Voros coefficients and the topological recursion for the hypergeometric differential equations

Yumiko Takei

National Institute of Technology(KOSEN), Ibaraki College

Frontiers in Nonlinear Differential Equations and Stokes Phenomena
OIST
12th September, 2023

Plan of this talk

Theorem (Iwaki - Koike - T)

For the family of the Gauss hypergeometric differential equations, we obtain the following results :

(1) They are realized as quantum curves for appropriate spectral curves.

(2) Voros coefficients are expressed as the difference values of the generating function of free energies with respect to parameters.

(3) The explicit forms of Voros coefficients and free energies are obtained.

<u>Remark:</u> The confluence diagram for the family of the Gauss hypergeometric differential equations is as follows:

$$\begin{tikzcd}
 & & \begin{matrix}\text{Weber}\\(1,3)\end{matrix} \arrow[dr] & \\
\begin{matrix}\text{Gauss}\\(1,1,1,1)\end{matrix} \arrow[r] & \begin{matrix}\text{Kummer}\\(1,1,2)\end{matrix} \arrow[ur] \arrow[dr] & & \begin{matrix}\text{Airy}\\(4)\end{matrix} \\
 & & \begin{matrix}\text{Bessel}\\(2,2)\end{matrix} \arrow[ur] &
\end{tikzcd}$$

Purpose

Purpose of this talk

To generalize these results to confluent hypergeometric differential equations associated with the 2-dimensional Garnier system.

Remark: The confluence diagram for the family of the hypergeometric differential systems associated with the 2-dimensional Garnier system is as follows ([OK]) :

$$
\begin{tikzcd}
 & & (1,1,3) \arrow[r] \arrow[ddr] & (1,4) \arrow[dr] & \\
(1,1,1,1,1) \arrow[r] & (1,1,1,2) \arrow[ur] \arrow[dr] & & & (5) \\
 & & (1,2,2) \arrow[r] \arrow[uur] & (2,3) \arrow[ur] &
\end{tikzcd}
$$

Purpose

Purpose of this talk

To generalize these results to confluent hypergeometric differential equations associated with the 2-dimensional Garnier system.

Remark: The confluence diagram for the family of the hypergeometric differential systems associated with the 2-dimensional Garnier system is as follows ([OK]) :

$$\begin{tikzcd}
 & & (1,1,3) \arrow[r] \arrow[dr] & (1,4) \arrow[dr] & \\
(1,1,1,1,1) \arrow[r] & (1,1,1,2) \arrow[ur] \arrow[dr] & & & (5) \\
 & & (1,2,2) \arrow[r] \arrow[ur] & (2,3) \arrow[ur] &
\end{tikzcd}$$

(1,2,2) equation

The (1,2,2) hypergeometric system with a small parameter $\hbar > 0$:

$$\left\{x_1\hbar^3\frac{\partial^3}{\partial {x_1}^3} - (x_1-\lambda_0-\lambda_1)\,\hbar^2\frac{\partial^2}{\partial {x_1}^2} - (x_2+\lambda_0)\,\hbar\frac{\partial}{\partial x_1} + x_2\right\}\psi = 0, \quad (1)$$

$$\left\{\hbar^2\frac{\partial^2}{\partial x_1\partial x_2} - 2\hbar\frac{\partial}{\partial x_1} + 1\right\}\psi = 0.$$

In what follows, setting $x_1 = x$ and $x_2 = t$ (fixed), we consider

$$\left\{x\hbar^3\frac{d^3}{dx^3} - (x-\lambda_0-\lambda_1)\,\hbar^2\frac{d^2}{dx^2} - (t+\lambda_0)\,\hbar\frac{d}{dx} + t\right\}\psi = 0. \qquad (2)$$

We call (2) the (1,2,2) equation.

Outline of our method

$$y^2 - \left(\frac{x^2}{4} - \lambda\right) = 0 \quad \xrightarrow[\text{([EO1], [CEO])}]{\text{Topological recursion}}$$

For $g \geq 0$, $n \geq 1$

- $W_{g,n}(z_1, \ldots, z_n)$
  : correlation function
- $F_g \in \mathbb{C}$
  : g-th free energy

Outline of our method

$$y^2 - \left(\frac{x^2}{4} - \lambda\right) = 0$$

Topological recursion ([EO1], [CEO]) $\longrightarrow$

For $g \geq 0$, $n \geq 1$

- $W_{g,n}(z_1, \ldots, z_n)$: correlation function
- $F_g \in \mathbb{C}$: g-th free energy

Quantization ([EO1], [DM], [BE]) $\downarrow$

$$\left[\hbar^2 \frac{d^2}{dx^2} - \left(\frac{x^2}{4} - \lambda + \frac{\nu}{2}\hbar\right)\right] \psi(x, \hbar) = 0$$

$$\psi(x, \hbar) = \exp\left[\sum_{g \geq 0,\, n \geq 1} \frac{\hbar^{2g+n-2}}{n!} \int_D \cdots \int_D W_{g,n}(z_1, \ldots, z_n)\right]\Bigg|_{z=z(x)}$$

