Realistic error bounds for asymptotic expansions arising from integrals via resurgence

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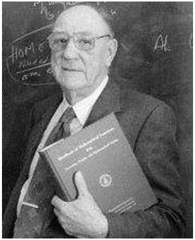
Olver's differential equation theory



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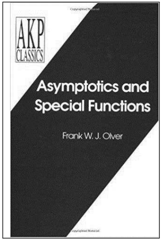

Figure 2: *Olver's book*

FRANK W. J. OLVER developed a general and rigorous theory for asymptotic expansions of solutions of linear second-order differential equations, summarized in his famous 1974 monograph ***Asymptotics and Special Functions***. Olver's theory provides sharp error bounds for the expansions, as well as recurrences for their coefficients.

On the other hand, establishing error bounds for asymptotic expansions arising from integrals has been a long-standing problem. In this talk, we shall discuss recent progress in this subject.

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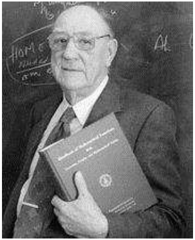

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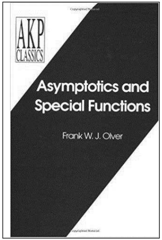

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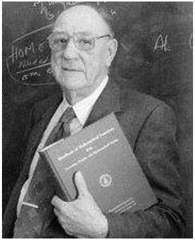

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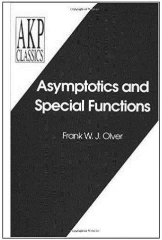

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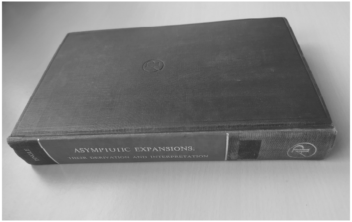
Dingle's interpretative theory

Figure 3: *Robert B. Dingle*



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In a series of papers and in a research monograph, ***Asymptotic Expansions: Their Derivation and Interpretation***, published in 1973, the theoretical physicist ROBERT B. DINGLE incorporated earlier and new, original ideas into a comprehensive theory which had a substantial impact on later developments in modern asymptotics. Dingle's intuition was that asymptotic expansions are exact coded representations of functions, and the main task of asymptotics is to decode them.

Dingle's basic terminants

The building blocks of Dingle's theory are the so-called basic terminants. The ***first basic terminant*** of order p and argument w is defined by

$$\Lambda_p(w) \stackrel{\text{def}}{=} w^p \mathrm{e}^w \Gamma(1-p,w) = \frac{1}{\Gamma(p)} \int_0^{+\infty} \frac{t^{p-1}\mathrm{e}^{-t}}{1+t/w} \mathrm{d}t,$$

for $\mathrm{Re}(p) > 0$ and $|\arg w| < \pi$, and by analytic continuation in w to the whole Riemann surface $\widehat{\mathbb{C}}$ of the logarithm.

Similarly, the ***second basic terminant*** of order p and argument w is defined by

$$\Pi_p(w) \stackrel{\text{def}}{=} \tfrac{1}{2}(\Lambda_p(w\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}) + \Lambda_p(w\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}})) = \frac{1}{\Gamma(p)} \int_0^{+\infty} \frac{t^{p-1}\mathrm{e}^{-t}}{1+(t/w)^2} \mathrm{d}t,$$

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The asymptotic expansion of the modified Bessel function

We shall briefly demonstrate Dingle's method through the example of the modified Bessel function of the second kind of order 0. It was shown by Ernst E. Kummer in 1837 that the modified Bessel function of the second kind of order 0 has an asymptotic expansion of the form

$$K_0(z) \sim \sqrt{\frac{\pi}{2z}} \mathrm{e}^{-z} \sum_{n=0}^{\infty} \frac{a_n}{z^n},$$

as $z \to \infty$ in the sector $|\arg z| \le \frac{3\pi}{2} - \delta \ (< \frac{3\pi}{2})$, with

$$a_n \stackrel{\text{def}}{=} (-1)^n \frac{(2n)!^2}{32^n n!^3}.$$

Dingle first noted that

$$a_n \sim \frac{(-1)^n}{\pi} \frac{\Gamma(n)}{2^n} \left(a_0 + \frac{2a_1}{n-1} + \frac{2^2 a_2}{(n-1)(n-2)} + \dots \right)$$

as $n \to +\infty$. The re-appearance of the early coefficients in this asymptotic expansion is a manifestation of ***resurgence***.

