrm{e}^{z} \sum_{\ell=0}^{\infty} (-1)^{\ell} \frac{a_\ell}{z^\ell}$$

is ***exponentially small*** in the common sector of validity.

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F. W. J. OLVER showed that from the numerical point of view the best asymptotic approximation to $K_0(z)$ is as follows:

$$K_0(z) \sim \sqrt{\frac{\pi}{2z}} e^{-z} \sum_{n=0}^{\infty} \frac{a_n}{z^n} + S(\arg z) \sqrt{\frac{\pi}{2z}} e^{z} \sum_{\ell=0}^{\infty} (-1)^{\ell} \frac{a_\ell}{z^\ell},$$

as $z \to \infty$, where

$$S(\arg z) \stackrel{\text{def}}{=} \begin{cases} 0 & \text{if } -\pi < \arg z < \pi, \\ \mathrm{i} & \text{if } \arg z = \pi, \\ 2\mathrm{i} & \text{if } \pi < \arg z < 2\pi, \end{cases}$$

is the so-called ***Stokes multiplier***. The discontinuous change in the form of the asymptotic expansion through the ray $\arg z = \pi$ is an example of the *Stokes phenomenon*. The ray $\arg z = \pi$ is called a *Stokes line*. A similar phenomenon happens when $\arg z = -\pi$.

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Laplace-Type Integrals

We consider integrals of the form

$$I^{(k)}(z) = \int_{\mathscr{C}^{(k)}(\theta)} \mathrm{e}^{-zf(t)} g(t) \mathrm{d}t, \quad z = |z|\mathrm{e}^{\mathrm{i}\theta},$$

where $\mathscr{C}^{(k)}(\theta)$ is the doubly infinite path of steepest descent passing through the ***simple saddle point*** $t^{(k)}$ of $f(t)$ (i.e., $f'(t^{(k)}) = 0$ but $f''(t^{(k)}) \neq 0$). The functions $f(t)$ and $g(t)$ are assumed to be analytic in a neighbourhood of the contour $\mathscr{C}^{(k)}(\theta)$. We also assume that $f(t)$ has several other simple saddle points in the complex t-plane situated at $t = t^{(p)}$ and indexed by $p \in \mathbb{Z}$.

The asymptotic expansion of $I^{(k)}(z)$ for large z follows from an application of ***the method of steepest descents***.

A hyperasymptotic theory for such integrals was developed by Sir M. V. Berry and C. J. Howls in the early 1990's.

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Picture Gallery

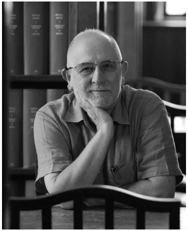

Figure 1: *Sir Michael V. Berry*

Figure 2: *Christopher J. Howls*

Figure 3: *Robert B. Dingle*

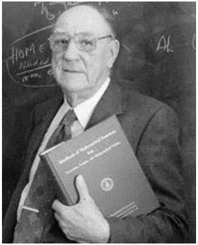

Figure 4: *Frank W. J. Olver*

Example: The Function $K_0(z)$

The modified Bessel function $K_0(z)$ has the integral representation

$$K_0(z) = \frac{1}{2}\int_{-\infty}^{+\infty} \mathrm{e}^{-z\cosh t}\mathrm{d}t,$$

provided $z > 0$. With our notation $f(t) = \cosh t$, $g(t) \equiv 1$ and the contour $\mathscr{C}^{(0)}(0)$ is the real axis running from $-\infty$ to $+\infty$ through the saddle point $t^{(0)} = 0$ of $f(t)$. The function $\cosh t$ has infinitely many other saddles at $t^{(p)} = \pi \mathrm{i} p$, $p \in \mathbb{Z}$.

For complex z we write

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$$\arg(z(\cosh t - \cosh(t^{(0)}))) = \arg(\mathrm{e}^{\mathrm{i}\theta}(\cosh t - 1)) = 0.$$

Example: The Function $K_0(z)$

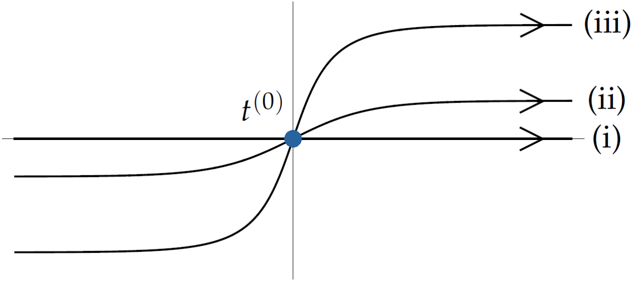

Figure 5: *The steepest descent contour $\mathscr{C}^{(0)}(\theta)$ associated with the modified Bessel function through the saddle point $t^{(0)} = 0$ when (i) $\theta = 0$, (ii) $\theta = -\frac{\pi}{4}$ and (iii) $\theta = -\frac{3\pi}{4}$.*

Example: The Function $K_0(z)$

We re-write our representation in the form

$$K_0(z) = \frac{e^{-z}}{2} \int_{\mathscr{C}^{(0)}(\theta)} e^{-z(\cosh t - 1)} dt.$$

By the definition of the steepest descent path, the function $z(\cosh t - 1)$ decreases strictly monotonically from $+\infty$ to 0 as t runs into the saddle $t^{(0)} = 0$ and then increases strictly monotonically from 0 to $+\infty$ as t moves away from the saddle $t^{(0)} = 0$. Thus, the equation $s = z(\cosh t - 1)$ has two solutions $t_\pm = t_\pm(s/z)$ for each positive s. Hence, we may write

$$\begin{aligned} K_0(z) &= \frac{e^{-z}}{2} \left(\int_{+\infty}^{0} e^{-s} \frac{dt_-}{ds} ds + \int_0^{+\infty} e^{-s} \frac{dt_+}{ds} ds \right) \\ &= \frac{e^{-z}}{2} \int_0^{+\infty} e^{-s} \left(\frac{dt_+}{ds} - \frac{dt_-}{ds} \right) ds. \end{aligned}$$

