Supersymmetric Quantum Mechanics and Morse theory

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Lectures 5,6,7 : Differential forms.

Contents

1 Differential forms

In previous section we discussed a simple method to describe the topology of smooth manifold M. We approximated M by the simplicial model Δ_M, which we refined in simplicial complex $C_\bullet(\Delta_M)$. The topology of M is captured by the homology of $C_\bullet(\Delta_M)$. We showed that the homology are independent on details of simplicial model, i.e that they are invariant of M itself. The simplicial models are useful in low dimensions or for manifolds with simple enough shape. The manifolds, relevant for various useful applications usually come with the smooth data, so we want to describe the topology of such manifolds using smooth geometric objects.

1.1 Differential forms on $\mathbb{R}^n$

Let $x^1, .., x^n$ be coordinates on $\mathbb{R}^n$, then dx^i are coordinates on cotangent space $T_0^*\mathbb{R}^n = \mathbb{R}^n$. We define Ω^* to be algebra over $\mathbb{R}$ generated by $dx^1, .., dx^n$ with relations

$$dx^i \wedge dx^j = -dx^j \wedge dx^i. \tag{1.1}$$

Such algebra is known as the *external algebra* and usually denoted as

$$\Omega^* = \Lambda^* T_0^*\mathbb{R}^n. \tag{1.2}$$

The Ω^* is a graded algebra ie it decomposed into direct sum

$$\Omega^* = \bigoplus_{k=0}^{n} \Omega^k, \tag{1.3}$$

such that the grading of Ω^k is k. The Ω^k is the liner space

$$\Omega^k = \mathbb{R}\langle dx^{i_1} \wedge dx^{i_2} \wedge ... \wedge dx^{i_k}\rangle, \quad 1 \leq i_1 < i_2 < ... < i_k \leq n \tag{1.4}$$

of dimension

$$\dim \Omega^k = C_n^k = \frac{n!}{k!(n-k)!}. \tag{1.5}$$

Definition: *Smooth differential forms* on $\mathbb{R}^n$ are elements of

$$\Omega^*(\mathbb{R}^n) = C^\infty(\mathbb{R}^n) \otimes_{\mathbb{R}} \Omega^* = \bigoplus_{k=0}^{n} \Omega^k(\mathbb{R}^n) \tag{1.6}$$

Elements of $\Omega^k(\mathbb{R}^n)$ are denoted as the *differential forms of degree* k or k*-forms* for short. In components

$$\Omega^k(\mathbb{R}^n) \ni \omega = \sum_{i_1,..,i_k} \frac{1}{k!} \omega_{i_1 \dots i_k}(x^1,..,x^n) dx^{i_1} \wedge .. \wedge dx^{i_k}, \tag{1.7}$$

with C_n^k functions $\omega_{i_1 \dots i_k}(x^1,..,x^n)$ being smooth functions on $\mathbb{R}^n$.

Example: 0-forms are smooth functions

$$\Omega^0(\mathbb{R}^n) = C^\infty(\mathbb{R}^n). \tag{1.8}$$

Example: The case of 1-form

$$A = \sum_{i=1}^{n} A_i(x) dx^i \in \Omega^1(\mathbb{R}^n) \tag{1.9}$$

and 2-form

$$F = \frac{1}{2} \sum_{i,j=1}^{n} F_{ij} dx^i \wedge dx^j = \sum_{i<j} F_{ij} dx^i \wedge dx^j \in \Omega^2(\mathbb{R}^n) \tag{1.10}$$

Remark: In some cases, especially in physics literature, the wedge $\wedge$ is dropped for simplicity.

1.2 Wedge product

We can extend the wedge product from generators dx^i to differential forms so that

$$\wedge : \Omega^k(\mathbb{R}^n) \times \Omega^l(\mathbb{R}^n) \to \Omega^{k+l}(\mathbb{R}^n), \tag{1.11}$$

which in components

$$\omega \wedge \mu = \sum_{i_1,..,i_n,j_1,..,j_l} \frac{1}{k!}\frac{1}{l!}\omega_{i_1...i_n}\mu_{j_1...j_l} dx^{i_1} \wedge .. \wedge dx^{i_k} \wedge dx^{j_1} \wedge .. \wedge dx^{j_l}. \tag{1.12}$$

