On the exact WKB analysis for difference equations

Yoshitsugu Takei
(Doshisha University)

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Plan of the talk

§1. Review of the exact WKB analysis for differential equations

§2. Stokes phenomena for the discrete Painlevé equation (alt-dP_{I})

§3. WKB analysis for the difference equation for Bessel functions

§4. Alternative approach to WKB analysis for difference equations

§5. Future problems

1 Exact WKB analysis for differential equations

(Voros, Pham-Delabaere, Aoki-Kawai-T.; 1980∼)

$$\left(\hbar^2 \frac{d^2}{dx^2} - Q(x)\right)\psi = 0 \quad \text{or} \quad \left(\frac{d^2}{dx^2} - \eta^2 Q(x)\right)\psi = 0$$

($\hbar$: Planck constant, $\eta = \hbar^{-1} > 0$: large parameter)

$$\psi_{\pm} = \exp(\pm\eta s(x)) \sum_{n=0}^{\infty} \eta^{-(n+1/2)} \psi_{\pm,n}(x) : \text{WKB solution}$$

(where $s(x) = \int^x \sqrt{Q(x)}\, dx$, $\psi_{\pm,n}(x)$ are recursively determined)

We give an analytic meaning to WKB solutions through the Borel-Laplace method w.r.t. the large parameter η:

Borel resummation

$$\psi_{\pm,B}(x,y) = \sum_{n=0}^{\infty} \frac{\psi_{\pm,n}(x)}{\Gamma(n+1/2)} (y \pm s(x))^{n-1/2}$$ **: Borel transform**

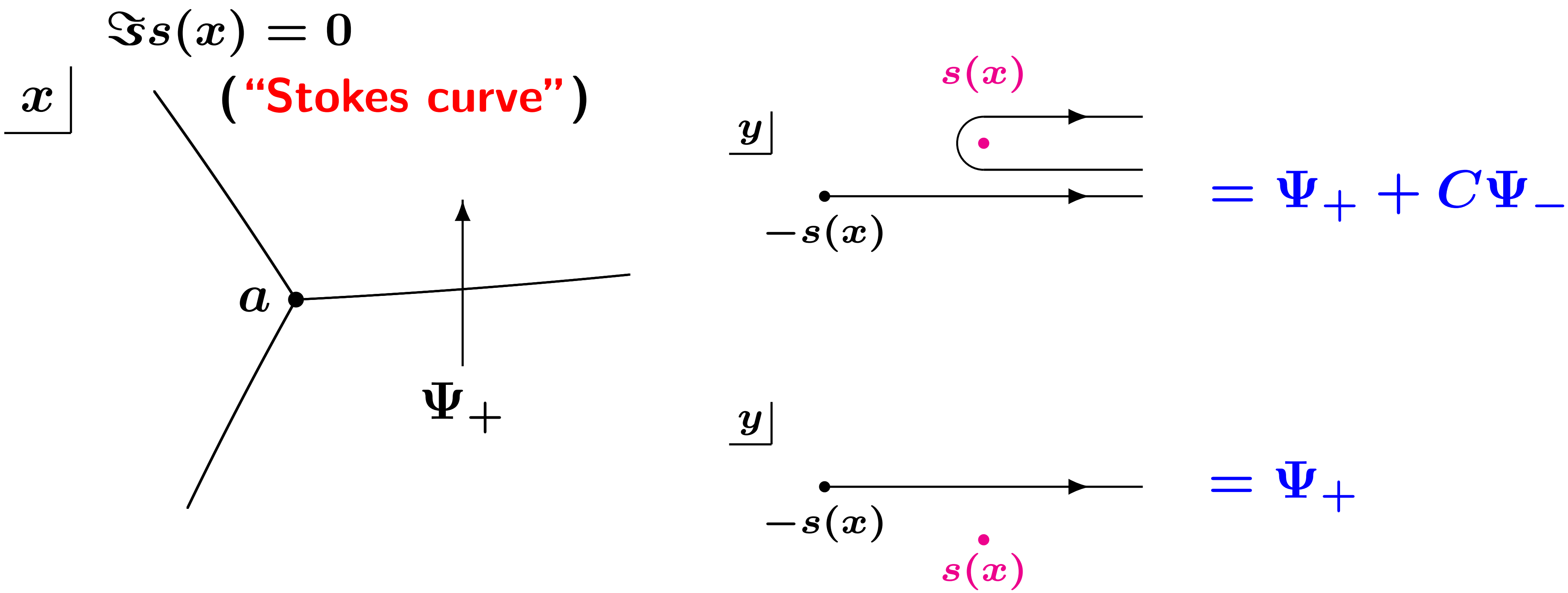
$$\Psi_{\pm}(x,\eta) = \int_{\mp s(x)}^{\infty} e^{-\eta y} \psi_{\pm,B}(x,y)\, dy$$ **: Borel sum**

