Supersymmetric Quantum Mechanics and Morse theory

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Lectures 10,11: Grassmann variables v1

Contents

1 Grassmann variables

Observation: We used the *exterior algebra* Λ^* to define the differential forms

$$\Lambda^*\mathbb{R}^n = T^*\mathbb{R}^n/\{e_i \otimes e_j + e_j \otimes e_i, \ \ \forall i, j = 1..n\}, \tag{1.1}$$

with T^* being tensor algebra

$$T^*\mathbb{R}^n = T^*\mathbb{R}\langle e_i\rangle = \mathbb{R} \oplus \mathbb{R}\langle e_i\rangle \oplus \mathbb{R}\langle e_i \otimes e_j\rangle \oplus \mathbb{R}\langle e_i \otimes e_j \otimes e_k\rangle \oplus .... \tag{1.2}$$

There is another commonly used name for the same algebra - *Grassmann algebra*, because we can describe the same algebra as algebra of polynomials of Grassmann variables ψ^i

$$\Lambda^*\mathbb{R}^n = \mathbb{R}[\psi^i] = \mathbb{R} \oplus \mathbb{R}\langle\psi_i\rangle \oplus \mathbb{R}\langle\psi_i \cdot \psi_j\rangle \oplus ... \oplus \mathbb{R}\langle\psi_1 \cdot ... \cdot \psi_n\rangle. \tag{1.3}$$

There are several other commonly used names for Grassmann variables - *fermionic variables* in physics literature and *Grassmann-odd variables* in math literature. Grassmann variables ψ^i obey exterior algebra relations

$$\{\psi^i, \psi^j\} = \psi^i\psi^j + \psi^j\psi^i = 0. \tag{1.4}$$

The functions on Grassmann variables ψ^i are finite polynomials since

$$\psi^i\psi^i = 0. \tag{1.5}$$

We can introduce Grassmann version of the n-dimensional Euclidean space, denoted by $\mathbb{R}^{0|n}$

and identify

$$\Lambda^*\mathbb{R}^n = \mathbb{R}[\psi^i] = C^\infty(\mathbb{R}^{0|n}). \tag{1.6}$$

Definition: Parity $|A|$ of the monomial A is an integer mod 2. Monomials with zero parity are denoted as *Grassmann-even* while the ones with parity 1 as *Grassmann-odd*. Sometimes the notation is simplified to *odd* and *even* variables. The generators of Grassmann algebra have parities

$$|\psi^i| = 1. \tag{1.7}$$

Parity of the product is the sum of parities

$$|A \cdot B| = |A| + |B|, \tag{1.8}$$

so the parity of the individual monomials is the total number of Grassmann variables mod 2

$$|\psi^i\psi^j| = 0, \quad |\psi^i\psi^j\psi^k| = 1, ... \tag{1.9}$$

The parity 0 expressions behave as usual variables i.e. commute among themselves. For example

$$[\psi^i\psi^j, \psi^k\psi^l] = 0, \tag{1.10}$$

while expressions with odd parity anti-commute between themselves

$$\{\psi^i\psi^j\psi^k, \psi^m\} = 0. \tag{1.11}$$

More generally for pair of expressions A and B with parities $|A|$ and $|B|$ the following relation holds

$$[A, B]_\pm = \{A, B] \equiv AB - (-1)^{|A|\cdot|B|}BA = 0. \tag{1.12}$$

The $\{\cdot,\cdot]$ is physics notation, while $[\cdot,\cdot]_\pm$ is math notation for the *graded commutator*, or *supercommutator* in physics literature. The graded commutator obey the graded version of the commutator properties

- Graded symmetry

$$\{A, B] = -(-1)^{|A|\cdot|B|}\{B, A] \tag{1.13}$$

- Graded Leibnitz

$$\{A, BC] = \{A, B]C + (-1)^{|A||B|}B\{A, C] \tag{1.14}$$

- Graded Jacobi

$$\{A,\{B,C]] + (-1)^{|A|(|B|+|C|)}\{B,\{C,A]] + (-1)^{|B|(|C|+|A|)}\{C,\{A,B]] = 0. \tag{1.15}$$

Example: The functions of single odd varible ψ are polynomials of degree 1

$$f(\psi) = f_0 + f_1\psi + f_2\psi^2 + ... = f_0 + f_1\psi, \quad f_0, f_1 \in \mathbb{R}. \tag{1.16}$$

Remark: Taylor series for functions on Grassmann spaces are always finite and we do not need to worry about the convergence issues.