Example: The wedge product of two 1-forms A and B is the 2-form C

$$\begin{aligned} A \wedge B &= \left(\sum_{i=1}^n A_i dx^i\right) \wedge \left(\sum_{j=1}^n B_j dx^j\right) = \sum_{ij} A_i B_j dx^i \wedge dx^j \\ &= \frac{1}{2}\sum_{i,j=1}^n C_{ij} dx^i \wedge dx^j, \;\; C_{ij} = A_i B_j - A_j B_i. \end{aligned} \tag{1.13}$$

in particular for two identical $\mathbb{R}$-valued 1-forms

$$A \wedge A = 0. \tag{1.14}$$

Example: A symplectic manifold $(\mathbb{R}^{2n}, \omega)$ has natural volume form μ know as the *symplectic volume form* or the *Liouville volume form* defined as

$$\mu = \frac{1}{n!}\omega^n = \frac{1}{n!}\omega \wedge \omega \wedge ... \wedge \omega \in \Omega^{2n}(\mathbb{R}^{2n}) \tag{1.15}$$

In particular for $n = 2$, the canonical symplectic form $\omega = dp^1 \wedge dq^1 + dp_2 \wedge dq_2$ squares to familiar volume form

$$\mu = \frac{1}{2}\omega \wedge \omega = \frac{1}{2}(dp^1 \wedge dq^1 \wedge dp_2 \wedge dq_2 + dp_2 \wedge dq_2 \wedge dp^1 \wedge dq^1) = dp^1 \wedge dq^1 \wedge dp_2 \wedge dq_2 \tag{1.16}$$

The wedge product is the graded multiplication on linear space of differential forms, so it obey the following properties:

1. *Associativity*

$$(\omega_1 \wedge \omega_2) \wedge \omega_3 = \omega_1 \wedge (\omega_2 \wedge \omega_3).$$

2. *Linearity*:

$$(\alpha\omega_1 + \beta\omega_2) \wedge \mu = \alpha\omega_1 \wedge \mu + \beta\omega_2 \wedge \mu, \quad \alpha, \beta \in \mathbb{R}.$$

3. *Graded symmetry*: Let $\omega \in \Omega^k(\mathbb{R}^n)$ and $\mu \in \Omega^l(\mathbb{R}^n)$

$$\omega \wedge \mu = (-)^{kl} \mu \wedge \omega.$$

1.3 Exterior derivative d

Definition: We can define an *exterior derivative*

$$d : \Omega^p(\mathbb{R}^n) \to \Omega^{p+1}(\mathbb{R}^n) \tag{1.17}$$

on functions

$$df = \sum_{i=1}^{n} \partial_i f(x) dx^i \in \Omega^1(\mathbb{R}^n) \tag{1.18}$$

and recursively on forms

$$d \sum_{i=1}^{n} f_i \wedge dx^i = \sum_{i=1}^{n} df_i \wedge dx^i. \tag{1.19}$$

Example: Let us take an external derivative of a 1-form

$$\begin{aligned} F =& dA = \left(\sum_{i=1}^{n} A_i dx^i \right) = \sum_{ij=1}^{n} \partial_j A_i dx^j \wedge dx^i = \frac{1}{2} \sum_{ij=1}^{n} (\partial_i A_j - \partial_j A_i) dx^i \wedge dx^j \\ =& \frac{1}{2} \sum_{ij=1}^{n} F_{ij} dx^i \wedge dx^j, \quad F_{ij} = \partial_i A_j - \partial_j A_i \end{aligned} \tag{1.20}$$

We can recognize components of a two form F as field strength for vector-potential A.

External derivative is the derivation on algebra of differential forms, i.e it obeys the following properties

1. *Linearity*:

$$d(\alpha\omega_1 + \beta\omega_2) = \alpha d\omega_1 + \beta d\omega_2, \quad \alpha, \beta \in \mathbb{R}.$$

2. *Graded Leibnitz rule*: Let $\omega \in \Omega^k(\mathbb{R}^n)$ and $\mu \in \Omega^m(\mathbb{R}^n)$

$$d(\omega \wedge \mu) = d\omega \wedge \mu + (-1)^k \omega \wedge d\mu.$$

3. *Differential*:

$$d(d\omega) = 0.$$

Remark: An algebra with differential d is the *differential graded algebra* DGA. Differential forms $\Omega^*(\mathbb{R}^n)$ is DGA.