Most important is the **Borel transform** $\psi_{\pm,B}(x,y)$:

- $\psi_{+,B}(x,y)$ has singularities at $y = \pm s(x) = \pm \int_a^x \sqrt{Q(x)}\,dx$.

(Here a is a zero of $Q(x)$, i.e., **"turning point"**.)

- **Singularities of the Borel transform induce Stokes phenomena.**



Thia enables us to analyze the global behavior of solutions.

2 Stokes phenomena for (alt-dP_{I})

(joint work with N. Joshi, 2018)

$$(\text{alt-}dP_{\mathrm{I}}) \qquad \frac{c+\eta^{-1}/2}{\overline{\lambda}+\lambda} + \frac{c-\eta^{-1}/2}{\underline{\lambda}+\lambda} + 2\lambda^2 + t = 0,$$

where $\overline{\lambda}$ **(or** $\underline{\lambda}$ **)** $= \lambda\Big|_{c \mapsto c \pm \eta^{-1}}.$

▶ **Appears through the Bäcklund transformation of**

$$(P_{\mathrm{II}}) \qquad \eta^{-2}\frac{d^2\lambda}{dt^2} = 2\lambda^3 + t\lambda + c.$$

▶ **Describes the compatibility condition of the following associated system (L_{II}) of linear differential-difference equations.**

$$
(L_{\mathrm{II}}) \qquad \begin{cases} \left(\eta^{-2}\dfrac{\partial^2}{\partial x^2} - Q_{\mathrm{II}}\right)\psi = 0, \\ \eta^{-1}\dfrac{\partial \psi}{\partial t} = A_{\mathrm{II}}\,\eta^{-1}\dfrac{\partial \psi}{\partial x} - \dfrac{\eta^{-1}}{2}\dfrac{\partial A_{\mathrm{II}}}{\partial x}\psi, \\ \overline{\psi} = g_{\mathrm{II}}\,\eta^{-1}\dfrac{\partial \psi}{\partial x} + f_{\mathrm{II}}\,\psi, \end{cases}
$$

where

$$
\begin{aligned}
Q_{\mathrm{II}} &= x^4 + tx^2 + 2cx + \nu^2 - (\lambda^4 + t\lambda^2 + 2c\lambda) \\
&\quad - \eta^{-1}\frac{\nu}{x-\lambda} + \eta^{-2}\frac{3}{4(x-\lambda)^2}, \quad A_{\mathrm{II}} = \frac{1}{2(x-\lambda)}, \\
g_{\mathrm{II}} &= \Big[(2\nu + 2\lambda^2 + t)(x-\lambda)(x-\overline{\lambda})\Big]^{-1/2}, \\
f_{\mathrm{II}} &= g_{\mathrm{II}}\left[x^2 - \lambda^2 - \nu + \eta^{-1}\frac{1}{2(x-\lambda)}\right].
\end{aligned}
$$