Outline of our method

$$y^2 - \left(\frac{x^2}{4} - \lambda\right) = 0 \xrightarrow[\text{([EO1], [CEO])}]{\text{Topological recursion}}$$

For $g \geq 0,\ n \geq 1$

- $W_{g,n}(z_1, \ldots, z_n)$
  : correlation function
- $F_g \in \mathbb{C}$
  : g-th free energy

Quantization ([EO1], [DM], [BE]) $\downarrow\uparrow$ Classical limit

$$\left[\hbar^2 \frac{d^2}{dx^2} - \left(\frac{x^2}{4} - \lambda + \frac{\nu}{2}\hbar\right)\right]\psi(x,\hbar) \;=\; 0$$

$$\psi(x,\hbar) \;=\; \exp\left[\sum_{g\geq 0,\ n\geq 1} \frac{\hbar^{2g+n-2}}{n!} \int_D \cdots \int_D W_{g,n}(z_1, \ldots, z_n)\right]\Bigg|_{z=z(x)}$$

Exact WKB analysis, I

Let us consider the following differential equation with a small parameter $\hbar$

$$P\left(x, \hbar\frac{d}{dx}\right)\psi = \left[p_0(x)\hbar^3\frac{d^3}{dx^3} + p_1(x)\hbar^2\frac{d^2}{dx^2} + p_2(x)\hbar\frac{d}{dx} + p_3(x)\right]\psi = 0 \tag{3}$$

and its WKB solutions

$$\psi(x,\hbar) = \exp\left(\int^x S(x,\hbar)\,dx\right), \tag{4}$$

where

$$S(x,\hbar) = \hbar^{-1}S_{-1}(x) + S_0(x) + \hbar S_1(x) + \cdots = \sum_{j \geqq -1} \hbar^j S_j(x) \tag{5}$$

is a solution of

$$\begin{aligned} &p_0(x)\hbar^3\left(\frac{d^2}{dx^2}S(x,\hbar) + 3S(x,\hbar)\frac{d}{dx}S(x,\hbar) + S(x,\hbar)^3\right) \\ &\quad + p_1(x)\hbar^2\left(\frac{d}{dx}S(x,\hbar) + S(x,\hbar)^2\right) + \hbar p_2(x)S(x,\hbar) + p_3(x) = 0. \end{aligned} \tag{6}$$

Exact WKB analysis, II

By substituting (5) into (6) and comparing like powers of both sides with respect to $\hbar$, we obtain

$$p_0(x)S_{-1}^3 + p_1(x)S_{-1}^2 + p_2(x)S_{-1} + p_3(x) = 0 \tag{7}$$

and

$$\begin{aligned}\left(3p_0(x){S_{-1}}^2 + 2p_1(x)S_{-1} + p_2(x)\right) S_{m+1} + \sum_{\substack{i+j+k=m-1\\ i,j,k\geq 0}} S_iS_jS_k + 3\sum_{j=0}^{m-1} S_{m-j-1}S_j \\ +3p_0(x)S_m\frac{dS_{-1}}{dx} + 3p_0(x)S_{-1}\frac{dS_m}{dx} + p_0(x)\frac{d^2S_{m-1}}{dx^2} + p_1(x)\sum_{j=0}^{m} S_{m-j}S_j \\ +p_1(x)\frac{dS_m}{dx} = 0 \quad (m \geq -1).\end{aligned} \tag{8}$$

Eq. (7) has three solutions, and once we fix one of them, we can determine S_m for $m \geq 0$ uniquely and recursively by (8).

Voros coefficients

Then, the Voros coefficient is defined by

$$\begin{aligned} V \text{``}=\text{''} & \int_\gamma S(x,\hbar)dx \\ &= \int_\gamma (S(x,\hbar) - \hbar^{-1}S_{-1}(x) - S_0(x))dx = \sum_{m=1}^{\infty} \hbar^m \int_\gamma S_m(x)dx, \end{aligned} \tag{9}$$

where γ is a path from a singular point to a singular point.

Asymptotic behaviors of characteristic roots

$$S_{-1}(x) = \begin{cases} \dfrac{\sqrt{t}}{x^{1/2}} - \dfrac{\lambda_0}{2x} - \dfrac{{\lambda_0}^2 + 4\lambda_1 t}{8\sqrt{t}} x^{-3/2} + O(x^{-2}) \quad (x \to \infty_0), \\ 1 - \dfrac{\lambda_1}{x} + O(x^{-2}) \quad (x \to \infty_1), \\ -\dfrac{\sqrt{t}}{x^{1/2}} - \dfrac{\lambda_0}{2x} + \dfrac{{\lambda_0}^2 + 4\lambda_1 t}{8\sqrt{t}} x^{-3/2} + O(x^{-2}) \quad (x \to \infty_2). \end{cases} \tag{10}$$