Example: The exponent function of four odd variables

$$e^{\sum A_{ij}\psi^i\psi^j} = 1 + \sum A_{ij}\psi^i\psi^j + \frac{1}{2}\sum A_{ij}A_{kl}\psi^i\psi^j\psi^k\psi^l. \tag{1.17}$$

1.1 Derivatives

We can define derivatives on monomials

$$\partial_\psi(1) = 0, \quad \partial_\psi(\psi) = 1, \tag{1.18}$$

so the derivative of arbitrary function

$$\partial_\psi f(\psi) = \partial_\psi(f_0 + f_1\psi) = f_0\partial_\psi(1) + f_1\partial_\psi(\psi) = f_1. \tag{1.19}$$

For multivariable case

$$\frac{\partial}{\partial\psi}\left(\psi^1\psi^2...\psi^k\psi\psi^{k+1}...\psi^n\right) = (-1)^k\psi^1...\psi^k\psi^{k+1}...\psi^n \tag{1.20}$$

the way to determine the overall sign is to move the relevant variable to the very front, take into account sign, from graded commutators and take a 1d derivative. The functions on Grassmann variables are identical to their Taylor series, so we can extend the derivatives on monomials to derivatives on functions. The derivative obey graded version of the usual derivative properties:

- Linearity

$$\partial_\psi(\alpha f + \beta g) = \alpha\partial_\psi f + \beta\partial_\psi g, \quad \alpha, \beta \in \mathbb{R}. \tag{1.21}$$

- Graded Leibnitz

$$\partial_\psi(A\cdot B)=\partial_\psi A\cdot B+(-1)^{|A|}A\cdot\partial_\psi B. \tag{1.22}$$

- For a single Grassmann variable ψ-derivative and multiplication by ψ obey

$$\{\partial_\psi,\psi\}=\partial_\psi\psi+\psi\partial_\psi=1. \tag{1.23}$$

- For multiple Grassmann variables ψ^i the derivative and multiplications obey

$$\{\partial_i,\partial_j\}=0,\ \ \{\partial_j,\psi^i\}=\delta^i_j. \tag{1.24}$$

1.2 Integration

The integration over Grassmann variables is performed using the Berezin rules

$$\int d\theta\ 1=0,\quad \int d\theta\ \theta=1. \tag{1.25}$$

The Berezin integral for function of singe odd variable

$$\int d\theta\ f(\theta)=\int d\theta\ (f_0+f_1\theta)=f_1\int d\theta\ \theta=f_1=\frac{\partial f}{\partial\theta}. \tag{1.26}$$

Let us list some properties of Berezin integration

- Integral of total derivative vanishes

$$\int d\theta\ \partial_\theta f(\theta)=\int d\theta\ f_0=0 \tag{1.27}$$

- Delta-function is the linear function

$$\delta(\theta)=\theta \tag{1.28}$$

  Indeed an explicit check

$$f(0)=\int d\theta\ \delta(\theta)f(\theta)=f_0=\int d\theta\ \theta f(\theta) \tag{1.29}$$

- Change of variables

$$d(a\theta)=\frac{1}{a}d\theta,\ \ d(\theta+\epsilon)=d\theta, \tag{1.30}$$

what follows from

$$1=\int d\theta'\theta'=\int d(a\theta)\ \ a\theta=\int\frac{1}{a}d\theta\ a\theta=\int d\theta\ \theta \tag{1.31}$$

and

$$f_1=\int d\theta'\ f(\theta')=\int d(\theta{+}\epsilon)\ f(\theta{+}\epsilon)=\int d(\theta{+}\epsilon)\ (f_1\theta{+}f_0{+}f_1\epsilon)=\int d\theta\ f_1\theta. \tag{1.32}$$