Transseries solution of (alt-dP_{I})

$$\lambda = \lambda^{(0)} + \eta^{-1/2}\alpha\,\lambda^{(1)} + (\eta^{-1/2}\alpha)^2\,\lambda^{(2)} + \cdots\,,$$

where

$$\lambda^{(0)} = \lambda_0 + \eta^{-1}\lambda_1 + \cdots\,, \quad 2\lambda_0^3 + t\lambda_0 + c = 0,$$

$$\lambda^{(1)} = \exp\int^c \omega\,dc, \quad \omega = \eta\omega_{-1} + \omega_0 + \eta^{-1}\omega_1 + \cdots\,,$$

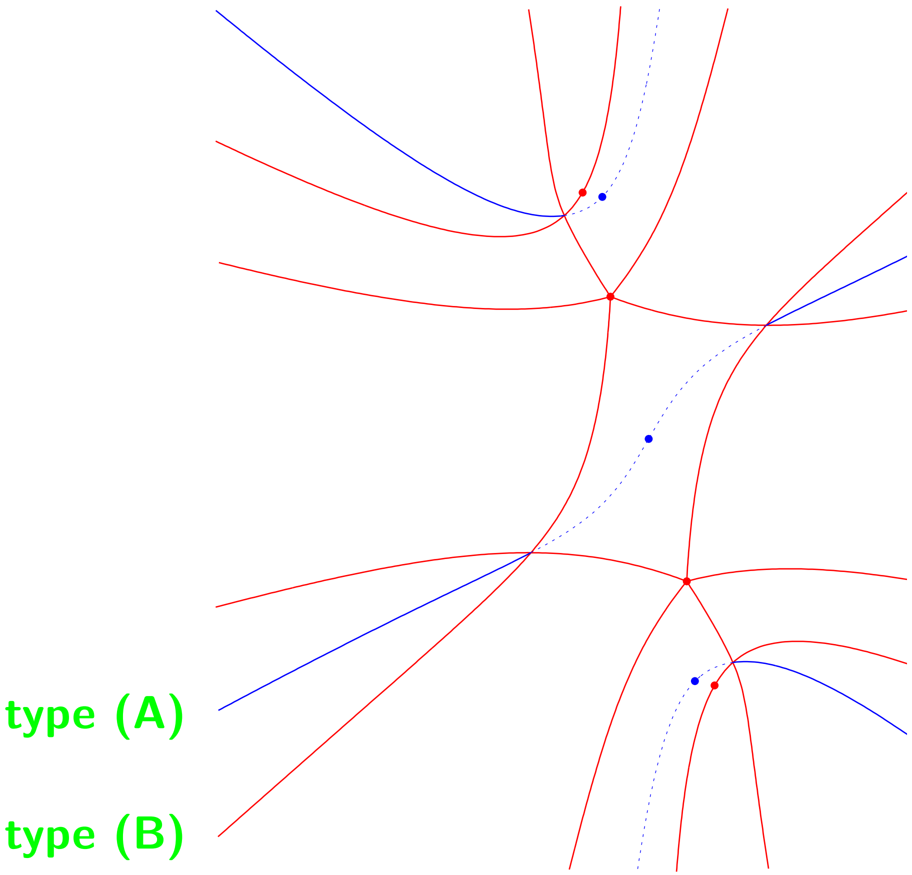
$$\omega_{-1} = \cosh^{-1}\left(\frac{8\lambda_0^3 - c}{c}\right),$$

$$\alpha = \sum_{l=0}^{\infty} a_l e^{2\pi i l \eta c} \quad (a_l\text{: free parameters})$$

Applying the results of §1 to (L_{II}) and computing its Stokes multipliers, we obtain the following connection formulas:

type (A) : $\quad \widetilde{\alpha} = \alpha(1 + e^{2\pi i \eta c}),$

type (B) : $\quad \widetilde{\alpha} = \alpha + \dfrac{i}{2\sqrt{\pi}} e^{2\pi i \eta c}.$



(Stokes curves of (alt-dP_{I}))

We want to generalize this result to other discrete Painlevé eq'ns.