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Dingle's interpretation

Then the divergent tail of the series is asymptotically given by

$$\sum_{n=N}^{\infty} \frac{a_n}{z^n} \sim \sum_{n=N}^{\infty} \frac{1}{z^n} \frac{(-1)^n}{\pi} \frac{\Gamma(n)}{2^n} \left(a_0 + \frac{2a_1}{n-1} + \frac{2^2 a_2}{(n-1)(n-2)} + \dots \right)$$
$$= \frac{1}{\pi} \sum_{n=N}^{\infty} \frac{(-1)^n}{z^n} \left(a_0 \frac{\Gamma(n)}{2^n} + a_1 \frac{\Gamma(n-1)}{2^{n-1}} + a_2 \frac{\Gamma(n-2)}{2^{n-2}} + \dots \right)$$

for large values of N. Introducing the Euler integral representation for the gamma function, changing the order of summation and integration and employing the summation formula for the geometric series yields

$$\sum_{n=N}^{\infty} \frac{a_n}{z^n} \sim \frac{(-1)^N}{\pi} \frac{\Gamma(N)}{(2z)^N} \left(a_0 \Lambda_N(2z) + \frac{2a_1 \Lambda_{N-1}(2z)}{N-1} + \frac{2^2 a_2 \Lambda_{N-2}(2z)}{(N-1)(N-2)} + \dots \right)$$

for large N. This is Dingle's interpretation (or termination) of the divergent tail of Kummer's expansion, the first step towards the theory of ***hyperasymptotics***.

We will show how resurgence and Dingle's terminants can be used to derive sharp bounds for the remainder terms of asymptotic expansions arising from integral representations, instead of approximating them.

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$$=\frac{1}{\pi}\sum_{n=N}^{\infty}\frac{(-1)^n}{z^n}\left(a_0\frac{\Gamma(n)}{2^n}+a_1\frac{\Gamma(n-1)}{2^{n-1}}+a_2\frac{\Gamma(n-2)}{2^{n-2}}+\ldots\right)$$

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Cauchy–Heine representation

Suppose that $|\arg z| < \pi$. By Cauchy's integral formula, we have

$$z^{-\frac{1}{2}} e^{z} K_0(z) = \frac{1}{2\pi i} \oint_{\Gamma} \frac{s^{-\frac{1}{2}} e^{s} K_0(s)}{s - z} ds.$$

Figure 5: *The "keyhole" contour.*

Here Γ is a "keyhole" contour omitting the non-positive real axis. If $|\arg s| \leq \pi$, the integrand is $\mathcal{O}(s^{-2})$ as $s \to \infty$ and is $\mathcal{O}(s^{-1/2} \log s)$ as $s \to 0$. Therefore, the integrals along the small and large circles vanish as their radii tend to zero and infinity, respectively.

Consequently, we deduce

$$z^{-\frac{1}{2}} e^{z} K_0(z) = \frac{1}{2\pi i} \int_{\infty e^{\pi i}}^{0} \frac{s^{-\frac{1}{2}} e^{s} K_0(s)}{s - z} ds + \frac{1}{2\pi i} \int_{0}^{\infty e^{-\pi i}} \frac{s^{-\frac{1}{2}} e^{s} K_0(s)}{s - z} ds.$$

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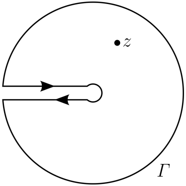

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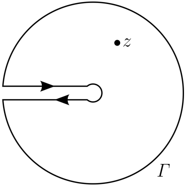

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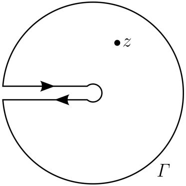

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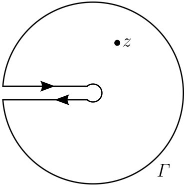

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Using a simple change of variables, we derive

$$z^{-\frac{1}{2}}e^{z}K_0(z) = \frac{1}{2\pi}\int_0^{+\infty}\frac{t^{-\frac{1}{2}}e^{-t}K_0(te^{\pi i})}{t+z}dt + \frac{1}{2\pi}\int_0^{+\infty}\frac{t^{-\frac{1}{2}}e^{-t}K_0(te^{-\pi i})}{t+z}dt.$$

This can be further simplified by an application of the connection formula $2K_0(t) = K_0(te^{\pi i}) + K_0(te^{-\pi i})$:

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provided $|\arg z| < \pi$. Equivalently,

$$K_0(z) = \sqrt{\frac{\pi}{2z}}e^{-z}\left(\frac{1}{\pi}\sqrt{\frac{2}{\pi}}\int_0^{+\infty}\frac{t^{-\frac{1}{2}}e^{-t}K_0(t)}{1+t/z}dt\right),$$

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$$K_0(z) = \sqrt{\frac{\pi}{2z}}\mathrm{e}^{-z}\left(\frac{1}{\pi}\sqrt{\frac{2}{\pi}}\int_0^{+\infty}\frac{t^{-\frac{1}{2}}\mathrm{e}^{-t}K_0(t)}{1+t/z}\mathrm{d}t\right),$$

provided $|\arg z| < \pi$.

Cauchy–Heine representation

Using a simple change of variables, we derive

$$z^{-\frac{1}{2}}\mathrm{e}^{z}K_0(z) = \frac{1}{2\pi}\int_0^{+\infty}\frac{t^{-\frac{1}{2}}\mathrm{e}^{-t}K_0(t\mathrm{e}^{\pi\mathrm{i}})}{t+z}\mathrm{d}t + \frac{1}{2\pi}\int_0^{+\infty}\frac{t^{-\frac{1}{2}}\mathrm{e}^{-t}K_0(t\mathrm{e}^{-\pi\mathrm{i}})}{t+z}\mathrm{d}t.$$

This can be further simplified by an application of the connection formula $2K_0(t) = K_0(t\mathrm{e}^{\pi\mathrm{i}}) + K_0(t\mathrm{e}^{-\pi\mathrm{i}})$:

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Exact remainder

For any non-negative integer N, $t > 0$ and $|\arg z| < \pi$, it holds that

$$\frac{1}{1+t/z} = \sum_{n=0}^{N-1} (-1)^n \frac{1}{z^n} t^n + (-1)^N \frac{1}{z^N} \frac{t^N}{1+t/z}.$$

Substitution into the above integral formula yields

$$K_0(z) = \sqrt{\frac{\pi}{2z}} \mathrm{e}^{-z} \left(\sum_{n=0}^{N-1} \frac{a_n}{z^n} + R_N(z) \right),$$

with

$$a_n = (-1)^n \frac{1}{\pi} \sqrt{\frac{2}{\pi}} \int_0^{+\infty} t^{n-\frac{1}{2}} \mathrm{e}^{-t} K_0(t) \mathrm{d}t$$

and

$$R_N(z) = (-1)^N \frac{1}{z^N} \frac{1}{\pi} \sqrt{\frac{2}{\pi}} \int_0^{+\infty} \frac{t^{N-\frac{1}{2}} \mathrm{e}^{-t} K_0(t)}{1+t/z} \mathrm{d}t.$$

A simple estimation of $R_N(z)$ shows that these identifications are indeed correct. This is the ***Cauchy–Heine representation*** of the remainder term $R_N(z)$ in Kummer's expansion.

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Error bounds: Boyd's approach

In 1990, William G. C. Boyd constructed error bounds for the asymptotic expansion of $K_0(z)$ (and more generally, for $K_\nu(z)$ with $|\nu| < \frac{1}{2}$) using the Cauchy–Heine representation of the remainder term. He found

$$|R_N(z)| \le \frac{|a_N|}{|z|^N} \times \begin{cases} 1 & \text{if} \quad |\arg z| \le \frac{\pi}{2}, \\ |\csc(\arg z)| & \text{if} \quad \frac{\pi}{2} < |\arg z| < \pi. \end{cases}$$

For the range $\frac{\pi}{2} < |\arg z| \le \pi$, he also gave

$$|R_N(z)| \le 2\sqrt{N}\frac{1}{\pi}\frac{\Gamma(N)}{2^N}\frac{1}{|z|^N} \quad \left(\sim 2\sqrt{N}\frac{|a_N|}{|z|^N}\right).$$

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Exact remainder: Dingle kernel