1.4 Pullback map f^*

Let $x^1, .., x^n$ and $y^1, .., y^m$ be coordinates on $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. A smooth map

$$f : \mathbb{R}^m \to \mathbb{R}^n \tag{1.21}$$

induces a *pullback map* on smooth functions

$$f^* : \Omega^0(\mathbb{R}^n) \to \Omega^0(\mathbb{R}^m) \ : \ g \mapsto f^*(g) = g \circ f. \tag{1.22}$$

We can extend the pullback map to all forms $f^* : \Omega^*(\mathbb{R}^n) \to \Omega^*(\mathbb{R}^m)$ in such a way that it commutes with d. The commutation with d defines f^* uniquely

$$f^* \left(\sum \omega_{i_1 \dots i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k} \right) = \sum (\omega_{i_1 \dots i_k} \circ f) df^{i_1} \wedge \dots \wedge df^{i_k} \in \Omega^*(\mathbb{R}^m) \tag{1.23}$$

with $f^i(y), i = 1, .., n$ being i-th component of a map f.

Example: We can describe electromagnetic field on $\mathbb{R}^n$ via the vector potential A_i in the form of the 1-form $A = A_i dx^i$. Point particle moving on M can be described using map

$$\gamma : I \to \mathbb{R}^n : t \mapsto x^i(t) \tag{1.24}$$

An interaction of point charged particle with electromagnetic field can be described using the pullback of vector potential on the world-line of particle I

$$\gamma^* A = \sum_{i=1}^{n} A_i(x(t)) \dot{x}^i(t) dt \in \Omega^1(I). \tag{1.25}$$

Example: Let $i_C : C \hookrightarrow \mathbb{R}^n$ be an embedding map then we can define the pullback map

$$i_C^* : \Omega^*(\mathbb{R}^n) \to \Omega^*(C) \ : \ \omega \mapsto i_C^* \omega = \omega|_C \tag{1.26}$$

which is simply a restriction of the form on $\mathbb{R}^n$ to submanifold $C \subset \mathbb{R}^n$.

The pullback map is the morphism between two DGAs, so it obeys the following relations

1. *Linearity*:
$$f^*(\alpha\omega_1 + \beta\omega_2) = \alpha f^*\omega_1 + \beta f^*\omega_2, \ \ \alpha, \beta \in \mathbb{R}.$$

2. *Multiplication*:
$$f^*(\omega \wedge \mu) = f^*\omega \wedge f^*\mu.$$

3. *Derivation*:
$$d(f^*\omega) = f^*(d\omega).$$

4. *Composition*
$$g^*(f^*\omega) = (f \circ g)^*\omega.$$

Remark: We can consider a diffeomorphism of $\mathbb{R}^n$ i.e. an invertible map $f : \mathbb{R}^n \to \mathbb{R}^n$ then the components of p-form transform as a covariant p-tensor

$$f^*\omega_{i_1 \dots i_p} = \omega_{j_1 \dots j_p} \frac{\partial f^{j_1}}{\partial x^{i_1}} \dots \frac{\partial f^{j_p}}{\partial x^{i_p}}. \tag{1.27}$$

Definition (*Homological algebra language*): Ω^* is contravriant functor from category of Euclidean spaces $\mathbb{R}^n$ and smooth maps to category of differential graded algebras.

1.5 Forms on manifold

Definition: A differential form ω on manifold M is a collection of forms ω_α for each U_α in open cover $\{U_\alpha\}$, which are compatible in the following sense: If i and j are inclusions

$$i : U_\alpha \cap U_\beta \hookrightarrow U_\alpha, \ \ j : U_\alpha \cap U_\beta \hookrightarrow U_\beta \tag{1.28}$$

then $i^*\omega_\alpha = j^*\omega_\beta$ in $\Omega^*(U_\alpha \cap U_\beta)$.