The multiple variable case integration picks up the top degree monomial form the function i.e.

$$\int d^n\theta f(\theta_i)=\int d\theta_1...d\theta_n\ f(\theta)=\int d^n\theta\ f_{11...11}\theta_1...\theta_n=f_{11...11}=\frac{\partial^n f(\theta_i)}{\partial\theta_1...\partial\theta_n}, \tag{1.33}$$

while the sign is determined by order of θ's, in a way similar to the derivative definition.

Example: The Berezin integral over $\mathbb{R}^{0|n}$

$$\int d^n\theta\ \theta_{i_1}...\theta_{i_n}=\epsilon_{i_1...i_n}. \tag{1.34}$$

There is an additional convention that we need is the integral of the form

$$\int_{\mathbb{R}^{0|n}} d\theta_1...d\theta_n\ \ \alpha\theta_1...\theta_n=\alpha \tag{1.35}$$

for α being monomial of Grassmann variables that does not depend on θ^i.

Example: The Grassmann-odd Fourier transform of 1 is

$$\int_{\mathbb{R}^{0|2}} d^2\theta\ \ e^{\eta_1\theta_1+\eta_2\theta_2}=\int_{\mathbb{R}^{0|2}} d^2\theta\ \eta_1\theta_1\eta_2\theta_2=-\int_{\mathbb{R}^{0|2}} d^2\theta\ \eta_1\eta_2\theta_1\theta_2=-\eta_1\eta_2. \tag{1.36}$$

1.3 Change of variables

In previous section we observed that change of variables differ for Grassmann-even and Grassmann-odd variables. Single variable case

$$d(ax)=adx,\quad d(a\theta)=\frac{1}{a}d\theta \tag{1.37}$$

The multivariable generalization for linear variable change

$$\begin{aligned} x'^i &= \sum_j A^i_j x^j, \quad d^n x' = \det A \; d^n x, \\ \theta'^\alpha &= \sum_\beta D^\alpha_\beta \theta^\beta, \quad d^m \theta' = \frac{1}{\det D} d^m \theta. \end{aligned} \tag{1.38}$$

with A and D being real matrices $A \in Mat_{n\times n}(\mathbb{R})$ and $D \in Mat_{m\times m}(\mathbb{R})$. We can generalize the linear change of variables to include mixed terms.

Definition The linear operator $W : \mathbb{R}^{n|m} \to \mathbb{R}^{n|m}$ can be written in block form

$$W = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{1.39}$$

with A and D being Grassmann-even while B and C being Grassmann-odd. We can use W to describe the mixed linear change of variables

$$\begin{aligned} x'^i &= \sum_{j=1}^{n} A^i_j x^i + \sum_{\beta=1}^{m} B^i_\beta \theta^\beta, \\ \theta^\alpha &= \sum_{\beta=1}^{m} D^\alpha_\beta \theta^\beta + \sum_{j=1}^{n} C^\alpha_j x^j. \end{aligned} \tag{1.40}$$

The change of mixed integration measure

$$d^n x' d^m \theta' = \mathrm{Ber}(W) \; d^n x d^m \theta, \tag{1.41}$$

with Ber(W) is the *Berezian*, or *superdeterminant* in physics literature, defined to be

$$\mathrm{Ber}(W) = \mathrm{sdet}(W) = \frac{\det(A - BD^{-1}C)}{\det D}. \tag{1.42}$$

The formula for Berezian fcan be derived using the multiplicativity under the composition i.e.

$$\mathrm{Ber}(W_1 \cdot W_2) = \mathrm{Ber}(W_1) \cdot \mathrm{Ber}(W_2) \tag{1.43}$$

and factorization

$$W = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A - BD^{-1}C & B \\ 0 & D \end{pmatrix} \begin{pmatrix} 1 & 0 \\ D^{-1}C & 1 \end{pmatrix}, \tag{1.44}$$