We substitute the Laplace transform

$$K_0(t) = \mathrm{e}^{-t} \int_0^{+\infty} \mathrm{e}^{-ts} \frac{\mathrm{d}s}{\sqrt{s(2+s)}}$$

into the explicit formula for the remainder and change the order of integration. In this way we obtain

$$\begin{aligned} R_N(z) =& (-1)^N \frac{1}{z^N} \frac{1}{\pi} \sqrt{\frac{2}{\pi}} \int_0^{+\infty} \frac{t^{N-\frac{1}{2}} \mathrm{e}^{-t} K_0(t)}{1+t/z} \mathrm{d}t \\ =& (-1)^N \frac{1}{z^N} \frac{1}{\pi} \sqrt{\frac{2}{\pi}} \Gamma\left(N+\tfrac{1}{2}\right) \int_0^{+\infty} \Lambda_{N+\frac{1}{2}}(z(2+s)) \frac{\mathrm{d}s}{\sqrt{s}(2+s)^{N+1}}. \end{aligned}$$

By an appeal to analytic continuation, this formula is valid in the wider range $|\arg z| < \frac{3\pi}{2}$. In a similar manner

$$a_n = (-1)^n \frac{1}{\pi} \sqrt{\frac{2}{\pi}} \Gamma\left(n+\tfrac{1}{2}\right) \int_0^{+\infty} \frac{\mathrm{d}s}{\sqrt{s}(2+s)^{n+1}}.$$

To estimate $R_N(z)$, we require estimates for the basic terminant.

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Bounds for the basic terminants

Proposition (G. N., 2017)

If $p > 0$ and $\chi(p) \overset{\text{def}}{=} \sqrt{\pi}\Gamma\left(\frac{p}{2}+1\right)/\Gamma\left(\frac{p}{2}+\frac{1}{2}\right)$, then

$$|\Lambda_p(w)| \le \begin{cases} 1 & \text{if } |\arg w| \le \frac{\pi}{2}, \\ \min(|\csc(\arg w)|, \chi(p)+1) & \text{if } \frac{\pi}{2} < |\arg w| \le \pi, \\ \dfrac{\sqrt{2\pi p}}{|\cos(\arg w)|^p} + \chi(p) + 1 & \text{if } \pi < |\arg w| < \frac{3\pi}{2}, \end{cases}$$

and

$$|\Pi_p(w)| \le \begin{cases} 1 & \text{if } |\arg w| \le \frac{\pi}{4}, \\ \min\left(|\csc(2\arg w)|, \frac{1}{2}\chi(p)+1\right) & \text{if } \frac{\pi}{4} < |\arg w| \le \frac{\pi}{2}, \\ \dfrac{\sqrt{2\pi p}}{2\,|\sin(\arg w)|^p} + \frac{1}{2}\chi(p) + 1 & \text{if } \frac{\pi}{2} < |\arg w| < \pi. \end{cases}$$

As $p \to +\infty$, $\chi(p) \sim \sqrt{\frac{\pi}{2}\left(p+\frac{1}{2}\right)}$.

Improved error bounds

For any non-negative integer N and for $|\arg z| < \frac{3\pi}{2}$, we have

$$K_0(z) = \sqrt{\frac{\pi}{2z}} \mathrm{e}^{-z} \left(\sum_{n=0}^{N-1} \frac{a_n}{z^n} + R_N(z) \right),$$

where the remainder $R_N(z)$ satisfies the estimates

$$|R_N(z)| \le \frac{|a_N|}{|z|^N} \times \begin{cases} 1 & \text{if } |\arg z| \le \frac{\pi}{2}, \\ \min\left(|\csc(\arg z)|, \chi\left(N+\frac{1}{2}\right)+1\right) & \text{if } \frac{\pi}{2} < |\arg z| \le \pi, \\ \dfrac{\sqrt{2\pi\left(N+\frac{1}{2}\right)}}{|\cos(\arg z)|^{N+\frac{1}{2}}} + \chi\left(N+\frac{1}{2}\right)+1 & \text{if } \pi < |\arg z| < \frac{3\pi}{2}. \end{cases}$$

We may compare this result with that of Olver ($N \ge 1$):

$$|R_N(z)| \le \frac{|a_N|}{|z|^N} \times \begin{cases} 2\exp\left(\frac{1}{4|z|}\right) & \text{if } |\arg z| \le \frac{\pi}{2}, \\ 2\chi(N)\exp\left(\frac{\pi}{8|z|}\right) & \text{if } \frac{\pi}{2} < |\arg z| \le \pi, \\ \dfrac{4\chi(N)}{|\cos(\arg z)|^N}\exp\left(\frac{\pi}{4|z\cos(\arg z)|}\right) & \text{if } \pi < |\arg z| < \frac{3\pi}{2}. \end{cases}$$