3 WKB analysis for the difference equation for Bessel functions

(joint work with Shun Ito, 2022)

Bessel functions

$$J_a(x) = \left(\frac{x}{2}\right)^a \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+a+1/2)n!} \left(\frac{x}{2}\right)^{2n}$$

- $x^2\psi'' + x\psi' + (x^2 - a^2)\psi = 0$
- **Contiguity relation**

$$\left(-\frac{\partial}{\partial x} + \frac{a}{x}\right)\psi = \psi\Big|_{a\mapsto a+1} = \sigma_a^1\psi$$

$$\Longrightarrow \quad \sigma_a^2\psi - \frac{2(a+1)}{x}\sigma_a^1\psi + \psi = 0$$

Introduce a large parameter η through

$$x \mapsto \eta x, \quad a+1 \mapsto \eta\gamma, \quad \sigma_\gamma f(\gamma) = f(\gamma + \eta^{-1})$$

$$\Longrightarrow \begin{cases} \left(x^2 \dfrac{\partial^2}{\partial x^2} + x\dfrac{\partial}{\partial x} + \eta^2 (x^2 - (\gamma - \eta^{-1})^2)\right)\psi = 0 \\ \left(\sigma_\gamma^2 - \dfrac{2\gamma}{x}\sigma_\gamma^1 + 1\right)\psi = 0 \qquad (\Delta) \end{cases}$$

Want to discuss the 2nd equation (Δ) mainly.

Basic idea

View σ_γ as an ∞-order differential operator via

$$\sigma_\gamma \psi = \psi(\gamma + \eta^{-1}) = \sum_{n=0}^{\infty} \frac{\eta^{-n}}{n!}\left(\frac{d}{d\gamma}\right)^n \psi(\gamma)$$

WKB solution

$$\psi = \exp\left(\eta \int^{\gamma} \phi(\gamma)\, d\gamma\right) \sum_{n=0}^{\infty} \psi_n(\gamma)\eta^{-n}$$

where $e^{2\phi} - \frac{2\gamma}{x} e^{\phi} + 1 = 0$, that is,

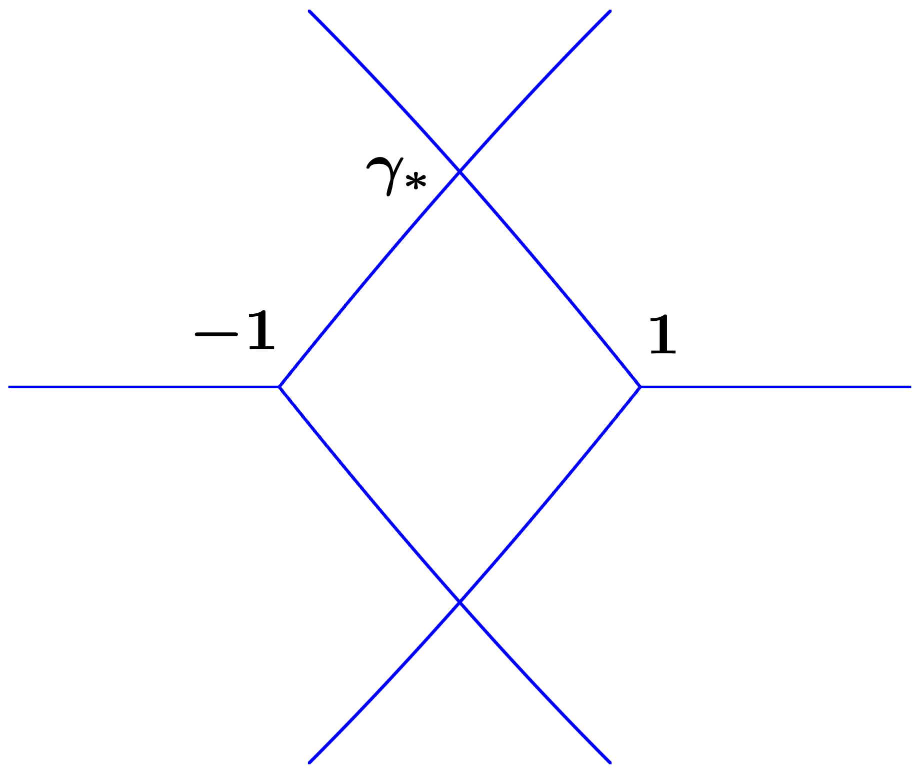
$$\phi = \phi_{\pm}(\gamma) = \log\left(\frac{\gamma}{x} \pm \sqrt{\left(\frac{\gamma}{x}\right)^2 - 1}\right) = \cosh^{-1}\left(\frac{\gamma}{x}\right)$$