$$S_{-1}(x) = \begin{cases} \dfrac{\lambda_0 + t - \sqrt{\lambda_0^2 + t^2 - 2\lambda_0 t - 4\lambda_1 t}}{2\,(\lambda_0 + \lambda_1)} + O(x) \quad (x \to 0_0), \\ \dfrac{\lambda_0 + t + \sqrt{\lambda_0^2 + t^2 - 2\lambda_0 t - 4\lambda_1 t}}{2\,(\lambda_0 + \lambda_1)} + O(x) \quad (x \to 0_1), \\ -\dfrac{\lambda_0 + \lambda_1}{x} + \dfrac{\lambda_1 - t}{\lambda_0 + \lambda_1} + O(x) \quad (x \to 0_2). \end{cases} \tag{11}$$

Spectral curve

$$\left\{x\hbar^3\frac{d^3}{dx^3} - (x-\lambda_0-\lambda_1)\,\hbar^2\frac{d^2}{dx^2} - (t+\lambda_0)\,\hbar\frac{d}{dx} + t\right\}\psi = 0. \quad : \text{ The (1,2,2) eq.}$$

Spectral curve

$$\left\{ x\hbar^3 \frac{d^3}{dx^3} - (x-\lambda_0-\lambda_1)\,\hbar^2\frac{d^2}{dx^2} - (t+\lambda_0)\,\hbar\frac{d}{dx} + t \right\}\psi = 0. \quad : \text{ The (1,2,2) eq.}$$

$$\Big\downarrow \; \hbar\frac{d}{dx} \to y,\ \hbar \to 0$$

$$P(x,y) = xy^3 - (x-\lambda_0-\lambda_1)\,y^2 - (t+\lambda_0)\,y + t = 0. \tag{12}$$

Spectral curve

$$\left\{ x\hbar^3 \frac{d^3}{dx^3} - (x - \lambda_0 - \lambda_1)\,\hbar^2 \frac{d^2}{dx^2} - (t + \lambda_0)\,\hbar \frac{d}{dx} + t \right\} \psi = 0. \quad : \text{ The (1,2,2) eq.}$$

Let us consider the following algebraic curve

$$P(x, y) = xy^3 - (x - \lambda_0 - \lambda_1)\, y^2 - (t + \lambda_0)\, y + t = 0. \tag{12}$$

For $z \in \mathbb{P}^1$ we choose

$$\begin{cases} x = x(z) = \dfrac{-(\lambda_0 + \lambda_1) z^2 + (t + \lambda_0) z - t}{z^2 (z - 1)}, \\ y = y(z) = z. \end{cases} \tag{13}$$

We call a pair $(x(z), y(z))$ a spectral curve and (13) the (1,2,2) curve.

Topological recursion (cf. [EO1])

Let $(x(z), y(z))$ be a spectral curve. We first define

$$W_{0,1}(z) = y(z)\frac{dx}{dz}(z)dz, \quad W_{0,2}(z_1, z_2) = B(z_1, z_2) = \frac{dz_1\, dz_2}{(z_1 - z_2)^2}.$$

For $g \geqq 0$, $n \geqq 0$ and $2g - 2 + n \geqq 0$, we construct meromorphic differentials $W_{g,n}(z_1, \ldots, z_n)$ on $(\mathbb{P}^1)^n$ by the following recursive formulas.

$$W_{g,n+1}(z_0, z_1, \ldots, z_n) = \sum_{a\,:\ \text{branch point}} \operatorname*{Res}_{z=a} \frac{\left(\frac{1}{z_0 - z}\right) dz_0}{(y(z) - y(\bar{z}))\, dx(z)}$$

$$\times \left\{ W_{g-1,n+2}(z, \bar{z}, z_1, \ldots, z_n) + \sum_{\substack{g_1+g_2=g \\ I \sqcup J = \{1,2,\ldots,n\}}}' W_{g_1,1+|I|}(z, z_I) W_{g_2,1+|J|}(\bar{z}, z_J) \right\}.$$

- Branch points are zeros of $dx(z)$ (assume that all branch points are simple);
- $\bar{z}$ is a local conjugate point of z near a branch point (i.e. $x(\bar{z}) = x(z)$);
- $z_I = (z_{i_1}, \ldots, z_{i_r})$ for $I = \{i_1, \ldots, i_r\} \subset \{1, \ldots, n\}$.

Free energy

We define $F_g = W_{g,0}$, called free energies, by the following ([EO1], [CEO]):

$$F_g = \frac{1}{2-2g} \sum_{a\,:\ \text{branch point}} \operatorname*{Res}_{z=a} \Phi(z) W_{g,1}(z) \quad (g \geqq 2),$$

where $\Phi(z)$ is any function satisfying $\frac{d\Phi}{dz}(z) = y(z)\frac{dx}{dz}(z)$.