We can define a graded generalization of trace also known as the *supertrace*

$$\text{Str } W = \text{Str} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \text{Tr}A - \text{Tr}D \tag{1.45}$$

The supertrace and superdeterminant are related is the same way as the trace and determinant

$$\text{sdet}(e^V) = \exp\left(\text{Str } V\right) \tag{1.46}$$

We can generalize the linear transformations to arbitrary smooth maps $f : \mathbb{R}^{n|m} \to \mathbb{R}^{n|m}$ with coordinate transformation of the integration measure being expressed through super-Jacobian J of coordinate transformation

$$d^m\theta' d^n x' = J d^m \theta d^n x, \tag{1.47}$$

with

$$J = \text{Ber}(J^A_B) \tag{1.48}$$

The Jacobian matrix

$$J^A_B = \begin{pmatrix} \frac{\partial x'^i}{\partial x^j} & \frac{\partial \theta'^i}{\partial \theta^\beta} \\ \frac{\partial \theta'^\alpha}{\partial x^j} & \frac{\partial \theta'^\alpha}{\partial \theta^\beta} \end{pmatrix}. \tag{1.49}$$

1.4 Gaussian integrals

In later sections we are going to use Gaussian integrals a lot, so let us summarize some properties of the Gaussian integrals for both even and odd variables. We will start with the most familiar single-variable case

$$\int_{\mathbb{R}} dx\, e^{-ax^2} = \sqrt{\frac{\pi}{a}}, \tag{1.50}$$

followed by the delta function inspired Gaussian integral

$$\int_{\mathbb{R}^2} dxdp\, e^{iapx} = 2\pi \int_{\mathbb{R}} dx\, \delta(ax) = \frac{2\pi}{a}. \tag{1.51}$$

The multi-varible generalization of the above integrals

$$\int_{\mathbb{R}^n} d^n x \;\; e^{-A_{ij}x^i x^j} = \frac{\pi^{n/2}}{(\det A)^{1/2}} \tag{1.52}$$

and

$$\int_{\mathbb{R}^{2n}} d^n p d^n x \;\; e^{iB_{kj}p^k x^j} = \frac{(2\pi)^n}{\det B}. \tag{1.53}$$

The Berezin version of the single variable

$$\int_{\mathbb{R}^{0|2}} d\theta d\bar{\theta}\; e^{a\theta\bar{\theta}} = \int_{\mathbb{R}^{0|2}} d\theta d\bar{\theta}\; (1 + a\theta\bar{\theta}) = \int_{\mathbb{R}^{0|2}} d\theta d\bar{\theta}\; a\theta\bar{\theta} = a \tag{1.54}$$

with multi-variable integrals

$$\int d^n\theta d^n\bar{\theta}\; e^{B_{ij}\theta^i\bar{\theta}^j} = \frac{1}{n!}\sum \epsilon^{i_1 \ldots i_n}\epsilon^{j_1 \ldots j_n} B_{i_1 j_1} \ldots B_{i_n j_n} = \det B. \tag{1.55}$$

Let us mention one more integral, commonly-used in physics literature

$$\int d^{2n}\theta\; e^{\frac{1}{2}A_{ij}\theta^i\theta^j} = \frac{1}{2^n n!}\sum \epsilon^{i_1 \ldots i_{2n}} A_{i_1 i_2} A_{i_3 i_4} \ldots . A_{i_{2n-1} i_{2n}} = \mathrm{Pf}(A). \tag{1.56}$$

with $\mathrm{Pf}(A)$ being *Pfaffian* of the matrix A_{ij} which obey

$$[\mathrm{Pf}(A)]^2 = \det A. \tag{1.57}$$

1.5 Differential forms as functions

The differential forms on $\mathbb{R}^n$ by construction are

$$\Omega^*(\mathbb{R}^n) = \Lambda^*\mathbb{R}^n \otimes C^\infty(\mathbb{R}^n). \tag{1.58}$$