Turning point & Stokes curve

turning point $\iff$ $\left(\frac{\gamma}{x}\right)^2 - 1 = 0 \iff \gamma = \pm x$

Stokes curve $\iff$ $\Im \int_{\pm x}^{\gamma} (\phi_+(\gamma) - \phi_-(\gamma)) d\gamma = 0$

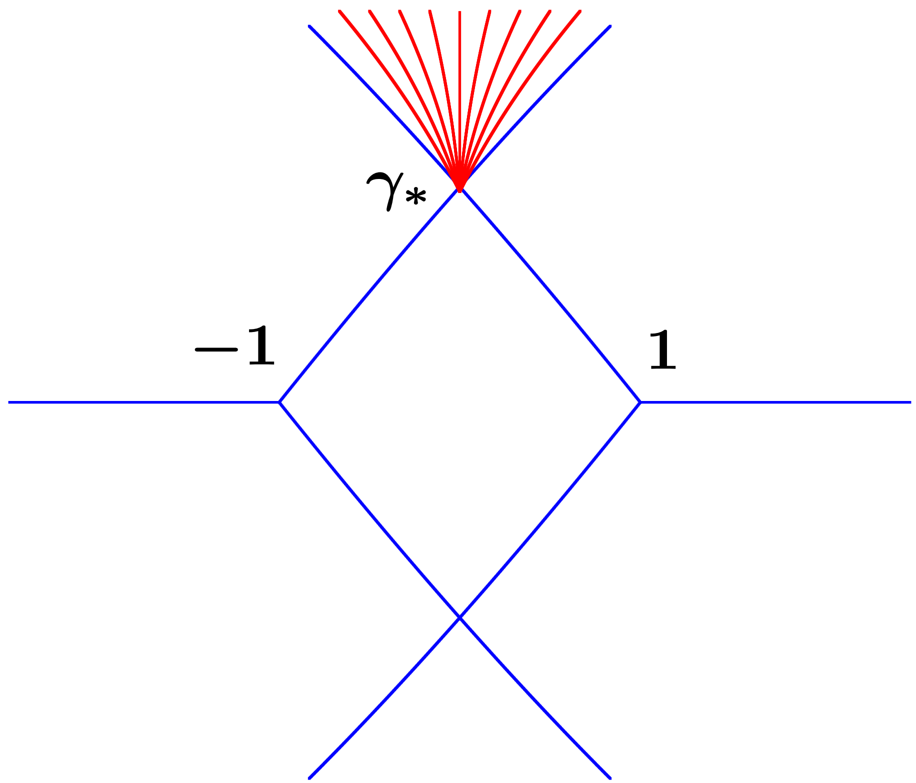
Stokes curves of (Δ) for $x = 1$



Cf. **BNR equation :** $\psi''' + 3\eta^2\psi' + 2ix\eta^3\psi = 0.$

Stokes curves are similar and a new Stokes curve appears from a crossing point of Stokes curves.

