# Knots and Modularity 

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## Knots

- A knot $K$ is an embedding of a circle in $\mathbb{R}^{3}$. For example, the right-handed trefoil knot is given by

- Trying to determine whether two given knots are equivalent can be a tricky problem...

... is equivalent to the unknot


## History

- The study of knots has a long history: Vandermonde, Gauss, Listing, Thomson (Lord Kelvin), Maxwell, Tait, Kirkman, Little, Artin, Alexander, Reidemeister, Schubert, Conway, Jones, Kauffman, Vassiliev, ...
- We focus on Peter Tait (1831-1901):



## Edinburgh, smoke rings and knot tables

- At age 17, Tait entered Peterhouse College in Cambridge. In January 1852, he became Senior Wrangler in the Tripos. In 1854, he became a professor at Queen's College (Belfast) and, in 1859, the Chair of Natural Philosophy in Edinburgh.
- The Courant reported: "there is another quality which is desirable in a Professor in a University like ours and that is the power of oral exposition proceeding on the supposition of imperfect knowledge or even total ignorance on the part of pupils."
- In January 1867, Tait shows Thomson an experiment:


- (Tait's conjecture) A reduced diagram of an alternating knot has the fewest possible crossings.


## Knots

- (Reidemeister, 1927) Let $K$ and $K^{\prime}$ be two knots with diagrams $D$ and $D^{\prime}$. Then $K$ is isotopic to $K^{\prime}$ in $\mathbb{R}^{3}$ if and only if $D$ is related to $D^{\prime}$ by a sequence of isotopies of $\mathbb{R}^{2}$ and the moves $R I, R I I$ and $R I I I$ given by the following:



## The Jones polynomial

- In 1984, Vaughan Jones discovered a new polynomial knot invariant. In 1990, he was awarded the Fields Medal for this work.
- The Jones polynomial $V(K)=V(K ; q)$ is given by

$$
V(K)=\left.\frac{1}{\left(-A^{2}-A^{-2}\right)}(-A)^{-3 w(D)}\langle D\rangle\right|_{A^{2}=q^{-1 / 2}}
$$

where

$$
w(D)=
$$

is the "writhe" of $D$.

- For example, $V($ trefoil $)=q^{-1}+q^{-3}-q^{-4}$.


## The Jones polynomial and colored Jones polynomial

- In 1987, Kauffman, Murasugi and Thistlethwaite proved the first two of Tait's conjectures using the Jones polynomial. Menasco and Thistlethwaite proved the third Tait conjecture in 1991.
- (Open question) Is there a non-trivial knot with Jones polynomial equal to 1 ?
- (Sikora, Tuzun, 2021) Let $K$ be a knot with trivial Jones polynomial. Then $K$ is the unknot or it has at least 25 crossings.
- K knot $\rightsquigarrow$ " $N$-th colored Jones polynomial" $J_{N}(K ; q)$. For example,

$$
J_{N}(\text { trefoil } ; q)=q^{1-N} \sum_{n \geq 0} q^{-n N}\left(q^{1-N}\right)_{n}
$$

where $(a)_{n}:=(1-a)(1-a q) \cdots\left(1-a q^{n-1}\right)$.

## Modular forms

"There are five fundamental operations of arithmetic: addition, subtraction, multiplication, division, and modular forms." (Martin Eichler)

- A modular form of weight $k$ is a holomorphic function $f: \mathbb{H} \rightarrow \mathbb{C}$ satisfying

$$
f\left(\frac{a z+b}{c z+d}\right)=(c z+d)^{k} f(z)
$$

for all $z \in \mathbb{H}$ and $\gamma=\left(\begin{array}{ll}a & b \\ c & d\end{array}\right) \in \mathrm{SL}_{2}(\mathbb{Z})$.

Modular forms play a pivotal role in the proof of Fermat's Last Theorem and occur in mathematical physics, algebraic geometry, combinatorics, ...
... 48 hours ago, Maryna Viazovska won the Fields Medal for her work on the sphere packing problem. A key aspect is the use of modular forms.