Example: The Function $K_0(z)$

We re-write our representation in the form

$$K_0(z) = \frac{\mathrm{e}^{-z}}{2} \int_{\mathscr{C}^{(0)}(\theta)} \mathrm{e}^{-z(\cosh t - 1)} \mathrm{d}t.$$

By the definition of the steepest descent path, the function $z(\cosh t - 1)$ decreases strictly monotonically from $+\infty$ to 0 as t runs into the saddle $t^{(0)} = 0$ and then increases strictly monotonically from 0 to $+\infty$ as t moves away from the saddle $t^{(0)} = 0$. Thus, the equation $s = z(\cosh t - 1)$ has two solutions $t_\pm = t_\pm(s/z)$ for each positive s. Hence, we may write

$$K_0(z) = \frac{\mathrm{e}^{-z}}{2} \left(\int_{+\infty}^{0} \mathrm{e}^{-s} \frac{\mathrm{d}t_-}{\mathrm{d}s} \mathrm{d}s + \int_{0}^{+\infty} \mathrm{e}^{-s} \frac{\mathrm{d}t_+}{\mathrm{d}s} \mathrm{d}s \right)$$

$$= \frac{\mathrm{e}^{-z}}{2} \int_{0}^{+\infty} \mathrm{e}^{-s} \left(\frac{\mathrm{d}t_+}{\mathrm{d}s} - \frac{\mathrm{d}t_-}{\mathrm{d}s} \right) \mathrm{d}s.$$

Example: The Function $K_0(z)$

We re-write our representation in the form

$$K_0(z) = \frac{\mathrm{e}^{-z}}{2} \int_{\mathscr{C}^{(0)}(\theta)} \mathrm{e}^{-z(\cosh t - 1)} \mathrm{d}t.$$

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$$\begin{aligned} K_0(z) &= \frac{\mathrm{e}^{-z}}{2} \left(\int_{+\infty}^{0} \mathrm{e}^{-s} \frac{\mathrm{d}t_-}{\mathrm{d}s} \mathrm{d}s + \int_0^{+\infty} \mathrm{e}^{-s} \frac{\mathrm{d}t_+}{\mathrm{d}s} \mathrm{d}s \right) \\ &= \frac{\mathrm{e}^{-z}}{2} \int_0^{+\infty} \mathrm{e}^{-s} \left(\frac{\mathrm{d}t_+}{\mathrm{d}s} - \frac{\mathrm{d}t_-}{\mathrm{d}s} \right) \mathrm{d}s. \end{aligned}$$

Example: The Function $K_0(z)$

We re-write our representation in the form

$$K_0(z) = \frac{\mathrm{e}^{-z}}{2} \int_{\mathscr{C}^{(0)}(\theta)} \mathrm{e}^{-z(\cosh t - 1)} \mathrm{d}t.$$

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Example: The Function $K_0(z)$

Since $s = z(\cosh t_\pm - 1)$, we have that

$$\frac{\mathrm{d}t_\pm}{\mathrm{d}s} = \frac{1}{z\sinh t_\pm}.$$

Thus,

$$K_0(z) = \frac{\mathrm{e}^{-z}}{2}\int_0^{+\infty} \mathrm{e}^{-s}\left(\frac{1}{z\sinh t_+} - \frac{1}{z\sinh t_-}\right)\mathrm{d}s.$$

This works if $\mathscr{C}^{(0)}(\theta)$ does not go through any of the other saddles $\pm\pi\mathrm{i}$, $\pm 2\pi\mathrm{i}$, $\pm 3\pi\mathrm{i}, \ldots$. We assume this to be the case. The apparent singularity at $t^{(0)} = 0$ is of a removable type. Now,

$$\pm\frac{1}{z\sinh t_\pm} = s^{-\frac{1}{2}}\left[\frac{\sqrt{z(\cosh u - 1)}}{\frac{\mathrm{d}}{\mathrm{d}u}(z(\cosh u - 1) - s)}\right]_{u=t_\pm}$$

$$= s^{-\frac{1}{2}}\operatorname*{Res}_{u=t_\pm}\frac{\sqrt{z(\cosh u - 1)}}{z(\cosh u - 1) - s} = \frac{s^{-\frac{1}{2}}}{2\pi\mathrm{i}}\oint_{(t_\pm +)}\frac{\sqrt{z(\cosh u - 1)}}{z(\cosh u - 1) - s}\mathrm{d}u.$$

Here, $\sqrt{\cosh u - 1} = 2^{-\frac{1}{2}}u + \mathcal{O}(u^2)$ as $u \to 0$.

Example: The Function $K_0(z)$

Since $s = z(\cosh t_\pm - 1)$, we have that

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This works if $\mathscr{C}^{(0)}(\theta)$ does not go through any of the other saddles $\pm\pi\mathrm{i}$, $\pm 2\pi\mathrm{i}$, $\pm 3\pi\mathrm{i}, \ldots$. We assume this to be the case. The apparent singularity at $t^{(0)} = 0$ is of a removable type. Now,

$$\begin{aligned}
\pm \frac{1}{z \sinh t_\pm} &= s^{-\frac{1}{2}} \left[\frac{\sqrt{z(\cosh u - 1)}}{\frac{\mathrm{d}}{\mathrm{d}u}(z(\cosh u - 1) - s)} \right]_{u=t_\pm} \\
&= s^{-\frac{1}{2}} \operatorname*{Res}_{u=t_\pm} \frac{\sqrt{z(\cosh u - 1)}}{z(\cosh u - 1) - s} = \frac{s^{-\frac{1}{2}}}{2\pi\mathrm{i}} \oint_{(t_\pm +)} \frac{\sqrt{z(\cosh u - 1)}}{z(\cosh u - 1) - s} \mathrm{d}u.
\end{aligned}$$

Here, $\sqrt{\cosh u - 1} = 2^{-\frac{1}{2}} u + \mathcal{O}(u^2)$ as $u \to 0$.