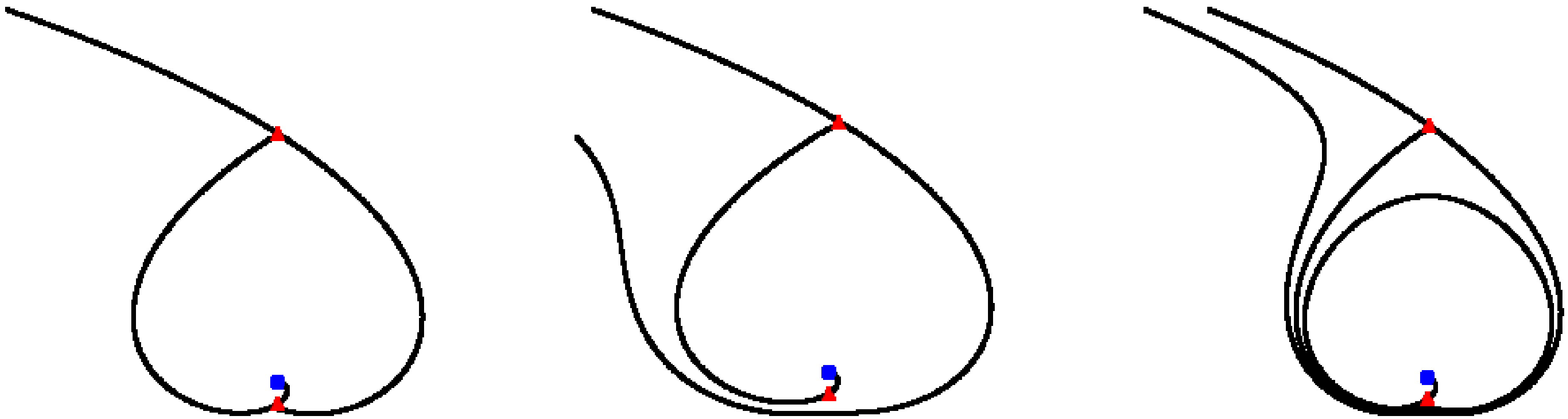
Prop. 1 ∞ number of new Stokes curves appear for (Δ).



This proposition is verified by using

$$\psi = \int \exp(\eta f(\gamma, u))\, du, \quad f(\gamma, u) = \frac{x}{2}\Big(u - \frac{1}{u}\Big) - \gamma \log u$$

- **saddle point :** $u = u_{\pm} = \dfrac{\gamma}{x} \pm \sqrt{\left(\dfrac{\gamma}{x}\right)^2 - 1}$
- γ **lies on a (new) Stokes curve if and only if saddle points** $u_{\pm}$ **are connected by a steepest descent path.**



4 Alternative approach to WKB analysis for difference equations

(work in progress with Yumiko Takei)

Idea **Use of the Laplace transform.**

Eample 1 (Hermite-Weber)

(1)
$$\begin{cases} \Big((\eta^{-1}\partial_x)^2 - x(\eta^{-1}\partial_x) + \lambda\Big)\psi = 0, \\ \sigma_\lambda\big(\lambda\psi\big) = \big(-\eta^{-1}\partial_x + x\big)\psi, \end{cases}$$

where $\sigma_\lambda \psi = \psi\Big|_{\lambda \mapsto \lambda + \eta^{-1}}$.

Apply the Laplace transform w.r.t. λ :

$$\psi = \int e^{\eta\lambda\mu}\widehat{\psi}\,d\mu, \quad \widehat{\psi} = \widehat{\psi}(x,\mu).$$

Then

$$\lambda \longleftrightarrow -\eta^{-1}\partial_\mu, \quad \eta^{-1}\partial_\lambda \longleftrightarrow \mu, \quad \sigma_\lambda \longleftrightarrow e^\mu,$$

and (1) is transformed to

$$(2) \qquad \begin{cases} \Big((\eta^{-1}\partial_x)^2 - x(\eta^{-1}\partial_x) - (\eta^{-1}\partial_\mu)\Big)\widehat{\psi} = 0, \\ \Big((\eta^{-1}\partial_x) - x - e^\mu(\eta^{-1}\partial_\mu)\Big)\widehat{\psi} = 0. \end{cases}$$

WKB solution

$$\widehat{\psi} = \exp\big(\eta\widehat{s}(x,\mu)\big)\sum_{n=0}^{\infty}\widehat{\psi}_n(x,\mu)\eta^{-n}$$

Prop. 2 $\exists\, \widehat{\psi} = \exp \eta \left(-\frac{1}{2}e^{-2\mu} + xe^{-\mu}\right)$

: **WKB solution of** (2) **consisting of finite terms.**

Corollary **By using a change of variables** $e^{-\mu} = t$**, we obtain**

$$\psi = \int e^{\eta\lambda\mu} e^{\eta(-\frac{1}{2}e^{-2\mu}+xe^{-\mu})}\, d\mu$$

$$= \int e^{\eta(-\frac{1}{2}t^2+xt-\lambda\log t)} \left(-\frac{dt}{t}\right)$$

This is the well-known integral representation for Hermite-Weber functions.

