Supersymmetric Quantum Mechanics and Morse theory

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Lectures 19-20: 1d Sigma model v0

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1 1-dimensional σ-model

Our goal is to study the topology of the manifold M, so let us consider a point particle moving on it. In mathematical terms we want to study maps $I \to M : t \mapsto x^i(t)$.

1.1 Classical theory

The naive generalization of the Newton equations

$$\ddot{x}^i = 0 \tag{1.1}$$

does not describe the consistent dynamics, since it is not invariant under the change of coordinate chart. Indeed, let us consider coordinate transformation $x_\alpha = g_{\alpha\beta}(x_\beta)$ between open charts U_α and U_β. The velocity in new coordinates

$$\dot{x}^i_\alpha = \sum_j \frac{\partial x^i_\alpha}{\partial x^j_\beta} \dot{x}^j = \sum_j \frac{\partial g^i_{\alpha\beta}}{\partial x^j_\beta} \dot{x}^j_\beta \tag{1.2}$$

The transformation law for the velocity is consistent with trivial dynamics

$$\dot{x}^i_\alpha = 0 \tag{1.3}$$

describing a point particle sitting at a same point at all times. The second derivative equation transforms under the change of coordinates via

$$\ddot{x}^i_\alpha = \partial_t \dot{x}^i_\alpha = \partial_t \left(\frac{\partial g^i_{\alpha\beta}}{\partial x^j_\beta} \dot{x}^j_\beta \right) = \frac{\partial g^i_{\alpha\beta}}{\partial x^j_\beta} \ddot{x}^j_\beta + \frac{\partial^2 g^i_{\alpha\beta}}{\partial x^j_\beta \partial x^k_\beta} \dot{x}^j_\beta \dot{x}^k_\beta \tag{1.4}$$

Indeed the $\ddot{x}^i_\alpha = 0$ does not equivalent to the $\ddot{x}^i_\beta = 0$. Readers, familiar with general relativity, know that the proper generalization of the particle moving on straight line with constant velocity to the curved space is the geodesic equation

$$\ddot{x}^i_\alpha + \Gamma^i_{jk}\dot{x}^j_\alpha \dot{x}^k_\alpha = 0 \tag{1.5}$$

with Γ being Christoffel symbol for Levi-Civita connection of the Riemann metric g on X

$$\Gamma^i_{jk} = \frac{1}{2}\sum_l g^{il}(\partial_j g_{lk} - \partial_k g_{jl} - \partial_l g_{jk}) = \frac{1}{2}g^{il}(g_{lk,j} - g_{jl,k} - g_{jk,l}) \tag{1.6}$$

The coordinate transformation for Christoffel symbol

$$\Gamma^i_{\alpha jk} = \Gamma^{i'}_{\beta j' k'} \frac{\partial x^i_\alpha}{\partial x^{i'}_\beta} \frac{\partial x^{j'}_\beta}{\partial x^j_\alpha} \frac{\partial x^{k'}_\beta}{\partial x^k_\alpha} - \frac{\partial^2 x^i_\alpha}{\partial x^{j'}_\beta \partial x^{k'}_\beta} \frac{\partial x^{j'}_\beta}{\partial x^j_\alpha} \frac{\partial x^{k'}_\beta}{\partial x^k_\alpha} \tag{1.7}$$

follows from the coordinate transform of the metric

$$g^\alpha_{ij} dx^i_\alpha dx^j_\alpha = g^\beta_{ij} dx^i_\beta dx^j_\beta \tag{1.8}$$

Geodesic is the curve with minimal length between two fixed points, i.e extremum of the length functional

$$L = \int_0^1 d\tau \sqrt{\dot{x}^i \dot{x}^j g_{ij}(x)} \tag{1.9}$$

where g_{ij} is the metric on the manifold. Using metric g_{ij} we can directly generalize the extremal action principle for the point particle

$$S = \frac{1}{2}\int_I dt\ \dot{x}^i \dot{x}^j g_{ij}(x) \tag{1.10}$$

The action above has extrema on geodesics as well since square root is the monotonic function.