Example: The Function $K_0(z)$

Therefore,

$$\begin{aligned}\frac{1}{z\sinh t_+} - \frac{1}{z\sinh t_-} &= \frac{s^{-\frac{1}{2}}}{2\pi\mathrm{i}} \oint_{(t_+ +)} \frac{\sqrt{z(\cosh u - 1)}}{z(\cosh u - 1) - s}\mathrm{d}u \\ &\quad + \frac{s^{-\frac{1}{2}}}{2\pi\mathrm{i}} \oint_{(t_- +)} \frac{\sqrt{z(\cosh u - 1)}}{z(\cosh u - 1) - s}\mathrm{d}u \\ &= \frac{s^{-\frac{1}{2}}}{2\pi\mathrm{i}} \oint_{\Gamma^{(0)}(\theta)} \frac{\sqrt{z(\cosh u - 1)}}{z(\cosh u - 1) - s}\mathrm{d}u,\end{aligned}$$

where $\Gamma^{(0)}(\theta)$ is an infinite contour encircling the steepest descent contour $\mathscr{C}^{(0)}(\theta)$ in the positive sense. Consequently,

$$\begin{aligned}K_0(z) &= \frac{\mathrm{e}^{-z}}{2} \int_0^{+\infty} \mathrm{e}^{-s} s^{-\frac{1}{2}} \frac{1}{2\pi\mathrm{i}} \oint_{\Gamma^{(0)}(\theta)} \frac{\sqrt{z(\cosh u - 1)}}{z(\cosh u - 1) - s}\mathrm{d}u\mathrm{d}s \\ &= \frac{\mathrm{e}^{-z}}{2z^{1/2}} \int_0^{+\infty} \mathrm{e}^{-s} s^{-\frac{1}{2}} \frac{1}{2\pi\mathrm{i}} \oint_{\Gamma^{(0)}(\theta)} \frac{\sqrt{\cosh u - 1}}{\cosh u - 1 - s/z}\mathrm{d}u\mathrm{d}s.\end{aligned}$$

Example: The Function $K_0(z)$

Therefore,

$$\begin{aligned}\frac{1}{z\sinh t_+} - \frac{1}{z\sinh t_-} &= \frac{s^{-\frac{1}{2}}}{2\pi \mathrm{i}} \oint_{(t_+ +)} \frac{\sqrt{z(\cosh u - 1)}}{z(\cosh u - 1) - s} \mathrm{d}u \\ &\quad + \frac{s^{-\frac{1}{2}}}{2\pi \mathrm{i}} \oint_{(t_- +)} \frac{\sqrt{z(\cosh u - 1)}}{z(\cosh u - 1) - s} \mathrm{d}u \\ &= \frac{s^{-\frac{1}{2}}}{2\pi \mathrm{i}} \oint_{\Gamma^{(0)}(\theta)} \frac{\sqrt{z(\cosh u - 1)}}{z(\cosh u - 1) - s} \mathrm{d}u,\end{aligned}$$

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Example: The Function $K_0(z)$

Therefore,

$$
\begin{aligned}
\frac{1}{z\sinh t_+} - \frac{1}{z\sinh t_-} &= \frac{s^{-\frac{1}{2}}}{2\pi\mathrm{i}} \oint_{(t_+ +)} \frac{\sqrt{z(\cosh u - 1)}}{z(\cosh u - 1) - s} \mathrm{d}u \\
&\quad + \frac{s^{-\frac{1}{2}}}{2\pi\mathrm{i}} \oint_{(t_- +)} \frac{\sqrt{z(\cosh u - 1)}}{z(\cosh u - 1) - s} \mathrm{d}u \\
&= \frac{s^{-\frac{1}{2}}}{2\pi\mathrm{i}} \oint_{\Gamma^{(0)}(\theta)} \frac{\sqrt{z(\cosh u - 1)}}{z(\cosh u - 1) - s} \mathrm{d}u,
\end{aligned}
$$

where $\Gamma^{(0)}(\theta)$ is an infinite contour encircling the steepest descent contour $\mathscr{C}^{(0)}(\theta)$ in the positive sense. Consequently,

$$
\begin{aligned}
K_0(z) &= \frac{\mathrm{e}^{-z}}{2} \int_0^{+\infty} \mathrm{e}^{-s} s^{-\frac{1}{2}} \frac{1}{2\pi\mathrm{i}} \oint_{\Gamma^{(0)}(\theta)} \frac{\sqrt{z(\cosh u - 1)}}{z(\cosh u - 1) - s} \mathrm{d}u\mathrm{d}s \\
&= \frac{\mathrm{e}^{-z}}{2z^{1/2}} \int_0^{+\infty} \mathrm{e}^{-s} s^{-\frac{1}{2}} \frac{1}{2\pi\mathrm{i}} \oint_{\Gamma^{(0)}(\theta)} \frac{\sqrt{\cosh u - 1}}{\cosh u - 1 - s/z} \mathrm{d}u\mathrm{d}s.
\end{aligned}
$$

Example: The Function $K_0(z)$

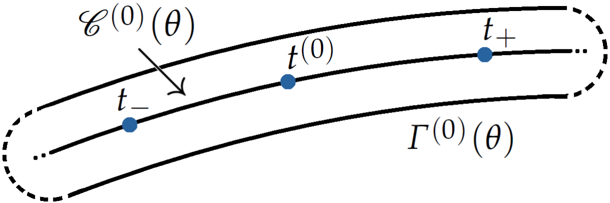

Figure 6: *The infinite "sausage" contour $\Gamma^{(0)}(\theta)$ encircling the steepest descent path $\mathscr{C}^{(0)}(\theta)$.*