Eample 2 (Kummer)

$$F(a,c;x) = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} \frac{x^n}{n!}$$

$$= \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 e^{xt} t^{a-1} (1-t)^{c-a-1}\, dt$$

: Kummer's confluent hypergeometric functions

$$x \mapsto \eta x, \quad a \mapsto \eta\gamma, \quad c \mapsto \eta\gamma,$$

$$\lambda_1 = \alpha, \quad \lambda_2 = \gamma - \alpha, \quad \sigma_j f = \sigma_{\lambda_j} f = f(\lambda_j + \eta^{-1})$$

$$\Longrightarrow \begin{cases} \Big(x(\eta^{-1}\partial_x)^2 + (\lambda_1 + \lambda_2 - x)(\eta^{-1}\partial_x) - \lambda_1\Big)\psi = 0, \\ \Big(x(\eta^{-1}\partial_x)\sigma_2 + \lambda_1\sigma_2 - \lambda_1\sigma_1\Big)\psi = 0, \\ \Big((\lambda_1 + \lambda_2)(\eta^{-1}\partial_x) - (\lambda_1 + \lambda_2) + \lambda_2\sigma_2\Big)\psi = 0. \end{cases}$$

Apply the 2-dimensional Laplace transform :

$$\psi = \iint e^{\eta(\lambda_1 \mu_1 + \lambda_2 \mu_2)} \widehat{\psi}\, d\mu_1 d\mu_2.$$

Then we find there exists a WKB solution of the Laplace transformed system consisting of finite terms, and obtain

$$\psi = \iint e^{-\eta(\lambda_1 \log t_1 + \lambda_2 \log t_2) + \eta x \frac{1-t_2}{t_1 - t_2}}\, x \frac{dt_1 dt_2}{(t_1 - t_2)^2}$$

(where we used a change of variables $e^{-\mu_j} = t_j$).

Furthermore, we make a change of unknown functions

$$\psi = \frac{\Gamma(\eta\gamma)}{\Gamma(\eta\alpha)\Gamma(\eta(\gamma - \alpha))} \varphi.$$

$$\Longrightarrow \begin{cases} \Big(x(\eta^{-1}\partial_x)^2 + (\lambda_1+\lambda_2-x)(\eta^{-1}\partial_x) - \lambda_1\Big)\varphi = 0, \\ \Big(x(\eta^{-1}\partial_x)\sigma_2 + \lambda_1\sigma_2 - \lambda_2\sigma_1\Big)\varphi = 0, \\ \Big((\eta^{-1}\partial_x) + \sigma_2 - 1\Big)\varphi = 0. \end{cases}$$

Finally we apply the Laplace transform, then we obtain

$$\widehat{\varphi} = e^{\eta x w_1}\delta_{\{w_1+w_2=1\}}$$

and

$$\varphi = \int e^{\eta(\lambda_1 \log t + \lambda_2 \log(1-t) + xt)}\,\frac{dt}{t(1-t)},$$

$$(e^{\mu_j} = w_j,\ t = w_1)$$

The final formula is the well-known integral representation for Kummer's confluent hypergeometric functions.

5 Future problems

▶ **To develop the exact WKB analysis for difference equations via the Laplace transform.**

($\sim$ exact WKB by using integral representations of solutions)

▶ **To clarify the relationship with the "exact steepest descent method".**

▶ **To analyze the Voros coefficients through difference equations.**

▶ **To generalize the results for discrete Painlevé equations.**