## Srinivasa Ramanujan (1887-1920)



## The first letter

- Ramanujan wrote to G.H. Hardy (Cambridge) on January 16, 1913. He states
"I beg to introduce myself to you as a clerk in the Accounts Department of the Port Trust Office at Madras ... I have not trodden through the conventional regular course which is followed in a University course, but I am striking out a new path for myself. I have made a special investigation of divergent series in general and the results I get are termed by the local mathematicians as "startling"."
- A correspondence began and on April 30, 1914, Ramanujan arrived at Trinity College.


## Cambridge and India

- From 1914 to 1919, Ramanujan wrote over 30 papers ( 7 with Hardy) on a variety of topics: hypergeometric series, elliptic functions, modular forms, the theory of partitions, $q$-series ...
- In 1918, he was finally recognized for his work as he was named a Fellow of Trinity College (Cambridge) and elected a Fellow of the Royal Society of London.
- He became ill, returned to India in Spring 1919 and died on April 26, 1920 at the age of 32 .


## Partitions

- In how many ways can we "partition" a number $n$ ? Let $p(n)$ be the number of partitions of $n$.
- Since

$$
4,3+1,2+2,2+1+1,1+1+1+1
$$

we say that $p(4)=5$.

- We have $p(5)=7$ since

$$
5,4+1,3+2,3+1+1,2+2+1,2+1+1+1,1+1+1+1+1 .
$$

- Check that $p(6)=11$.


## Larger values

- Percy Alexander MacMahon, a major in the British Royal Artillery, computed the values of $p(n)$ for all $n$ up to 200 . He found that

$$
p(200)=3,972,999,029,388 .
$$

- How? He used the recursion formula

$$
\begin{aligned}
p(n) & =p(n-1)+p(n-2)-p(n-5) \\
& -p(n-7)+p(n-12)+\ldots
\end{aligned}
$$

## Table of values

| 1 | 1 | 2 | 3 | 5 |
| :---: | :---: | :---: | :---: | :---: |
| 7 | 11 | 15 | 22 | 30 |
| 42 | 56 | 77 | 101 | 135 |
| 176 | 231 | 297 | 385 | 490 |
| 627 | 792 | 1002 | 1255 | 1575 |
| 1958 | 2436 | 3010 | 3718 | 4565 |

Ramanujan noticed that $p(4), p(9), p(14), p(19), p(24), p(29)$ are all divisible by 5 .

## Ramanujan's congruences

In 1919, he stated:
"I have proved a number of arithmetic properties of $p(n) \ldots$ in particular that

$$
\begin{aligned}
& p(5 n+4) \text { is divisible by } 5, \\
& p(7 n+5) \text { is divisible by } 7 .
\end{aligned}
$$

... I have since found another method which enables me to prove all of these properties and a variety of others, of which the most striking is

$$
p(11 n+6) \text { is divisible by } 11 \text {. }
$$

... It appears that there are no equally simple properties for any moduli involving primes other than these three."

In 2003, Ahlgren and Boylan proved that Ramanujan's conjecture is true.

The last letter on January 12, 1920
(A) $1+\frac{v^{2}}{(1-2)^{2}}+\frac{v^{4}}{(1-2)^{2}\left(1-v^{2}\right)^{2}}+\frac{\varepsilon^{2}}{\left.\left.(1-2)^{2}(i)-v^{2}\right)^{2}(-2)^{2}\right)}$





when $q=e-t a c>0$

$$
\begin{aligned}
& \left.q=\sqrt{\frac{c}{2 \pi} e^{\frac{\pi}{6}}-\frac{c}{2 z}}+\frac{e^{2}}{\frac{\pi 5^{2}}{6}}-\frac{c}{60}+1\right)^{+} \\
& (B)=\frac{\sqrt{3}}{2}
\end{aligned}
$$





$$
\begin{aligned}
& \text { of le }
\end{aligned}
$$

$$
\begin{aligned}
& \left\{(1-8)\left(1+2^{2}\right)\left(1-थ^{\circ}\right)-\right\}^{120}
\end{aligned}
$$