Example: The Function $K_0(z)$

We now expand the inner integral using the formula ($N \geq 0$)

$$\frac{\sqrt{\cosh u - 1}}{\cosh u - 1 - t/z} = \frac{1}{\sqrt{\cosh u - 1}} \frac{1}{1 - t/z(\cosh u - 1)}$$
$$= \sum_{n=0}^{N-1} \frac{t^n}{z^n (\cosh u - 1)^{n+\frac{1}{2}}} + \frac{t^N}{z^N (\cosh u - 1)^{N+\frac{1}{2}}} \frac{1}{1 - t/z(\cosh u - 1)},$$

to obtain

$$K_0(z) = \sqrt{\frac{\pi}{2z}} \mathrm{e}^{-z} \left(\sum_{n=0}^{N-1} \frac{a_n}{z^n} + R_N(z) \right),$$

with

$$a_n = \frac{\Gamma(n+\frac{1}{2})}{\sqrt{2\pi}} \frac{1}{2\pi \mathrm{i}} \oint_{\Gamma^{(0)}(\theta)} \frac{\mathrm{d}u}{(\cosh u - 1)^{n+\frac{1}{2}}} = \frac{\Gamma(n+\frac{1}{2})}{\sqrt{2\pi}} \frac{1}{2\pi \mathrm{i}} \oint_{(0+)} \frac{\mathrm{d}u}{(\cosh u - 1)^{n+\frac{1}{2}}}$$

and

$$R_N(z) = \frac{1}{\sqrt{2\pi}} \frac{1}{z^N} \int_0^{+\infty} \mathrm{e}^{-t} t^{N-\frac{1}{2}} \frac{1}{2\pi \mathrm{i}} \oint_{\Gamma^{(0)}(\theta)} \frac{1}{(\cosh u - 1)^{N+\frac{1}{2}}} \frac{1}{1 - t/z(\cosh u - 1)} \mathrm{d}u \mathrm{d}t.$$

Example: The Function $K_0(z)$

We now expand the inner integral using the formula ($N \geq 0$)

$$
\begin{aligned}
\frac{\sqrt{\cosh u - 1}}{\cosh u - 1 - t/z} &= \frac{1}{\sqrt{\cosh u - 1}} \frac{1}{1 - t/z(\cosh u - 1)} \\
&= \sum_{n=0}^{N-1} \frac{t^n}{z^n (\cosh u - 1)^{n+\frac{1}{2}}} + \frac{t^N}{z^N (\cosh u - 1)^{N+\frac{1}{2}}} \frac{1}{1 - t/z(\cosh u - 1)},
\end{aligned}
$$

to obtain

$$
K_0(z) = \sqrt{\frac{\pi}{2z}} \mathrm{e}^{-z} \left(\sum_{n=0}^{N-1} \frac{a_n}{z^n} + R_N(z) \right),
$$

with

$$
a_n = \frac{\Gamma\left(n + \frac{1}{2}\right)}{\sqrt{2\pi}} \frac{1}{2\pi \mathrm{i}} \oint_{\Gamma^{(0)}(\theta)} \frac{\mathrm{d}u}{(\cosh u - 1)^{n+\frac{1}{2}}} = \frac{\Gamma\left(n + \frac{1}{2}\right)}{\sqrt{2\pi}} \frac{1}{2\pi \mathrm{i}} \oint_{(0+)} \frac{\mathrm{d}u}{(\cosh u - 1)^{n+\frac{1}{2}}}
$$

and

$$
R_N(z) = \frac{1}{\sqrt{2\pi}} \frac{1}{z^N} \int_0^{+\infty} \mathrm{e}^{-t} t^{N-\frac{1}{2}} \frac{1}{2\pi \mathrm{i}} \oint_{\Gamma^{(0)}(\theta)} \frac{1}{(\cosh u - 1)^{N+\frac{1}{2}}} \frac{1}{1 - t/z(\cosh u - 1)} \mathrm{d}u \mathrm{d}t.
$$

Example: The Function $K_0(z)$

We now expand the inner integral using the formula ($N \geq 0$)

$$\frac{\sqrt{\cosh u - 1}}{\cosh u - 1 - t/z} = \frac{1}{\sqrt{\cosh u - 1}} \frac{1}{1 - t/z(\cosh u - 1)}$$

$$= \sum_{n=0}^{N-1} \frac{t^n}{z^n (\cosh u - 1)^{n+\frac{1}{2}}} + \frac{t^N}{z^N (\cosh u - 1)^{N+\frac{1}{2}}} \frac{1}{1 - t/z(\cosh u - 1)},$$

to obtain

$$K_0(z) = \sqrt{\frac{\pi}{2z}} \mathrm{e}^{-z} \left(\sum_{n=0}^{N-1} \frac{a_n}{z^n} + R_N(z) \right),$$

with

$$a_n = \frac{\Gamma\left(n + \frac{1}{2}\right)}{\sqrt{2\pi}} \frac{1}{2\pi \mathrm{i}} \oint_{\Gamma^{(0)}(\theta)} \frac{\mathrm{d}u}{(\cosh u - 1)^{n+\frac{1}{2}}} = \frac{\Gamma\left(n + \frac{1}{2}\right)}{\sqrt{2\pi}} \frac{1}{2\pi \mathrm{i}} \oint_{(0+)} \frac{\mathrm{d}u}{(\cosh u - 1)^{n+\frac{1}{2}}}$$

and

$$R_N(z) = \frac{1}{\sqrt{2\pi}} \frac{1}{z^N} \int_0^{+\infty} \mathrm{e}^{-t} t^{N-\frac{1}{2}} \frac{1}{2\pi \mathrm{i}} \oint_{\Gamma^{(0)}(\theta)} \frac{1}{(\cosh u - 1)^{N+\frac{1}{2}}} \frac{1}{1 - t/z(\cosh u - 1)} \mathrm{d}u \mathrm{d}t.$$