Remark: The free energies F_0 and F_1 for $g = 0$ and 1 are also defined, but in a different manner. For the (1,2,2) curve F_0 is given by

$$F_0 = \frac{1}{4}\left\{3{\lambda_0}^2 + 6\lambda_0\lambda_1 + 4\lambda_1 t + {\lambda_0}^2 \log t + 2{\lambda_1}^2 \log \lambda_1 \right.$$
$$\left. -2(\lambda_0+\lambda_1)^2 \log(\lambda_0+\lambda_1) - 2\lambda_0(\lambda_0+\lambda_1)\log(-1)\right\}.$$

Variational formula (cf. [EO2])

From the variational formula, $W_{g,n}(z_1,\ldots,z_n)$ and F_g satisfy

$$\frac{\partial W_{g,n}}{\partial \lambda_j}(z_1,\ldots,z_n) = \int_{\zeta\in\gamma_j} W_{g,n+1}(z_1,\ldots,z_n,\zeta) \quad (2g+n\geq 2), \tag{14}$$

$$\frac{\partial F_g}{\partial \lambda_j} = \int_{\zeta\in\gamma_j} W_{g,1}(\zeta) \quad (g\geq 1), \tag{15}$$

$$\frac{\partial F_g}{\partial t} = -\operatorname*{Res}_{z=0} \frac{1}{z} W_{g,1}(z) \quad (g\geq 1). \tag{16}$$

Here, γ_j ($j=0,1$) is a path from $z=\infty$ to $z=j$.

x	0_2	$\infty_0,\ \infty_2$	∞_1
z	∞	0	1

Table: Correspondence of points for the (1,2,2) curve

Quantization ([BE])

We define

$$\psi(x,\hbar) = \exp\left[\hbar^{-1}\int_D W_{0,1}(z) + \frac{1}{2!}\int_D\int_D \left\{W_{0,2}(z_1,z_2) - \frac{dx(z_1)\,dx(z_2)}{(x(z_1)-x(z_2))^2}\right\}\right.$$
$$\left.\left. + \sum_{m=1}^{\infty}\hbar^m\left\{\sum_{\substack{2g+n-2=m\\ g\geq 0,\, n\geq 1}} \frac{1}{n!}\int_D\cdots\int_D W_{g,n}(z_1,\ldots,z_n)\right\}\right]\right|_{z=z(x)}, \qquad (17)$$

where $z = z(x)$ is an inverse function of $x = x(z)$ and

$$\int_D = \nu_0\int_0^z + \nu_1\int_1^z .$$

Here ν_j $(j = 0,1)$ satisfy $\nu_0 + \nu_1 = 1$. Then, $\psi(x,\hbar)$ is a WKB solution of

$$\left\{x\hbar^3\frac{d^3}{dx^3} - \left(x - \tilde{\lambda}_0 - \tilde{\lambda}_1 - 3\hbar\right)\hbar^2\frac{d^2}{dx^2} - \left(t + \tilde{\lambda}_0 + 2\hbar\right)\hbar\frac{d}{dx} + t\right\}\psi = 0, \qquad (18)$$

where $\tilde{\lambda}_j = \lambda_j - \nu_j\hbar$ $(j = 0,1)$. We call (18) the quantum (1,2,2) curve.

1 Introduction

2 Exact WKB analysis

3 Topological recursion

4 Main results

Theorem 1

Let $F_g(\lambda_0, \lambda_1, t)$ be free energies for the spectral (1,2,2) curve and

$$F(\underline{\lambda}, t; \hbar) = F(\lambda_0, \lambda_1, t; \hbar) = \sum_{g=0}^{\infty} F_g(\lambda_0, \lambda_1, t)\hbar^{2g-2}$$

be the generating function of $F_g(\lambda_0, \lambda_1, t)$. Then, we obtain

$$V^{(0,\infty)} = F\left(\tilde{\lambda}_0 + \hbar, \tilde{\lambda}_1, t; \hbar\right) - F\left(\tilde{\lambda}_0, \tilde{\lambda}_1, t; \hbar\right) - \frac{\partial F_0}{\partial \lambda_0}\hbar^{-1} + \nu_1 \frac{\partial^2 F_0}{\partial \lambda_0 \partial \lambda_1} + \frac{2\nu_0 - 1}{2}\frac{\partial^2 F_0}{\partial {\lambda_0}^2},$$

where $V^{(j,\infty)}$ are Voros coefficients for the quantum (1,2,2) curve whose path is from 0_2 to ∞_j $(j = 0, 1)$.

x	0_2	∞_0, ∞_2	∞_1
z	∞	0	1

Another Voros coefficient $V^{(1,\infty)}$ can be expressed similarly.