Example: Let us describe form on S^1 using two open sets $U_0 = (-\epsilon, \pi + \epsilon)$ and $U_1 = (-\epsilon, \pi + \epsilon)$ as an open cover. The intersection $U_1 \cap U_0$ is a disjoint union of two open intervals U_{01} and U_{10}. Let us use coordinates x_0 and x_1 on U_0 and U_1, so the 1-form

$$\alpha = \begin{cases} \alpha_0 = \alpha_0(x_0)dx_0 \in \Omega^1(U_0) \\ \alpha_1 = \alpha_1(x_1)dx_1 \in \Omega^1(U_1). \end{cases} \tag{1.29}$$

The intersection component $U_{01} = I_t = (\pi - \epsilon, \partial + \epsilon)$ with coordinate t is embedded into U_0 and U_1 via

$$i_t : I_t \to U_0 \ : \ t \mapsto x^0(t) = t, \ \ j_t : I_t \to U_1 \ : \ t \mapsto x^1(t) = t - \pi. \tag{1.30}$$

The compatibility of restrictions

$$i_t^* \alpha_0 = \alpha_0(t)dt = j_t^* \alpha_1 = \alpha_1(t - \pi)dt \in \Omega^1(I_t). \tag{1.31}$$

Similar analysis for other component of the intersection $U_{10} = I_s = (-\epsilon, \epsilon)$

$$i_t : I_s \to U_0 \ : \ s \mapsto x^0(s) = s, \ \ j_s : I_t \to U_1 \ : \ t \mapsto x^1(s) = s + \pi \tag{1.32}$$

The compatibility of restrictions

$$i_s^* \alpha_0 = \alpha_0(s)ds = j_s^* \alpha_1 = \alpha_1(s + \pi)ds \in \Omega^1(I_s). \tag{1.33}$$

Both compatibility conditions relate $\alpha_0(s)$ and $\alpha_1(s)$ what leads to constraints for $\alpha_1(s)$ itself

$$\alpha_1(t - \pi) = \alpha_0(t) = \alpha_1(t + \pi) \ \Longrightarrow \alpha_1(t + 2\pi) = \alpha_1(t). \tag{1.34}$$

Summarizing all consistency arguments we can describe generic 1-form on S^1

$$\alpha = \begin{cases} \alpha_0 = \alpha(x_0)dx_0 \in \Omega^1(U_0) \\ \alpha_1 = \alpha(x_1 + \pi)dx_1 \in \Omega^1(U_1) \end{cases} \ \alpha(x) = \alpha(x + 2\pi). \tag{1.35}$$

Remark: We can describe vector fields on manifold M as global sections of tangent bundle TM, with fiber over point p being tangent space T_pM. The tangent space $V = T_pM$ is the vector space, so we can canonically define its dual $V^* = T_p^*M$. Similarly, the gluing data for tangent bundle can be canonically dualized, so it becomes gluing data for cotangent bundle T^*M. For a vector space V we can define vector space of k-forms $\Lambda^k V$, as well as the linear maps on V can be extended to linear maps on $\Lambda^k V$. The collection of vector spaces $\Lambda^k T_p^* M$ with gluing maps inherited from the ones on TM allows us to define $\Lambda^k T^* M$-bundle on M. Differential k-forms on M are global sections of $\Lambda^k T^* M$ bundle over M.

1.6 Integration of differential forms

We can integrate the p-form ω on M over the p-dimensional submanifold $C \subset M$ to get a real number i.e. there is a map

$$\Omega^p(M) \times C_p(M) \to \mathbb{R} \ : \ (\omega, C) \mapsto \int_C \omega. \tag{1.36}$$

Zero forms $\Omega^0(M)$ are just functions on M while the 0d submanifolds are collection of points $C = \{p_1\} \sqcup ... \sqcup \{p_k\}$. The integration formula is the evaluation of the function on a point.

$$\int_p \omega^0 = \omega(p), \ \ \int_C \omega^0 = \sum_{i=1}^{k} \omega^0(p_i). \tag{1.37}$$

The case of 1-forms is more interesting. Let us restrict our attention to the connected 1d submanifolds, also knows as curves. We can describe curve γ as a map

$$\gamma : I = [0,1] \to M. \tag{1.38}$$

We can define the integral of a 1-form ω over the curve γ as a limit of the Riemann sum

$$\int_\gamma \omega = \lim_{\Delta \to 0} \sum_{\alpha=1}^{N} \omega(p_\alpha; \xi_\alpha)|\Delta_\alpha|. \tag{1.39}$$

with summation data being defined in the following way. We split $I = [0,1]$ into parts

$$\Delta_\alpha = [t_\alpha, t_{\alpha+1}], \ \ I = \cup \Delta_\alpha. \tag{1.40}$$