We can use Grassmann-odd variables to represent $\Lambda^*\mathbb{R}^n$ so that

$$\Omega^*(\mathbb{R}^n) = C^\infty(\mathbb{R}^{0|n}) \otimes C^\infty(\mathbb{R}^{n|0}) = C^\infty(\mathbb{R}^{n|n}). \tag{1.59}$$

Let us introduce a map

$$F : \Omega^*(\mathbb{R}^n) \to C^\infty(\mathbb{R}^{n|n}) : \omega \mapsto F_\omega. \tag{1.60}$$

Using coordinates x^i and ψ^i on $\mathbb{R}^{n|n}$ we can write the map F in components

$$\omega = \frac{1}{k!}\sum \omega_{i_1\ldots i_k}(x)dx^{i_1}\wedge\ldots\wedge dx^{i_k} \mapsto F_\omega = \frac{1}{k!}\sum \omega_{i_1\ldots i_k}(x)\psi^{i_1}\cdot\ldots\cdot\psi^{i_k} \tag{1.61}$$

Proposition: The map

$$F : \Omega^*(\mathbb{R}) \to C^\infty(\mathbb{R}^{n|n}) \tag{1.62}$$

is the isomorphism of differential graded algebras.

Proof: We need to check whether F preserves the structures defining the DGA:

- Vector space structure

$$F_{\alpha\omega+\beta\mu} = \alpha F_\omega + \beta F_\mu, \quad \forall\alpha,\beta\in\mathbb{R}, \quad \forall\omega,\mu\in\Omega^*(\mathbb{R}^n) \tag{1.63}$$

- Multiplication structure

$$F_{\omega\wedge\mu} = F_\omega\cdot F_\mu. \tag{1.64}$$

- Grading

$$|F_\omega| = |\omega| \tag{1.65}$$

- the external derivative is the first-order differential operator

$$F_{d\omega} = D_d F_\omega \tag{1.66}$$

  with

$$D_d = \sum_{i=1}^n \psi^i\partial_i, \tag{1.67}$$

- D_d is differential

$$D_d^2 = \sum\psi^i\psi^j\partial_i\partial_j = -\sum\psi^j\psi^i\partial_i\partial_j = -\sum\psi^j\psi^i\partial_j\partial_i = -D_d^2 = 0. \tag{1.68}$$

- D_d obey graded Leibnitz

$$D_d(FG) = D_dF\cdot G + (-1)^{|F||G|}F\cdot D_dG \tag{1.69}$$

1.6 Differential forms as functions on supermanifold

Let $\{U_\alpha\}$ be an *open covering* of the topological space M. We can endow M with the structure of $n|m$-dimensional suprmanifold with the following information. Let $\phi_\alpha : U_\alpha \to \mathbb{R}^{n|m}$ be a coordinate chart on U_α with *local coordinates* $(x_\alpha, \theta_\alpha)$. On $U_\alpha \cap U_\beta$ we can relate $(x_\alpha, \theta_\alpha)$ and (x_β, θ_β) by $(x_\alpha, \theta_\alpha) = \phi_\alpha \circ \phi_\beta^{-1}(x_\beta, \theta_\beta)$. The map $G_{\alpha\beta} = \phi_\alpha \circ \phi_\beta^{-1}$ is a *transition function.* By construction

$$G_{\beta\alpha} G_{\alpha\beta} = 1, \quad G_{\alpha\beta} G_{\beta\gamma} G_{\alpha\gamma} = 1 \tag{1.70}$$

Alternatively, we can describe the manifold M purely by specifying the transition functions $g_{\alpha\beta}$ that obey (1.70). A manifold is called *differentiable* if $g_{\alpha\beta}$ are differentiable, *smooth* if $g_{\alpha\beta}$ are smooth.