Example: The Function $K_0(z)$

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$$R_N(z) = \frac{1}{\sqrt{2\pi}} \frac{1}{z^N} \int_0^{+\infty} \mathrm{e}^{-t} t^{N-\frac{1}{2}} \frac{1}{2\pi \mathrm{i}} \oint_{\Gamma^{(0)}(\theta)} \frac{1}{(\cosh u - 1)^{N+\frac{1}{2}}} \frac{1}{1 - t/z(\cosh u - 1)} \mathrm{d}u \mathrm{d}t.$$

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Example: The Function $K_0(z)$

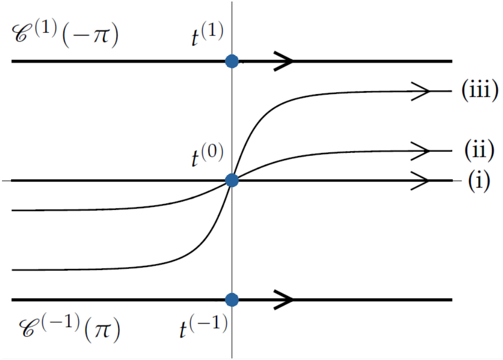

Figure 7: *The steepest descent contour $\mathscr{C}^{(0)}(\theta)$ associated with the modified Bessel function through the saddle point $t^{(0)} = 0$ when (i) $\theta = 0$, (ii) $\theta = -\frac{\pi}{4}$ and (iii) $\theta = -\frac{3\pi}{4}$. The paths $\mathscr{C}^{(1)}(-\pi)$ and $\mathscr{C}^{(-1)}(\pi)$ are the adjacent contours for $t^{(0)}$. The domain $\Delta^{(0)}$ comprises all points between $\mathscr{C}^{(1)}(-\pi)$ and $\mathscr{C}^{(-1)}(\pi)$.*

Example: The Function $K_0(z)$

By expanding $\Gamma^{(0)}(\theta)$ to the boundary of $\Delta^{(0)}$, we obtain

$$R_N(z) = \frac{1}{\sqrt{2\pi}}\frac{1}{z^N}\int_0^{+\infty} \mathrm{e}^{-t} t^{N-\frac{1}{2}} \frac{1}{2\pi \mathrm{i}}\int_{\mathscr{C}^{(-1)}(\pi)} \frac{(\cosh u - 1)^{-N-\frac{1}{2}}}{1 - t/z(\cosh u - 1)} \mathrm{d}u\mathrm{d}t$$
$$- \frac{1}{\sqrt{2\pi}}\frac{1}{z^N}\int_0^{+\infty} \mathrm{e}^{-t} t^{N-\frac{1}{2}} \frac{1}{2\pi \mathrm{i}}\int_{\mathscr{C}^{(1)}(-\pi)} \frac{(\cosh u - 1)^{-N-\frac{1}{2}}}{1 - t/z(\cosh u - 1)} \mathrm{d}u\mathrm{d}t,$$

provided that $|\arg z| < \pi$.

- The domain $\Delta^{(0)}$ is defined as $\Delta^{(0)} \stackrel{\text{def}}{=} \bigcup_{|\theta|<\pi} \mathscr{C}^{(0)}(\theta)$.
- On $\mathscr{C}^{(0)}(\theta)$ we have $\arg(\mathrm{e}^{\mathrm{i}\theta}(\cosh u - 1)) = 0$, i.e., $\arg(\cosh u - 1) = -\theta$.
- Also, $\sqrt{\cosh u - 1} = 2^{-\frac{1}{2}} u + \mathcal{O}(u^2)$ as $u \to 0$.
- Geometrically, $\mathscr{C}^{(0)}(\theta) = \mathscr{C}^{(0)}(\theta \pm 4\pi) = \mathscr{C}^{(0)}(\theta \pm 8\pi) = \cdots$.

Therefore, we define $\sqrt{\cosh u - 1}$ to be positive when u is positive and we define it by continuity elsewhere.

Example: The Function $K_0(z)$

By expanding $\Gamma^{(0)}(\theta)$ to the boundary of $\Delta^{(0)}$, we obtain

$$\begin{aligned} R_N(z) = {} & \frac{1}{\sqrt{2\pi}} \frac{1}{z^N} \int_0^{+\infty} \mathrm{e}^{-t} t^{N-\frac{1}{2}} \frac{1}{2\pi\mathrm{i}} \int_{\mathscr{C}^{(-1)}(\pi)} \frac{(\cosh u - 1)^{-N-\frac{1}{2}}}{1 - t/z(\cosh u - 1)} \mathrm{d}u\mathrm{d}t \\ & - \frac{1}{\sqrt{2\pi}} \frac{1}{z^N} \int_0^{+\infty} \mathrm{e}^{-t} t^{N-\frac{1}{2}} \frac{1}{2\pi\mathrm{i}} \int_{\mathscr{C}^{(1)}(-\pi)} \frac{(\cosh u - 1)^{-N-\frac{1}{2}}}{1 - t/z(\cosh u - 1)} \mathrm{d}u\mathrm{d}t, \end{aligned}$$

provided that $|\arg z| < \pi$.