Sketch of Proof of Theorem 1

We can express $V^{(0,\infty)}$ in terms of $W_{g,n}$ as follows:

$$V^{(0,\infty)} = \sum_{m=1}^{\infty} \hbar^m \int_{\infty}^{0} \left\{ \sum_{2g+n-2=m} \frac{1}{n!} \frac{d}{dz} \int_D \cdots \int_D W_{g,n}(z_1, \ldots, z_n) \right\} dz$$

$$= \sum_{m=1}^{\infty} \hbar^m \sum_{2g+n-2=m} \frac{1}{n!} \left\{ \left((1-\nu_0) \int_{\gamma_0} - \nu_1 \int_{\gamma_1} \right)^n - \left(-\nu_0 \int_{\gamma_0} - \nu_1 \int_{\gamma_1} \right)^n \right\} W_{g,n}$$

$$= \sum_{m=1}^{\infty} \hbar^m \sum_{\substack{2g+n-2=m \\ g \geqq 0, n \geqq 1}} \left[\sum_{k_0+k_1=n} \frac{\left\{ (1-\nu_0)^{k_0} - (-\nu_0)^{k_0} \right\} (-\nu_1)^{k_1}}{k_0! k_1!} \left(\int_{\gamma_0} \right)^{k_0} \left(\int_{\gamma_1} \right)^{k_1} W_{g,n} \right]$$

where we use the notation $\left(\int_{\gamma} \right)^n W_{g,n} = \int_{\zeta_1 \in \gamma} \cdots \int_{\zeta_n \in \gamma} W_{g,n}(\zeta_1, \ldots, \zeta_n)$.

Note: $\displaystyle \int_D = \nu_0 \int_0^z + \nu_1 \int_1^z, \quad \int_{\gamma_0} = \int_{\infty}^{0}, \quad \int_{\gamma_1} = \int_{\infty}^{1}.$

$$
\begin{aligned}
V^{(0,\infty)} &= \sum_{m=1}^{\infty} \hbar^m \sum_{\substack{2g+n-2=m \\ g\geqq 0, n\geqq 1}} \left[\sum_{k_0+k_1=n} \frac{\left\{(1-\nu_0)^{k_0} - (-\nu_0)^{k_0}\right\}(-\nu_1)^{k_1}}{k_0!k_1!} \left(\int_{\gamma_0}\right)^{k_0} \left(\int_{\gamma_1}\right)^{k_1} W_{g,n} \right], \\
&= \sum_{m=1}^{\infty} \hbar^m \sum_{\substack{2g+n-2=m \\ g\geqq 0, n\geqq 1}} \left[\sum_{k_0+k_1=n} \frac{\left\{(1-\nu_0)^{k_0} - (-\nu_0)^{k_0}\right\}(-\nu_1)^{k_1}}{k_0!k_1!} \frac{\partial^n F_g}{\partial\lambda_0^{k_0}\partial\lambda_1^{k_1}} \right] \\
&= \sum_{n=1}^{\infty} \hbar^n \left[\sum_{k_0+k_1=n} \frac{\left\{(1-\nu_0)^{k_0} - (-\nu_0)^{k_0}\right\}(-\nu_1)^{k_1}}{k_0!k_1!} \frac{\partial^n}{\partial\lambda_0^{k_0}\partial\lambda_1^{k_1}} \left\{ \sum_{g\geq 0} \hbar^{2g-2} F_g \right\} \right] \\
&\quad - \frac{1}{\hbar}\frac{\partial F_0}{\partial\lambda_0} + \nu_1 \frac{\partial^2 F_0}{\partial\lambda_0\partial\lambda_1} + \frac{2\nu_0-1}{2}\frac{\partial^2 F_0}{\partial{\lambda_0}^2} \\
&= F\left(\tilde{\lambda}_0+\hbar, \tilde{\lambda}_1, t; \hbar\right) - F\left(\tilde{\lambda}_0, \tilde{\lambda}_1, t; \hbar\right) - \frac{1}{\hbar}\frac{\partial F_0}{\partial\lambda_0} + \nu_1 \frac{\partial^2 F_0}{\partial\lambda_0\partial\lambda_1} + \frac{2\nu_0-1}{2}\frac{\partial^2 F_0}{\partial{\lambda_0}^2}.
\end{aligned}
$$

where we use

$$
\frac{\partial W_{g,n}}{\partial\lambda_j}(z_1,\ldots,z_n) = \int_{\zeta\in\gamma_j} W_{g,n+1}(z_1,\ldots,z_n,\zeta), \tag{14}
$$

$$
\frac{\partial F_g}{\partial\lambda_j} = \int_{\zeta\in\gamma_j} W_{g,1}(\zeta). \tag{15}
$$

Explicit form of free energies

From Theorem 1 and contiguity relations we obtain

$$F(\lambda_0 + \hbar, \lambda_1, t; \hbar) - 2F(\lambda_0, \lambda_1, t; \hbar) + F(\lambda_0 - \hbar, \lambda_1, t; \hbar) = \frac{\partial^2 F_0}{\partial {\lambda_0}^2}, \tag{19}$$

$$F(\lambda_0, \lambda_1 + \hbar, t; \hbar) - 2F(\lambda_0, \lambda_1, t; \hbar) + F(\lambda_0, \lambda_1 - \hbar, t; \hbar) = \frac{\partial^2 F_0}{\partial {\lambda_1}^2}. \tag{20}$$