Example: Let X be the n-dimensional smooth manifold then we can construct *odd tangent bundle* ΠTX, a supermanifold of dimension $n|n$. Let $\{U_\alpha\}$ be open covering of X and with local coordinates x^α on U_α. Using transition function $g_{\alpha\beta}$ for X we can describe transition function for ΠTX

$$\begin{aligned} x^i_\alpha &= g^i_{\alpha\beta}(x_\beta) \\ \theta^i_\alpha &= \theta^j_\beta \partial_j g^i_{\alpha\beta}(x_\beta) \\ (x_\alpha, \theta_\alpha) &= G_{\alpha\beta}(x_\beta, \theta_\beta) = (g^i_{\alpha\beta}(x_\beta), \theta^j_\beta \partial_j g^i_{\alpha\beta}(x_\beta)) \end{aligned} \tag{1.71}$$

In previous sections we discussed that the differential forms on $\mathbb{R}^n$ are the same as functions on $\mathbb{R}^{n|n} = \Pi T\mathbb{R}^n$. This correspondence can be further generalized to

$$F : \Omega^*(X) \to C^\infty(\Pi TX) : \omega \mapsto F_\omega \tag{1.72}$$

Proposition: Map F is an isomorphism of DGA.

Proof: We already showed that F is an isomorphism of DGA for each open set U_α of the covering $\{U_\alpha\}$ of X. What we left to check is the transition for forms on X and functions on ΠTX as we across the intersections $U_\alpha \cap U_\beta$. Let us label coordinates on ΠTX as x^i for even part and ψ^i for odd part. The same coordinates x^i can be used as coordinates on X. The two differential forms ω_α and ω_β on U_α and U_β are related by the pullback map

$$\omega_\beta = g^*_{\alpha\beta} \omega_\alpha \tag{1.73}$$

where $g_{\alpha\beta}$ are the transition functions from x_β to x_α

$$x^i_\alpha = g^i_{\alpha\beta}(x_\beta) \tag{1.74}$$

The corresponding functions

$$\omega_\alpha = \frac{1}{k!}\sum \omega_{i_1\dots i_k}(x_\alpha)dx^{i_1}_\alpha \wedge \dots \wedge dx^{i_k}_\alpha \mapsto F_{\omega_\alpha} = \frac{1}{k!}\sum \omega_{i_1\dots i_k}(x_\alpha)\psi^{i_1}_\alpha \cdot \dots \cdot \psi^{i_k}_\alpha \tag{1.75}$$

are related via

$$\begin{aligned} F_{\omega_\beta} = F_{g^*_{\alpha\beta}\omega_\alpha} &= \frac{1}{k!}\sum (g^*_{\alpha\beta}\omega)_{i_1\dots i_k}\psi^{i_1}_\beta \cdot \dots \cdot \psi^{i_k}_\beta \\ &= \frac{1}{k!}\sum_{i_1 j_1\dots i_k j_k} \partial_{j_1} g^{i_1}_{\alpha\beta}\dots\partial_{j_k} g^{i_k}_{\alpha\beta}\, \omega_{i_1\dots i_k}(g_{\alpha\beta}(x_\beta))\psi^{j_1}_\beta \cdot \dots \cdot \psi^{j_k}_\beta \\ &= \frac{1}{k!}\sum_{i_1 j_1\dots i_k j_k} \omega_{i_1\dots i_k}(g_{\alpha\beta}(x_\beta))\partial_{j_1} g^{i_1}_{\alpha\beta}\psi^{j_1}_\beta \cdot \dots\partial_{j_k} g^{i_k}_{\alpha\beta} \cdot \psi^{j_k}_\beta \\ &= G^*_{\alpha\beta}F_{\omega_\alpha}. \end{aligned} \tag{1.76}$$

Thus we conclude that the functions F_{ω_α} and F_{ω_β} are related by the pullback map $G^*_{\alpha\beta}$ on ΠTX, hence F_ω is well-defined function on ΠTX.

The integral over X of a top form ω is identical to the Grassmann integral of the corresponding function

$$\int_X \omega = \int_{\Pi TX} d^n x d^n \psi\; F_\omega(x,\psi). \tag{1.77}$$