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A numerical example

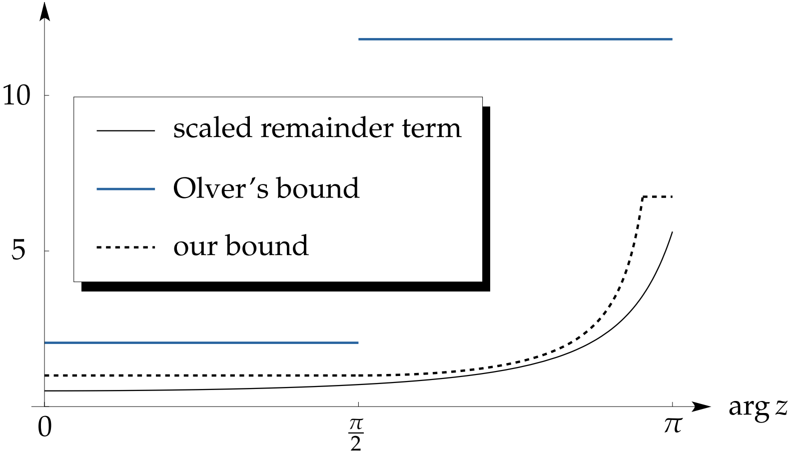

Figure 6: *Numerical comparison of different bounds for the scaled remainder term* $|R_N(z)|/\frac{|a_N|}{|z|^N}$ *with* $N = 20$, $|z| = 10$ *and* $0 \leq \arg z \leq \pi$.

Another example: the logarithm of the gamma function

For any positive integer N and for $|\arg z| < \pi$, we have

$$\log \Gamma(z) = \left(z - \tfrac{1}{2}\right)\log z - z + \tfrac{1}{2}\log(2\pi) + \sum_{n=1}^{N-1} \frac{B_{2n}}{2n(2n-1)z^{2n-1}} + R_N(z),$$

where B_{2n} stands for the Bernoulli numbers and the remainder $R_N(z)$ satisfies

$$\begin{aligned} |R_N(z)| &\le \frac{|B_{2N}|}{2N(2N-1)\,|z|^{2N-1}} \sup_{r\ge 1} |\Pi_{2N-1}(2\pi z r)| \\ &\le \frac{|B_{2N}|}{2N(2N-1)\,|z|^{2N-1}} \\ &\quad \times \begin{cases} 1 & \text{if } |\arg z| \le \frac{\pi}{4}, \\ \min\left(|\csc(2\arg z)|, \frac{1}{2}\chi(2N-1)+1\right) & \text{if } \frac{\pi}{4} < |\arg z| \le \frac{\pi}{2}, \\ \dfrac{\sqrt{2\pi(2N-1)}}{2\,|\sin(\arg z)|^{2N-1}} + \frac{1}{2}\chi(2N-1)+1 & \text{if } \frac{\pi}{2} < |\arg z| < \pi. \end{cases} \end{aligned}$$

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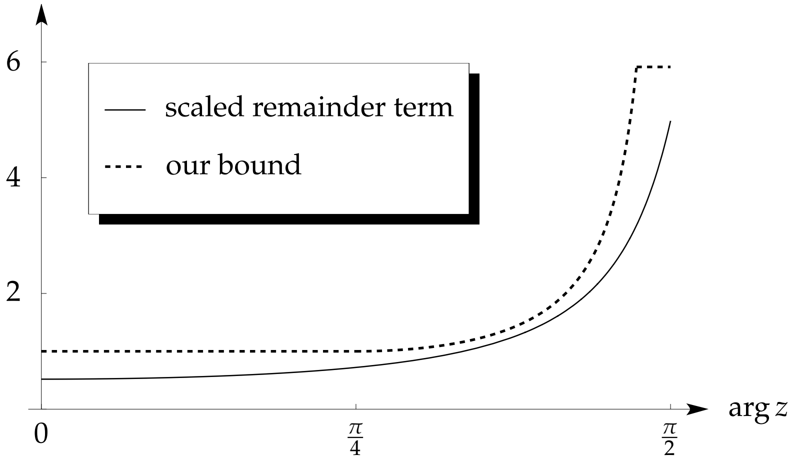

Figure 7: *Numerical comparison of our bound with the scaled remainder term* $|R_N(z)| / \frac{|B_{2N}|}{2N(2N-1)|z|^{2N-1}}$ *with* $N = 31$, $|z| = 10$ *and* $0 \le \arg z \le \frac{\pi}{2}$.