## Mock theta functions

- He states
"I am extremely sorry for not writing you a single letter up to now... I discovered very interesting functions recently which I call "Mock" $\vartheta$-functions ... they enter into mathematics as beautifully as the ordinary $\vartheta$-functions. I am sending you with this letter some examples ..."
- In this last letter, Ramanujan lists 17 "mock theta functions". For example,

$$
f(q)=1+\frac{q}{(1+q)^{2}}+\frac{q^{4}}{(1+q)^{2}\left(1+q^{2}\right)^{2}}+\ldots=\sum_{n \geq 0} \frac{q^{n^{2}}}{(-q)_{n}^{2}}
$$

- On November 14, 1935, G. N. Watson considers three more. In the spring of 1976, Andrews found two more in Ramanujan's "lost notebook".
- It was believed that these functions were related to the theory of modular forms.


## A Dutchman enters ...

- Thanks to Sander Zwegers' Ph.D. thesis (2002), we now know that each of the 22 mock theta functions $f$ is a piece of a general type of modular form.
- This means

$$
\underbrace{g}_{\text {WMF }}=\underbrace{f}_{\text {MMF }}+\underbrace{f^{*}}_{\text {NH integral }}
$$

- Since 2007, there have been $\sim 200$ papers and 17 international conferences (AIM, MPIM, ICTP, ICERM, Simons Institute, KITP, INIMS, ... ).
- A. Folsom, "Perspectives on mock modular forms", J. Number Theory 176 (2017), 500-540.


## Quantum modular forms



- (Zagier, 2010) A quantum modular form of weight $k$ is a function $g: \mathbb{Q} \rightarrow \mathbb{C}$ such that for all $\gamma=\left(\begin{array}{ll}a & b \\ c & d\end{array}\right) \in \mathrm{SL}_{2}(\mathbb{Z})$,

$$
r_{\gamma}(\alpha):=g(\alpha)-(c \alpha+d)^{-k} g\left(\frac{\partial \alpha+b}{c \alpha+d}\right)
$$

has "nice" properties (e.g., continuity or analyticity).

From knots to modularity

- For the right-handed trefoil and its mirror image

we have

$$
\begin{gathered}
J_{N}(\text { trefoil; } q) \xrightarrow{q=\zeta_{N}} F(q):=\sum_{n \geq 0}(q)_{n} \\
J_{N}\left(\text { trefoil }^{*} ; q\right) \xrightarrow{q=\zeta_{N}} U(q):=\sum_{n \geq 0}(-q)_{n}^{2} q^{n+1} .
\end{gathered}
$$

- (Zagier, 2010) $F(q)$ is a weight $3 / 2$ quantum modular form.
- (Hikami and Lovejoy, 2015) $U(q)$ is a weight $1 / 2$ mock modular form.


## Our result

- Consider the family of torus knots $T(p, q)$. For example, the trefoil knot $T(3,2)$ is given by

- For $t \geq 1$, we have

$$
J_{N}\left(T\left(3,2^{t}\right) ; q\right) \xrightarrow{q=\zeta_{N}} \mathcal{F}_{t}(q) .
$$

Theorem (Goswami, -, 2021)
$\mathcal{F}_{t}(q)$ is a weight $3 / 2$ quantum modular form.

## Future work

- Consider the picture:

- $T(3,2)$ : Zagier $\rightsquigarrow F \checkmark$, Hikami, Lovejoy $\rightsquigarrow U \checkmark$ $T(2,2 m+1)$ : Hikami $\rightsquigarrow F \checkmark$, Mortenson, Zwegers $\rightsquigarrow U \checkmark$ Today! $T\left(3,2^{t}\right)$ : Goswami, $-\rightsquigarrow F \checkmark$, NO U yet!! Satellite knots? Hyperbolic knots?

ありがとうございます