- The domain $\Delta^{(0)}$ is defined as $\Delta^{(0)} \stackrel{\text{def}}{=} \bigcup_{|\theta|<\pi} \mathscr{C}^{(0)}(\theta)$.
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By expanding $\Gamma^{(0)}(\theta)$ to the boundary of $\Delta^{(0)}$, we obtain

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Example: The Function $K_0(z)$

The remainder $R_N(z)$ is given by

$$R_N(z) = \frac{1}{\sqrt{2\pi}}\frac{1}{z^N}\int_0^{+\infty} \mathrm{e}^{-t} t^{N-\frac{1}{2}} \frac{1}{2\pi\mathrm{i}}\int_{\mathscr{C}^{(-1)}(\pi)} \frac{(\cosh u - 1)^{-N-\frac{1}{2}}}{1 - t/z(\cosh u - 1)} \mathrm{d}u\mathrm{d}t$$
$$- \frac{1}{\sqrt{2\pi}}\frac{1}{z^N}\int_0^{+\infty} \mathrm{e}^{-t} t^{N-\frac{1}{2}} \frac{1}{2\pi\mathrm{i}}\int_{\mathscr{C}^{(1)}(-\pi)} \frac{(\cosh u - 1)^{-N-\frac{1}{2}}}{1 - t/z(\cosh u - 1)} \mathrm{d}u\mathrm{d}t,$$

provided that $|\arg z| < \pi$. We perform the change of variable from t and u to s and u by

$$t = s(\cosh u - 1)\mathrm{e}^{\pi\mathrm{i}} = s((\cosh u - \cosh(-\pi\mathrm{i}))\mathrm{e}^{\pi\mathrm{i}} + 2)$$

in the first integral, and by

$$t = s(\cosh u - 1)\mathrm{e}^{-\pi\mathrm{i}} = s((\cosh u - \cosh(\pi\mathrm{i}))\mathrm{e}^{-\pi\mathrm{i}} + 2)$$

in the second integral.

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$$- \frac{1}{\sqrt{2\pi}} \frac{1}{z^N} \int_0^{+\infty} \mathrm{e}^{-t} t^{N-\frac{1}{2}} \frac{1}{2\pi \mathrm{i}} \int_{\mathscr{C}^{(1)}(-\pi)} \frac{(\cosh u - 1)^{-N-\frac{1}{2}}}{1 - t/z(\cosh u - 1)} \mathrm{d}u \mathrm{d}t,$$

provided that $|\arg z| < \pi$. We perform the change of variable from t and u to s and u by

$$t = s(\cosh u - 1)\mathrm{e}^{\pi \mathrm{i}} = s((\cosh u - \cosh(-\pi \mathrm{i}))\mathrm{e}^{\pi \mathrm{i}} + 2)$$

in the first integral, and by

$$t = s(\cosh u - 1)\mathrm{e}^{-\pi \mathrm{i}} = s((\cosh u - \cosh(\pi \mathrm{i}))\mathrm{e}^{-\pi \mathrm{i}} + 2)$$

in the second integral.

Example: The Function $K_0(z)$

In this way, we arrive at

$$R_N(z) = (-1)^N \frac{1}{\pi} \frac{1}{\sqrt{2\pi}} \frac{1}{z^N} \int_0^{+\infty} \frac{s^{N-\frac{1}{2}} \mathrm{e}^{-2s}}{1+s/z} \frac{1}{2} \int_{\mathscr{C}^{(-1)}(\pi)} \mathrm{e}^{-s\mathrm{e}^{\pi \mathrm{i}}(\cosh u - \cosh(-\pi \mathrm{i}))} \mathrm{d}u \mathrm{d}s$$

$$+ (-1)^N \frac{1}{\pi} \frac{1}{\sqrt{2\pi}} \frac{1}{z^N} \int_0^{+\infty} \frac{s^{N-\frac{1}{2}} \mathrm{e}^{-2s}}{1+s/z} \frac{1}{2} \int_{\mathscr{C}^{(1)}(-\pi)} \mathrm{e}^{-s\mathrm{e}^{-\pi \mathrm{i}}(\cosh u - \cosh(\pi \mathrm{i}))} \mathrm{d}u \mathrm{d}s,$$

provided that $|\arg z| < \pi$. Finally, we shift $\mathscr{C}^{(-1)}(\pi)$ up by $\pi\mathrm{i}$ and $\mathscr{C}^{(1)}(-\pi)$ down by $\pi\mathrm{i}$:

$$R_N(z) = (-1)^N \frac{1}{\pi} \frac{1}{\sqrt{2\pi}} \frac{1}{z^N} \int_0^{+\infty} \frac{s^{N-\frac{1}{2}} \mathrm{e}^{-2s}}{1+s/z} \frac{1}{2} \int_{-\infty}^{+\infty} \mathrm{e}^{-s(\cosh t - 1)} \mathrm{d}t \mathrm{d}s$$

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$$= (-1)^N \frac{1}{z^N} \frac{1}{\pi} \sqrt{\frac{2}{\pi}} \int_0^{+\infty} \frac{s^{N-\frac{1}{2}} \mathrm{e}^{-s}}{1+s/z} \frac{1}{2} \int_{-\infty}^{+\infty} \mathrm{e}^{-s\cosh t} \mathrm{d}t \mathrm{d}s$$

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$$= (-1)^N \frac{1}{z^N}\frac{1}{\pi}\sqrt{\frac{2}{\pi}}\int_0^{+\infty}\frac{s^{N-\frac{1}{2}}\mathrm{e}^{-s}}{1+s/z}\frac{1}{2}\int_{-\infty}^{+\infty}\mathrm{e}^{-s\cosh t}\mathrm{d}t\mathrm{d}s$$

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In this way, we arrive at

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provided that $|\arg z| < \pi$. Finally, we shift $\mathscr{C}^{(-1)}(\pi)$ up by $\pi\mathrm{i}$ and $\mathscr{C}^{(1)}(-\pi)$ down by $\pi\mathrm{i}$:

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Stokes Phenomenon Again

In summary, for any non-negative integer N, we have

$$K_0(z) = \sqrt{\frac{\pi}{2z}} e^{-z} \left(\sum_{n=0}^{N-1} \frac{a_n}{z^n} + R_N(z) \right)$$

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As $\arg z$ increases beyond π, the pole arising from the denominator is entrapped: consequently, the analytic continuation of $R_N(z)$ to the sector $\pi < \arg z < 3\pi$ is given by the formula