The point p_α is the image of t_α, while ξ_α is the tangent vector at t_α

$$p_\alpha = \gamma(t_\alpha), \ \ \xi_\alpha = \dot{\gamma}(t_\alpha) \in T_{p_\alpha} M. \tag{1.41}$$

We can take the continuous $\Delta \to 0$ limit in terms of 1d integral over I

$$\int_\gamma \omega = \lim_{\Delta\to 0} \sum_{\alpha=1}^{N} \omega(p_\alpha; \xi_\alpha)|\Delta_\alpha| = \int_0^1 dt\ \omega_i(\gamma(t))\dot{\gamma}^i(t). \tag{1.42}$$

Let us notice that right-handside of the integral is identical to the

$$\int_I \gamma^*\omega = \int_0^1 \gamma^*\omega(t; \partial_t)dt. \tag{1.43}$$

Furthermore, let us notice that

$$\gamma^*\omega \in \Omega^1(I) = \Omega^{top}(I) \tag{1.44}$$

Any top degree differential form on interval I with coordinate t we can represent as

$$\gamma^*\omega = (\gamma^*\omega)_t(t)dt, \tag{1.45}$$

so we can use the definition of the integral of function over the interval to define the 1-form integral

$$\int_\gamma \omega = \int_I \gamma^*\omega = \int_I (\gamma^*\omega)_t(t) = \int_0^1 dt\ (\gamma^*\omega)_t(t). \tag{1.46}$$

We can generalize the 1d integral formula to the higher dimensional integral. Suppose we want to evaluate the integral of a p-form ω on manifold M over p-chain C on M. We can represent C as the image of the p-chain $\Delta^p \subset \mathbb{R}^p$ under the trivialization map $\phi : \mathbb{R}^p \to M$ and use the pullback map to rewrite the integral in the form

$$\int_C \omega = \int_{\phi(\Delta^p)} \omega = \int_{\Delta^p} \phi^*\omega \tag{1.47}$$

The pullback $\phi^*\omega$ is the top form on $\Delta^p \subset \mathbb{R}^p$, so can identify it with the function using the canonical volume form on $\mathbb{R}^p$

$$\phi^*\omega = \phi^*\omega_{top} \cdot \mathrm{vol}_{\mathbb{R}^p} \tag{1.48}$$

We can evaluate the last integral as integral of function $\phi^*\omega_{top}$ over region $\Delta^p \subset \mathbb{R}^p$

$$\int_C \omega = \int_{\phi(\Delta^p)} \omega = \int_{\Delta^p} \phi^*\omega = \int_{\Delta^p\subset\mathbb{R}^p} d\mu_{\mathbb{R}^p}\ \ \phi^*\omega_{top}. \tag{1.49}$$

Using the definition for the integral of a differential form over the chain of the same dimension we can prove the *Newton-Leibniz-Gauss-Green-Ostrogradskii-Stokes-Poincare formula*

$$\int_{\partial C} \omega = \int_C d\omega \tag{1.50}$$

Example: Let us consider 0-form $\omega = f(x)$ and smooth curve $\gamma : I \to M : t \mapsto x^i(t)$. Then the integral

$$\begin{aligned}\int_\gamma d\omega &= \int_I \gamma^* d\omega = \int_I \gamma^*(\partial_i f dx^i) = \int_I \partial_i f(x(t))\dot{x}^i(t)dt \\ &= \int_0^1 dt\ \partial_i f(x(t))\dot{x}^i(t) = \int_0^1 dt\ \partial_t f(x(t)) = f(x(1)) - f(x(0)) \\ &= [\partial\gamma]^*\omega(1) - [\partial\gamma]^*\omega(0) = \int_{\partial I} [\partial\gamma]^*\omega = \int_{\partial\gamma} \omega\end{aligned} \tag{1.51}$$