1.2 Phase space

Let us perform the Legendre transform

$$\begin{aligned} p_i = p_i(x, \dot{x}) = \frac{\partial L}{\partial \dot{x}^i} = g_{ij}(x)\dot{x}^j, \\ H(p, x) = p_i \dot{x}^i - L(x, \dot{x}) = \frac{1}{2} g^{ij}(x) p_i p_j \end{aligned} \tag{1.11}$$

with g^{ij} being inverse metric i.e.

$$g_{ij}(x)g^{jk}(x) = \delta_i^k, \tag{1.12}$$

The local description of the phase space near some point $x \in X$ consist of coordinates x^i on some open chart and momentum vector p_i. The coordinate transformation acts on p_i in the following way

$$x^i = x^i(y), \quad p_i = g_{ij}(x)\dot{x}^j = g_{ij}(x)\frac{\partial x^j}{\partial y^k}\dot{y}^k = \frac{\partial y^l}{\partial x^i}\tilde{g}_{kl}\dot{y}^k = \frac{\partial y^l}{\partial x^i}\tilde{p}_l \tag{1.13}$$

where we used the fact that the distance between points is invariant i.e.

$$g_{ij}dx^i dx^j = g_{ij}\frac{\partial x^i}{\partial y^k}\frac{\partial x^j}{\partial y^l}dy^k dy^l = \tilde{g}_{kl}dy^k dy^l \tag{1.14}$$

Observation: The phase space for the particle moving on X is *cotangent bundle* of X denoted as T^*X, which is symplectic manifold (T^*X, ω^{can}) with canonical symplectic structure.

Definition: Let X be an n-dimensional smooth manifold and denote $M = T^*X$ its cotangent bundle. Let

$$\pi : \ T^*X \to X, \ \ (x, \xi) \mapsto x \tag{1.15}$$

be the bundle projection map. T^*X is a smooth manifold of dimension $2n$ and any local coordinate system on X induces a coordinate system on T^*X as follows: From any coordinate patch $(U; x^1, ..., x^n)$ of X we can construct, for $x \in U$ a basis $\{\partial_1, ..., \partial_n\}$ of T_xX, which depends smoothly on x. We denote the dual basis in T_x^*M by $\{dx^1, ..., dx^n\}$. This basis gives the induced coordinate system, $(x^1, ..., x^n, p_1, ..., p_n)$ on $M_U = \pi^{-1}(U)$. Namely, of $p \in T_x^*X$ then

$$p = \sum_{i=1}^{n} p_i(dx^i)_x \tag{1.16}$$

In this coordinates coordinates there is a canonical 1-form

$$\theta = \sum p_i dx^i. \tag{1.17}$$

The symplectic form on T^*X is

$$\omega^{can} = d\theta = \sum dp_i \wedge dx^i \tag{1.18}$$

1.3 Canonical quantization for cotangent bundle

The manifold structure on nontrivial topological space X is defined by trivializations $\phi_i : U_i \to \mathbb{R}^n$ over open covering $\{U_i\}$. On intersections $U_i \cap U_j$ we specify the transition functions $g_{ij} = \phi_i \circ \phi_j^{-1}$. The cotangent direction in T^*X is topologically trivial so we can complete any diffeomorphism $f : X \to X$ to the canonical transformation $F : T^*X \to T^*X$

$$f : x^i \to f^i(x), \quad F : (p_i, x^j) \to (J_i^j(x) p_j, f^j(x)), \quad J_j^i(x) \partial_i f^k(x) = \delta_j^k \tag{1.19}$$

so that

$$\omega = dp_i \wedge dx^i \to F^*\omega = d(J_i^j p_j) \wedge df^i = \partial_k J_i^j p_j dx^k \wedge df^i + J_i^j dp_j \wedge df^i \tag{1.20}$$

The change in symplectic potential

$$F^*\theta - \theta = P_i dX^i - p_i dx^i = p_j J_i^j df^i - p_i dx^i = p_j J_i^j \partial_k f^i dx^k - p_i dx^i = 0 \tag{1.21}$$

Thus the wave-functions in new coordinates

$$\psi(X) = \psi(X(x)) \tag{1.22}$$

Such transformation law is the one for the functions on X, so we can conclude that

$$\mathcal{H}_{(T^*X,\, \omega^{can})} = C^\infty(X) \tag{1.23}$$

The Hilbert space requires paring

$$\langle \psi, \phi \rangle = \int \sqrt{g} d^n x \; \psi(x) \phi^*(x) \tag{1.24}$$