(19) is rewritten as

$$\left\{ e^{\hbar \frac{\partial}{\partial \lambda_0}} - 2 + e^{-\hbar \frac{\partial}{\partial \lambda_0}} \right\} F(\underline{\lambda}, t, \hbar) = e^{-\hbar \frac{\partial}{\partial \lambda_0}} \left(e^{\hbar \frac{\partial}{\partial \lambda_0}} - 1 \right)^2 F(\underline{\lambda}, t, \hbar) = \frac{\partial^2 F_0}{\partial {\lambda_0}^2}.$$

Explicit form of free energies

From Theorem 1 and contiguity relations we obtain

$$F(\lambda_0+\hbar,\lambda_1,t;\hbar)-2F(\lambda_0,\lambda_1,t;\hbar)+F(\lambda_0-\hbar,\lambda_1,t;\hbar)=\frac{\partial^2 F_0}{\partial{\lambda_0}^2}, \tag{19}$$

$$F(\lambda_0,\lambda_1+\hbar,t;\hbar)-2F(\lambda_0,\lambda_1,t;\hbar)+F(\lambda_0,\lambda_1-\hbar,t;\hbar)=\frac{\partial^2 F_0}{\partial{\lambda_1}^2}. \tag{20}$$

(19) is rewritten as

$$\left\{e^{\hbar\frac{\partial}{\partial\lambda_0}}-2+e^{-\hbar\frac{\partial}{\partial\lambda_0}}\right\}F(\underline{\lambda},t,\hbar)=e^{-\hbar\frac{\partial}{\partial\lambda_0}}\left(e^{\hbar\frac{\partial}{\partial\lambda_0}}-1\right)^2F(\underline{\lambda},t,\hbar)=\frac{\partial^2 F_0}{\partial{\lambda_0}^2}.$$

Explicit form of free energies

From Theorem 1 and contiguity relations we obtain

$$F(\lambda_0+\hbar,\lambda_1,t;\hbar)-2F(\lambda_0,\lambda_1,t;\hbar)+F(\lambda_0-\hbar,\lambda_1,t;\hbar)=\frac{\partial^2 F_0}{\partial {\lambda_0}^2}, \tag{19}$$

$$F(\lambda_0,\lambda_1+\hbar,t;\hbar)-2F(\lambda_0,\lambda_1,t;\hbar)+F(\lambda_0,\lambda_1-\hbar,t;\hbar)=\frac{\partial^2 F_0}{\partial {\lambda_1}^2}. \tag{20}$$

(19) is rewritten as

$$F(\underline{\lambda},t,\hbar)=e^{\hbar\frac{\partial}{\partial\lambda_0}}\left(e^{\hbar\frac{\partial}{\partial\lambda_0}}-1\right)^{-2}\frac{\partial^2 F_0}{\partial {\lambda_0}^2}.$$

$$\frac{e^w}{(e^w-1)^2}=\frac{1}{w^2}-\sum_{g=1}^{\infty}\frac{B_{2g}}{2g(2g-2)!}w^{2g-2} \qquad \downarrow \qquad \frac{\partial^2 F_0}{\partial {\lambda_0}^2}=\frac{1}{2}\log t-\log(\lambda_0+\lambda_1)$$

$$F(\underline{\lambda},t,\hbar)=F_0\hbar^{-2}+F_1+\sum_{g=2}^{\infty}\left[\frac{B_{2g}}{2g(2g-2)}\left\{-(\lambda_0+\lambda_1)^{2-2g}\right\}+\tilde{G}(\lambda_1,t)\right]\hbar^{2g-2}.$$

Explicit form of free energies

From Theorem 1 and contiguity relations we obtain

$$F(\lambda_0+\hbar,\lambda_1,t;\hbar)-2F(\lambda_0,\lambda_1,t;\hbar)+F(\lambda_0-\hbar,\lambda_1,t;\hbar)=\frac{\partial^2 F_0}{\partial {\lambda_0}^2}, \tag{19}$$

$$F(\lambda_0,\lambda_1+\hbar,t;\hbar)-2F(\lambda_0,\lambda_1,t;\hbar)+F(\lambda_0,\lambda_1-\hbar,t;\hbar)=\frac{\partial^2 F_0}{\partial {\lambda_1}^2}. \tag{20}$$