1.7 Interior product ι_v

Definition: Let $v \in Vect(M)$ and $\Omega^k(M)$ then we can define and *interior product*

$$\iota_v : \Omega^k(M) \times Vect(M) \to \Omega^{k-1}(M) : \quad (\omega, v) \mapsto \iota_v\omega, \tag{1.52}$$

via the following relation

$$\iota_v\omega(v_1, ..., v_{k-1}) = \sum_{r=1}^{k} (-1)^{r-1}\omega(v_1, v_2, .., v_{r-1}, v, v_{r+1}, ..., v_{k-1}). \tag{1.53}$$

For every v the interior product ι_v is the derivation of the algebra of differential forms, so it obey the relations

1. *Linearity in the form*:

$$\iota_v(\alpha\omega_1 + \beta\omega_2) = \alpha\iota_v\omega_1 + \beta\iota_v\omega_2, \quad \alpha, \beta \in \mathbb{R}.$$

2. *Linearity in the vector field*:

$$\iota_{v+w}\omega = \iota_v\omega + \iota_w\omega.$$

3. *Graded Leibnitz*: Let $\omega \in \Omega^k(M)$ and $\mu \in \Omega^m(M)$

$$\iota_v(\omega \wedge \mu) = \iota_v\omega \wedge \mu + (-1)^k\omega \wedge \iota_v\mu.$$

4. *Antisymmetry*:

$$\iota_v \iota_w \omega = -\iota_w \iota_v \omega.$$

1.8 Lie derivative $\mathcal{L}_v$

Definition: *Diffeomorphism* $\Phi : M \to M$ is an invertible map from smooth manifold M to itself. Diffeormorphisms on M form a (Lie) group $Diff(M)$ with group multiplication being composition of maps, inverse being the inverse map.

The Lie algebra of $Diff(M)$ is a Lie algebra of vector fields on M

$$T_e(Diff(M), \circ) = (Vect(M), [\cdot, \cdot]) \tag{1.54}$$

with bracket being the commutator of vector fields. In local coordinates $x^1, ..., x^n$ the commutator

$$[v, w]^i \partial_i = v^j \partial_j w^i \partial_i - w^j \partial_j v^i \partial_i. \tag{1.55}$$

An exponentiation of the element $v \in Vect(M)$ is the one parameter subgroup of diffeomorphisms $\Phi_t : M \to M$. Geometrically such diffeomorphisms describe flow along the vector field i.e.

$$\Phi_t : x \to x'(x, t) \quad \dot{x}' = v(x'), \ \ x'(x, 0) = x. \tag{1.56}$$

It is useful to study the behavior of the geometric objects under such map.

Definition: A Lie derivative is a map $\mathcal{L}_v : \Omega^k(M) \to \Omega^k(M)$ defined via

$$\mathcal{L}_v \omega = \lim_{t \to 0} \frac{\Phi_t^* \omega - \omega}{t}. \tag{1.57}$$

Example: In case of functions

$$\begin{aligned} \mathcal{L}_v f(p) &= \lim_{t \to 0} \frac{\Phi_t^* f(p) - f(p)}{t} = \lim_{t \to 0} \frac{f(x(t)) - f(p)}{t} \\ &= \lim_{t \to 0} \frac{f(p) + t v^i \partial_i f(p) - f(p) + \mathcal{O}(t^2)}{t} = v^i \partial_i f(p). \end{aligned} \tag{1.58}$$

Lie derivative is the derivation of the algebra of differential forms so it obey the following relations:

1. *Cartan formula*:

$$\mathcal{L}_v = d\iota_v + \iota_v d$$

2. *Linearity in the form*:

$$\mathcal{L}_v(\alpha\omega_1 + \beta\omega_2) = \alpha\mathcal{L}_v\omega_1 + \beta\mathcal{L}_v\omega_2, \quad \alpha, \beta \in \mathbb{R}.$$

3. *Linearity in the vector field*:

$$\mathcal{L}_{v+w}\omega = \mathcal{L}_v\omega + \mathcal{L}_w\omega.$$

4. *Leibnitz rule*: Let $\omega \in \Omega^k(M)$ and $\mu \in \Omega^m(M)$

$$\mathcal{L}_v(\omega \wedge \mu) = \mathcal{L}_v\omega \wedge \mu + \omega \wedge \mathcal{L}_v\mu.$$

5. *Lie algebra representation*:

$$(\mathcal{L}_v\mathcal{L}_w - \mathcal{L}_w\mathcal{L}_v)\omega = \mathcal{L}_{[v,w]}\omega.$$

   and

$$(\mathcal{L}_v\iota_w - \iota_w\mathcal{L}_v)\omega = \iota_{[v,w]}\omega.$$

Remark: Lie derivative $\mathcal{L}$ is the (infinite-dimensional) representation of the algebra $Vect(M)$ on differential forms.