According to the canonical quantization

$$\{p_k, x^j\} = \delta_k^j \quad \Rightarrow [\hat{p}_k, \hat{x}^j] = -i\hbar \delta_k^j \tag{1.25}$$

Let us recall that the coordinate transformation on X leads to canonical transformation

$$(p_i, x^j) \to (J_i^j(x) p_j, f^j(x)) \tag{1.26}$$

which transforms p into product of p and some function of x. Hence we should expect possible corrections to the momentum operator. The quantum operators $\hat{x}^i$ and $\hat{p}_i$

$$\hat{x}^i = x^i\cdot, \quad \hat{p}_j = -i\hbar g^{-1/4}\frac{\partial}{\partial x^j}g^{1/4} = -i\hbar\partial_j - \frac{i\hbar}{4}\partial_j \log g \tag{1.27}$$

are hermitian operators with respect to the inner product

$$\langle(\hat{x}^j)^\dagger\psi, \phi\rangle = \langle\psi, \hat{x}^j\phi\rangle = \int_X d^n x\ x^j\psi^*(x)\phi(x)\sqrt{g} = \langle\hat{x}^j\psi, \phi\rangle \tag{1.28}$$

$$\begin{aligned}\langle(\hat{p}_j)^\dagger\psi, \phi\rangle = \langle\psi, \hat{p}_j\phi\rangle &= \int_X d^n x\ \psi^*(x)[-i\hbar g^{-1/4}\partial_j g^{1/4}\phi(x)]\sqrt{g} \\ &= -i\hbar\int_X d^n x\ \psi^*(x)\partial_j[g^{1/4}\phi(x)]g^{1/4} = -i\hbar\int_X d^n x\ \partial_j[g^{1/4}\psi^*(x)]\phi(x)g^{1/4} \\ &= \int_X d^n x\ [-i\hbar g^{-1/4}\partial_j g^{1/4}\psi]^*(x)\phi(x)\sqrt{g} = \langle\hat{p}_j\psi, \phi\rangle\end{aligned} \tag{1.29}$$

Remark: Operators $\hat{p}_i$ are covariant derivatives on half-densities

$$-i\hbar\nabla_j(g^{1/4}\psi) = g^{1/4}\left(-i\hbar\partial_j\psi - \frac{i\hbar}{4}\psi\partial_j\log g\right) \tag{1.30}$$

The quantum Hamiltonian depends on ordering, since $\hat{p}, \hat{x}$ does not commute.The most general hermitian operator, invariant under the coordinate transformation is of the form

$$\hat{H} = -\frac{1}{2}\Delta + \xi R, \quad \Delta = \frac{1}{\sqrt{g}}\partial_i g^{ij}\sqrt{g}\partial_j, \quad R = g^{ij}R_{ij} \tag{1.31}$$

for some real constant ξ. Indeed we can evaluate

$$\begin{aligned}\langle H^\dagger\psi, \phi\rangle = \langle\psi, \hat{H}\phi\rangle &= \int_X \sqrt{g}d^n x\ \psi^*\left(-\frac{1}{2}\Delta + \xi R\right)\phi \\ &= \int_X d^n x\ \psi^*\left(-\frac{1}{2}\partial_i(g^{ij}\sqrt{g}\partial_j\phi) + \xi R\sqrt{g}\phi\right) \\ &= \frac{1}{2}\int_X d^n x\ \sqrt{g}\left(g^{ij}\partial_i\psi^*\partial_j\phi + \xi R\psi^*\phi\right) = \langle H\psi, \phi\rangle\end{aligned} \tag{1.32}$$

The coefficient ξ being different for different quantization approaches

- Canonical quantization in λ-ordering: $\xi = \frac{2\lambda^2+1}{6}$
- Lagrangian Path integral $\xi = 0$
- Geometric quantization: $\xi = \frac{1}{12}$

- $N = 2$ SUSY, Hodge-de Rham theory $\xi = 0$

$$2H = \{Q, Q^\dagger\} = \{d, d^*\} = \Delta \tag{1.33}$$

- $N = 1$ SUSY, equivalently Dirac operator representation $\xi = 1/4$

$$2\hat{H} = \hat{Q}^2 = (\gamma^i D_i)^2 = \Delta + \frac{1}{2}R \tag{1.34}$$