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The First Hyperterminant

Let $p > 0$ be a real number and let σ be an element on $\widehat{\mathbb{C}}$. The ***first hyperterminant*** is defined by

$$\mathbf{F}^{(1)}\left(w;\ \begin{matrix} p \\ \sigma \end{matrix}\right) \stackrel{\text{def}}{=} \frac{1}{\Gamma(p)} \int_0^{+\infty} \frac{t^{p-1}\mathrm{e}^{-\sigma t}}{1 + t/w} \mathrm{d}t$$

provided $|\arg(\sigma w)| < \pi$, and by analytic continuation elsewhere. It can also be represented in terms of the incomplete gamma function as

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The simple observation $a_N = z^N(R_N(z) - R_{N+1}(z))$ and the definition of the first hyperterminant leads to the formula

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for any $N \geq 0$. Consequently,

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$$|R_N(z)| \leq \frac{|a_N|}{|z|^N} \sup_{r>0} \left| \mathbf{F}^{(1)}\left(r\mathrm{e}^{\mathrm{i}\arg z}; \begin{array}{c} N+\frac{1}{2} \\ 1 \end{array} \right) \right|$$

provided $|\arg z| < \frac{3\pi}{2}$. We need some simple estimates for the first hyperterminant to make this result useful.

Bounds for the First Hyperterminant

Proposition (G. N., 2016)

If $p > 0$ and $|\arg w| < \frac{3\pi}{2}$, then

$$\left|\mathbf{F}^{(1)}\left(w;\, \begin{matrix} p \\ 1 \end{matrix}\right)\right| \le \begin{cases} 1 & \text{if } \ |\arg w| \le \frac{\pi}{2}, \\ \min(|\csc(\arg w)|, \chi(p) + 1) & \text{if } \ \frac{\pi}{2} < |\arg w| \le \pi, \\ \frac{\sqrt{2\pi p}}{|\cos(\arg w)|^p} + \chi(p) + 1 & \text{if } \ \pi < |\arg w| < \frac{3\pi}{2}, \end{cases}$$

with

$$\chi(p) \stackrel{\text{def}}{=} \frac{\sqrt{\pi}\,\Gamma\left(\frac{p}{2} + 1\right)}{\Gamma\left(\frac{p}{2} + \frac{1}{2}\right)}.$$

We remark that G. N. Watson showed that

$$\sqrt{\frac{\pi}{2}\left(p + \frac{1}{2}\right)} < \chi(p) < \sqrt{\frac{\pi}{2}\left(p + \frac{2}{\pi}\right)},$$

for any $p > 0$.

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Error Bounds

For any non-negative integer N and for $|\arg z| < \frac{3\pi}{2}$, we have

$$K_0(z) = \sqrt{\frac{\pi}{2z}}\mathrm{e}^{-z}\left(\sum_{n=0}^{N-1}\frac{a_n}{z^n} + R_N(z)\right),$$

where the remainder $R_N(z)$ satisfies the estimates

$$|R_N(z)| \le \frac{|a_N|}{|z|^N} \times \begin{cases} 1 & \text{if } |\arg z| \le \frac{\pi}{2}, \\ \min\left(|\csc(\arg z)|, \chi\left(N+\frac{1}{2}\right)+1\right) & \text{if } \frac{\pi}{2} < |\arg z| \le \pi, \\ \frac{\sqrt{2\pi(N+\frac{1}{2})}}{|\cos(\arg z)|^{N+\frac{1}{2}}} + \chi\left(N+\frac{1}{2}\right)+1 & \text{if } \pi < |\arg z| < \frac{3\pi}{2}. \end{cases}$$

We may compare this result with that of F. W. J. Olver ($N \ge 1$):

$$|R_N(z)| \le \frac{|a_N|}{|z|^N} \times \begin{cases} 2\exp\left(\frac{1}{4|z|}\right) & \text{if } |\arg z| \le \frac{\pi}{2}, \\ 2\chi(N)\exp\left(\frac{\pi}{8|z|}\right) & \text{if } \frac{\pi}{2} < |\arg z| \le \pi, \\ \frac{4\chi(N)}{|\cos(\arg z)|^N}\exp\left(\frac{\pi}{4|z\cos(\arg z)|}\right) & \text{if } \pi < |\arg z| < \frac{3\pi}{2}. \end{cases}$$

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We have shown that for any non-negative integer N, it holds that

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provided $|\arg z| < \pi$. Re-iterating this formula once, we find that

$$\begin{aligned}
R_N(z) &= (-1)^N\frac{1}{z^N}\frac{1}{\pi}\sum_{\ell=0}^{L-1}a_\ell\int_0^{+\infty}\frac{s^{N-\ell-1}\mathrm{e}^{-2s}}{1+s/z}\mathrm{d}s \\
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&= (-1)^N\frac{1}{z^N}\frac{1}{\pi}\sum_{\ell=0}^{L-1}a_\ell\Gamma(N-\ell)\mathbf{F}^{(1)}\left(z;\ \begin{matrix}N-\ell\\ 2\end{matrix}\right) \\
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The numerically least term in Kummer's asymptotic expansion occurs when $N \approx 2|z|$. It can be shown that if L is fixed, $N \approx 2|z|$, and $z \to \infty$, then

$$R_N(z) = (-1)^N \frac{1}{z^N}\frac{1}{\pi}\sum_{\ell=0}^{L-1} a_\ell \Gamma(N-\ell)\mathbf{F}^{(1)}\left(z;\ \frac{N-\ell}{2}\right) + \begin{cases} \mathcal{O}\left(\dfrac{\mathrm{e}^{-2|z|}}{|z|^L}\right) & \text{if } |\arg z| \le \pi, \\ \mathcal{O}\left(\dfrac{\mathrm{e}^{2\operatorname{Re}(z)}}{|z|^L}\right) & \text{if } \pi \le |\arg z| \le \frac{5\pi}{2} - \delta \ (< \frac{5\pi}{2}). \end{cases}$$