1.9 Cartan formula via homotopy*

The infinitesimal diffeomorphism Φ_t is homotopic to the indentity map. Let γ be a p-dimensional submanifold of M then there exists a homotopy

$$H_t : X \times [0,t] \to X \ : \ (x,s) \mapsto (x^i + sv^i(x)) \tag{1.59}$$

such that

$$\Phi_t(\gamma) - \gamma = \partial(H_t([0,t],\gamma)) - H_t([0,t],\partial\gamma) \tag{1.60}$$

Let us integrate a differential p-form ω over the the relation above i.e.

$$\int_{\Phi_t(\gamma)-\gamma} \omega = \int_{\partial(H_t\gamma)+H_t(\partial\gamma)} \omega \tag{1.61}$$

and use various properties of the integrals

$$\int_{\Phi_t(\gamma)-\gamma} \omega = \int_\gamma (\Phi_t^*\omega - \omega) = \int_{\partial(H_t\gamma)+H_t(\partial\gamma)} \omega = \int_{H_t\gamma} d\omega + \int_{H_t\partial\gamma} \omega \tag{1.62}$$

The left integral can be used to evaluate the Lie derivative

$$\int_\gamma \mathcal{L}_v\omega = \lim_{t\to 0} \frac{1}{t} \int_\gamma (\Phi_t^*\omega - \omega) \tag{1.63}$$

so let us look at the integrals to the right

$$\int_{H_t\gamma} d\omega = \int_{[0,t]\times\gamma} H_t^* d\omega \tag{1.64}$$

The pullback map

$$\begin{aligned} H_t^*\alpha &= H_t^*\left(\frac{1}{(p+1)!}\alpha_{i_1\ldots i_{p+1}}(x)dx^{i_1}\ldots dx^{i_{p+1}}\right) \\ &= \frac{1}{(p+1)!}\alpha_{i_1\ldots i_{p+1}}(x+sv)d(x^{i_1}+sv^{i_1})\ldots d(x^{i_{p+1}}+sv^{i_{p+1}}) \\ &= \frac{1}{(p+1)!}\alpha_{i_1\ldots i_{p+1}}(x+sv)(dx^{i_1}+v^{i_1}ds+s\partial_j v^{i_1}dx^j)\ldots(dx^{i_{p+1}}+v^{i_{p+1}}ds+s\partial_j v^{i_{p+1}}dx^j) \\ &= \omega^{p+1} + ds\wedge\iota_v\alpha + \mathcal{O}(s) \end{aligned} \tag{1.65}$$

We want to integrate $H_t^* d\omega^p$ over $\gamma\times[0,t]$ so the only nonzero contribution is from the middle term i.e.

$$\int_{H_t\gamma} d\omega = \int_{[0,t]\times\gamma} H_t^* d\omega = \int_0^t ds \int_\gamma \iota_v d\omega + \mathcal{O}(t^2) = t\int_\gamma \iota_v d\omega + \mathcal{O}(t^2) \tag{1.66}$$

Similarly

$$\int_{H_t\partial\gamma} \omega = \int_{[0,t]\times\partial\gamma} H_t^*\omega = \int_0^t ds \int_{\partial\gamma} \iota_v\omega + \mathcal{O}(t^2) = t\int_{\partial\gamma} \iota_v\omega + \mathcal{O}(t^2) \tag{1.67}$$

which allows us to arrive into relation

$$\int_\gamma \mathcal{L}_v\omega = \int_\gamma \iota_v d\omega + \int_{\partial\gamma} \iota_v\omega = \int_\gamma (\iota_v d + d\iota_v)\omega \tag{1.68}$$

Our choice of γ and ω are arbitrary so the relation above should hold for all γ and ω what concludes the proof of Cartan formula.