(19) is rewritten as

$$F(\underline{\lambda},t,\hbar)=e^{\hbar\frac{\partial}{\partial\lambda_0}}\left(e^{\hbar\frac{\partial}{\partial\lambda_0}}-1\right)^{-2}\frac{\partial^2 F_0}{\partial {\lambda_0}^2}.$$

$$\frac{e^w}{(e^w-1)^2}=\frac{1}{w^2}-\sum_{g=1}^{\infty}\frac{B_{2g}}{2g(2g-2)!}w^{2g-2} \quad \Big\downarrow \quad \frac{\partial^2 F_0}{\partial {\lambda_0}^2}=\frac{1}{2}\log t-\log(\lambda_0+\lambda_1)$$

$$F(\underline{\lambda},t,\hbar)=F_0\hbar^{-2}+F_1+\sum_{g=2}^{\infty}\left[\frac{B_{2g}}{2g(2g-2)}\left\{{\lambda_1}^{2-2g}-(\lambda_0+\lambda_1)^{2-2g}\right\}+G(t)\right]\hbar^{2g-2}.$$

$$\left(\frac{\partial^2 F_0}{\partial {\lambda_1}^2}=\log\lambda_1-\log(\lambda_0+\lambda_1)\right)$$

Then we obtain that for $g \geq 2$

$$F_g = \frac{B_{2g}}{2g(2g-2)} \left\{ \frac{1}{{\lambda_1}^{2g-2}} - \frac{1}{(\lambda_0+\lambda_1)^{2g-2}} \right\} + G(t),$$

where $G(t)$ is an unknown function. In the following, we will prove $G(t) = 0$.

By substituting $\nu_1 = 0$ in (17), we get

$$\sum_{m=-1}^{\infty} \hbar^m \int^{x(z)} S_m dx = \sum_{m=-1}^{\infty} \hbar^m \left\{ \sum_{\substack{2g+n-2=m \\ g\geq 0,\, n\geq 1}} \frac{1}{n!} \int_{z_1=0}^{z} \cdots \int_{z_n=0}^{z} W_{g,n}(z_1,\ldots,z_n) \right\}. \quad (21)$$

$\downarrow$ We differentiate the both sides with respect to z.

$$\sum_{m=-1}^{\infty} \hbar^m S_m(x(z)) dx(z)$$

$$= \sum_{g\geq 0} \hbar^{2g-1} W_{g,1}(z) + \sum_{\substack{2g+n-2=m \\ g\geq 0,\, n\geq 2}} \frac{\hbar^{2g+n-2}}{(n-1)!} \int_{z_2=0}^{z} \cdots \int_{z_n=0}^{z} W_{g,n}(z,z_2,\ldots,z_n). \quad (22)$$

$$\operatorname*{Res}_{z=0} \frac{1}{z} \sum_{m=-1}^{\infty} \hbar^m S_m(x(z)) dx(z)$$

$$= \operatorname*{Res}_{z=0} \frac{1}{z} \sum_{g \geq 0} \hbar^{2g-1} W_{g,1}(z) + \operatorname*{Res}_{z=0} \frac{1}{z} \sum_{\substack{2g+n-2=m \\ g\geq 0,\, n\geq 2}} \frac{\hbar^{2g+n-2}}{(n-1)!} \int_0^z \cdots \int_0^z W_{g,n}(z, z_2, \ldots, z_n). \tag{23}$$

$$\operatorname*{Res}_{z=0} \frac{1}{z} \sum_{m=-1}^{\infty} \hbar^m S_m(x(z))dx(z)$$

$$= \operatorname*{Res}_{z=0} \frac{1}{z} \sum_{g \geq 0} \hbar^{2g-1} W_{g,1}(z) + \operatorname*{Res}_{z=0} \frac{1}{z} \sum_{\substack{2g+n-2=m \\ g \geq 0,\, n \geq 2}} \frac{\hbar^{2g+n-2}}{(n-1)!} \int_0^z \cdots \int_0^z W_{g,n}(z, z_2, \ldots, z_n). \tag{23}$$

From the variational formula

$$\frac{\partial F_g}{\partial t} = -\operatorname*{Res}_{z=0} \frac{1}{z} W_{g,1}(z) \quad (g \geq 1), \tag{16}$$

we obtain

$$\sum_{g \geq 1} \hbar^{2g-1} \frac{\partial F_g}{\partial t} = \operatorname*{Res}_{z=0} \frac{1}{z} \hbar^{-1} W_{0,1} - \operatorname*{Res}_{z=0} \frac{1}{z} \sum_{m=-1}^{\infty} \hbar^m S_m(x(z))dx(z)$$

$$+ \operatorname*{Res}_{z=0} \frac{1}{z} \sum_{\substack{2g+n-2=m \\ g \geq 0,\, n \geq 2}} \frac{\hbar^{2g+n-2}}{(n-1)!} \int_{z_2=0}^z \cdots \int_{z_n=0}^z W_{g,n}(z, z_2, \ldots, z_n). \tag{24}$$

Then, we compare the odd degree terms with respect to $\hbar$ of both-sides.