This is called an ***exponentially improved asymptotic expansion***. This can be further approximated by

$$R_N(z) \approx 2\mathrm{i}\mathrm{e}^{2z} \sum_{\ell=0} (-1)^\ell \frac{a_\ell}{z^\ell} \times \tfrac{1}{2}\operatorname{erfc}\left((\pi - \arg z)\sqrt{|z|}\right)$$

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The Formula of Berry and Howls

In general, we have

$$\int_{\mathscr{C}^{(k)}(\theta)} \mathrm{e}^{-zf(t)} g(t)\mathrm{d}t = z^{-\frac{1}{2}} \mathrm{e}^{-zf(t^{(k)})} \left(\sum_{n=0}^{N-1} \frac{a_n^{(k)}}{z^n} + R_N^{(k)}(z) \right)$$

with

$$a_n^{(k)} \stackrel{\text{def}}{=} \frac{\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{2\pi}} \oint_{(t^{(k)}+)} \frac{g(t)}{\left(f(t)-f(t^{(k)})\right)^{n+\frac{1}{2}}} \mathrm{d}t$$

and

$$R_N^{(k)}(z) \stackrel{\text{def}}{=} \frac{1}{2\pi\mathrm{i}} \frac{1}{z^N} \sum_m \frac{(-1)^{\gamma_{km}}}{\mathrm{e}^{\left(N+\frac{1}{2}\right)\mathrm{i}\sigma_{km}}} \int_0^{+\infty} \frac{s^{N-\frac{1}{2}} \mathrm{e}^{-|\mathcal{F}_{km}|s}}{1-s/z\mathrm{e}^{\mathrm{i}\sigma_{km}}}$$
$$\times \int_{\mathscr{C}^{(m)}(-\sigma_{km})} \mathrm{e}^{-s\mathrm{e}^{-\mathrm{i}\sigma_{km}}\left(f(t)-f(t^{(m)})\right)} g(t)\mathrm{d}t\mathrm{d}s.$$

The so-called ***singulants*** are defined by

$$\mathcal{F}_{km} \stackrel{\text{def}}{=} f(t^{(m)}) - f(t^{(k)}), \quad \arg \mathcal{F}_{km} = \sigma_{km},$$

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The least term truncation is

$$N \approx |\mathcal{F}_{km^*}|\, |z|\,,$$

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Late Coefficients

As a consequence, we have

$$a_n^{(k)} \sim \frac{1}{2\pi \mathrm{i}} \sum_m (-1)^{\gamma_{km}} \sum_{\ell=0}^{\infty} \frac{a_\ell^{(m)}}{\mathcal{F}_{km}^{n-\ell}} \Gamma(n-\ell)$$

$$\sim \frac{1}{2\pi \mathrm{i}} \sum_m (-1)^{\gamma_{km}} \frac{\Gamma(n)}{\mathcal{F}_{km}^{n}} \left(a_0^{(m)} + \frac{\mathcal{F}_{km} a_1^{(m)}}{(n-1)} + \frac{\mathcal{F}_{km}^2 a_2^{(m)}}{(n-1)(n-2)} + \cdots \right)$$

as $n \to +\infty$. This formula links the ***late coefficients*** of an asymptotic expansion of the integral to the ***early coefficients*** of the asymptotic expansion of integrals over the adjacent saddles.

If there is a value m^* for which $|\mathcal{F}_{km^*}|$ is less than the corresponding quantities for all the other adjacent saddles, then at leading order

$$a_n^{(k)} \sim (-1)^{\gamma_{km^*}} \frac{g(t^{(m^*)})}{\sqrt{2\pi f''(t^{(m^*)})\mathrm{i}}} \frac{\Gamma(n)}{\mathcal{F}_{km^*}^{n}}$$

as $n \to +\infty$ (with the extra assumption that $g(t^{(m^*)}) \neq 0$).

Late Coefficients

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$$a_n^{(k)} \sim \frac{1}{2\pi \mathrm{i}} \sum_m (-1)^{\gamma_{km}} \sum_{\ell=0}^{\infty} \frac{a_\ell^{(m)}}{\mathcal{F}_{km}^{n-\ell}} \Gamma(n-\ell)$$

$$\sim \frac{1}{2\pi \mathrm{i}} \sum_m (-1)^{\gamma_{km}} \frac{\Gamma(n)}{\mathcal{F}_{km}^{n}} \left(a_0^{(m)} + \frac{\mathcal{F}_{km} a_1^{(m)}}{(n-1)} + \frac{\mathcal{F}_{km}^2 a_2^{(m)}}{(n-1)(n-2)} + \cdots \right)$$

as $n \to +\infty$. This formula links the ***late coefficients*** of an asymptotic expansion of the integral to the ***early coefficients*** of the asymptotic expansion of integrals over the adjacent saddles.

If there is a value m^* for which $|\mathcal{F}_{km^*}|$ is less than the corresponding quantities for all the other adjacent saddles, then at leading order

$$a_n^{(k)} \sim (-1)^{\gamma_{km^*}} \frac{g(t^{(m^*)})}{\sqrt{2\pi f''(t^{(m^*)})\mathrm{i}}} \frac{\Gamma(n)}{\mathcal{F}_{km^*}^{n}}$$

as $n \to +\infty$ (with the extra assumption that $g(t^{(m^*)}) \neq 0$).

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Concluding Remarks

- An exact formula for the remainder can be found for the asymptotic expansion of an integral with simple saddles.
- The exact remainder incorporates the Stokes phenomenon.
- The remainder can be estimated in a sharp and realistic manner.
- An exponentially improved expansion can be obtained for an integral with simple saddles.
- Late coefficients diverge like a “factorial divided by power”.
- The method extends to integrals with finite endpoints and with arbitrary order saddles.

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Thank you for your attention!