From $S_m(x) \sim O(1/x^2)$ $(m \geq 1)$, we find that

$$\operatorname*{Res}_{z=0} \frac{1}{z} \sum_{m=1}^{\infty} \hbar^m S_m(x(z)) dx(z) = 0. \tag{25}$$

Because $W_{g,n}(z_1, z_2, \ldots, z_n)$ is holomorphic at $z_i = 0$ $(1 \leq i \leq n)$,

$$W_{g,n}(z_1, z_2, \ldots, z_n) = \sum_{k_1, \ldots, k_n \geqq 0} a_{k_1,\ldots,k_n} {z_1}^{k_1} \cdots {z_n}^{k_n}, \tag{26}$$

$$\int_0^{z_2} \cdots \int_0^{z_n} W_{g,n}(z, z_2, \ldots, z_n)\Big|_{z_2,\ldots z_n = z} = z^{n-1} \left\{a_{0,\ldots,0} + O\left(z\right)\right\}. \tag{27}$$

By using this we obtain

$$\operatorname*{Res}_{z=0} \frac{1}{z} \sum_{\substack{2g+n-2=m \\ g \geq 0,\, n \geq 2}} \frac{\hbar^{2g+n-2}}{(n-1)!} \int_{z_2=0}^{z} \cdots \int_{z_n=0}^{z} W_{g,n}(z, z_2, \ldots, z_n) = 0. \tag{28}$$

Therefore, $\dfrac{\partial F_g}{\partial t} = \dfrac{\partial G}{\partial t} = 0$ holds for $g \geq 1$.

Explicit forms of Voros coefficients

Using the explicit form of the free energy

$$F_g = \frac{B_{2g}}{2g(2g-2)} \left\{ \frac{1}{{\lambda_1}^{2g-2}} - \frac{1}{(\lambda_0+\lambda_1)^{2g-2}} \right\} \tag{29}$$

and Theorem 1, we get the explicit forms of the Voros coefficients.

Explicit forms of the Voros coefficients

$$V^{(0,\infty)} = \sum_{m=1}^{\infty} \frac{\hbar^m}{m(m+1)} \left\{ -\frac{(-1)^{m+1} B_{m+1}}{(\lambda_0+\lambda_1)^m} \right\}, \tag{30}$$

$$V^{(1,\infty)} = \sum_{m=1}^{\infty} \frac{\hbar^m}{m(m+1)} \left\{ \frac{B_{m+1}(\nu_1)}{{\lambda_1}^m} - \frac{(-1)^{m+1} B_{m+1}}{(\lambda_0+\lambda_1)^m} \right\}, \tag{31}$$

where $B_m(X)$ designates the m-th Bernoulli polynomial defined by

$$\frac{we^{Xw}}{e^w-1} = \sum_{m=0}^{\infty} \frac{B_m(X)}{m!} w^m.$$

Summary

For confluent hypergeometric differential equations of third order, we also obtain the following results:

- Voros coefficients are expressed as the difference values of the generating function of the free energies with respect to parameters.
$\longrightarrow$ It means that the Voros coefficients are controlled by the free energy, in other words, the free energy is more essential quantity.
- As its applications, we get the explicit forms of the Voros coefficients and free energies.

References I

[BE] V. Bouchard and B. Eynard, Reconstructing WKB from topological recursion, *Journal de l'Ecole polytechnique – Mathematiques*, **4** (2017), 845–908.

[CEO] L. Chekhov, B.Eynard and N. Orantin, Free energy topological expansion for the 2-matrix model, *JHEP12* (2006), 053.

[DM] O. Dumitrescu and M. Mulase, Quantum curves for Hitchin fibrations and the Eynard-Orantin theory, Lett. Math. Phys., **104** (2014), 635–671.

[EO1] B. Eynard and N. Orantin, Invariants of algebraic curves and topological expansion, Comm. in Number Theory and Phys., **1** (2007), 347–452.

[EO2] B. Eynard and N. Orantin, Algebraic methods in random matrices and enumerative geometry, arXiv:0811.3531 [math-ph].

[HZ] J. Harer and D. Zagier : The Euler characteristic of the moduli space of curves, Invent. Math., **85** (1986), 457–485.

[IKo] K. Iwaki and T. Koike, On the computation of Voros coefficients via middle convolutions, *RIMS Kôkyûroku Bessatsu*, **B52** (2014), 55–70.

References II

[IKoT] K. Iwaki, T. Koike and Y.-M. Takei, Voros coefficients for the hypergeometric differential equations and Eynard-Orantin's topological recursion – Part I : For the Weber Equation –, *Annales Henri Poincaré* **24** (2023), 1305–1353.

[IKoT2] K. Iwaki, T. Koike and Y.-M. Takei, Voros coefficients for the hypergeometric differential equations and Eynard-Orantin's topological recursion – Part II : For Confluent Family of Hypegeometric Equations –, Journal of Integrable Systems **3** (2019), 1–46.

[OK] K. Okamoto and H. Kimura : On particular solutions of the Garnier systems and the hypergeometric functions of several variables, Quarterly J. Math., 37 (1986), 61–80.

[T] Y.-M.Takei, Voros coefficients for a class of the hypergeometric differential equations associated with the degeneration of the 2-dimensional Garnier system and the topological recursion; arXiv:2005.08957.

Thank you for your attention !