Integrals with simple saddles

Consider the integral

$$I^{(k)}(z) \stackrel{\text{def}}{=} \int_{\mathscr{C}^{(k)}(\theta)} \mathrm{e}^{-zf(t)} g(t) \mathrm{d}t,$$

where $z = |z|\mathrm{e}^{\mathrm{i}\theta}$ and $\mathscr{C}^{(k)}(\theta)$ is the doubly-infinite path of steepest descent passing through the simple saddle point $t^{(k)}$ of $f(t)$ along the two valleys of $\mathrm{Re}\big[-\mathrm{e}^{-\mathrm{i}\theta}(f(t) - f(t^{(k)}))\big]$. The functions f and g are assumed to be analytic in a neighbourhood of the contour $\mathscr{C}^{(k)}(\theta)$.

It is convenient to consider instead of the integral $I^{(k)}$, its slowly varying part, defined by

$$T^{(k)}(z) \stackrel{\text{def}}{=} z^{\frac{1}{2}} \mathrm{e}^{zf(t^{(k)})} I^{(k)}(z) = z^{\frac{1}{2}} \int_{\mathscr{C}^{(k)}(\theta)} \mathrm{e}^{-z(f(t) - f(t^{(k)}))} g(t) \mathrm{d}t.$$

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It is convenient to consider instead of the integral $I^{(k)}$, its slowly varying part, defined by

$$T^{(k)}(z) \stackrel{\text{def}}{=} z^{\frac{1}{2}} \mathrm{e}^{zf(t^{(k)})} I^{(k)}(z) = z^{\frac{1}{2}} \int_{\mathscr{C}^{(k)}(\theta)} \mathrm{e}^{-z(f(t) - f(t^{(k)}))} g(t) \mathrm{d}t.$$

Asymptotics of the slowly varying part

The asymptotic expansion of the slowly varying part can be deduced by an application of the ***method of steepest descents***:

$$T^{(k)}(z) \sim \sum_{n=0}^{\infty} \frac{a_n^{(k)}}{z^n},$$

as $z \to \infty$ in a suitable sectoral region of $\widehat{\mathbb{C}}$.

The coefficients $a_n^{(k)}$ can be described via the local properties of f and g at the simple saddle point $t^{(k)}$ using ***Perron's formula***:

$$a_n^{(k)} = \frac{\Gamma\left(n+\frac{1}{2}\right)}{2\pi\mathrm{i}} \oint_{(t^{(k)}+)} \frac{g(t)}{\left(f(t)-f(t^{(k)})\right)^{n+\frac{1}{2}}}\mathrm{d}t$$

$$= \frac{\sqrt{\pi}}{4^n n!}\left[\frac{\mathrm{d}^{2n}}{\mathrm{d}t^{2n}}\left(g(t)\left(\frac{(t-t^{(k)})^2}{f(t)-f(t^{(k)})}\right)^{n+\frac{1}{2}}\right)\right]_{t=t^{(k)}}.$$

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Exact remainder: the theory of Berry and Howls

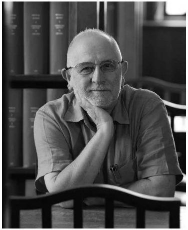

Figure 8: *Sir Michael V. Berry*

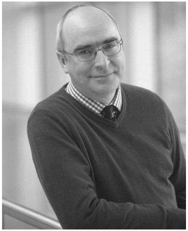

Figure 9: *Christopher J. Howls*

For any non-negative integer N, we introduce the remainder term $R_N^{(k)}(z)$ via

$$T^{(k)}(z) = \sum_{n=0}^{N-1} \frac{a_n^{(k)}}{z^n} + R_N^{(k)}(z).$$

A theory for obtaining an exact formula for this remainder was developed by Sir Michael V. Berry and Christopher J. Howls in 1991.

Adjacent saddles

Definition

A saddle point $t^{(m)} \neq t^{(k)}$ of f is said to be ***adjacent*** to $t^{(k)}$ iff it lies on a path of steepest descent issuing from the saddle point $t^{(k)}$.

Definition

The ***singulant*** $\mathcal{F}_{km}$ corresponding to the saddle $t^{(k)}$ and its adjacent saddle $t^{(m)}$ is defined by

$$\mathcal{F}_{km} \stackrel{\text{def}}{=} f(t^{(m)}) - f(t^{(k)}), \quad \sigma_{km} \stackrel{\text{def}}{=} \arg \mathcal{F}_{km}.$$

We assume that $\mathscr{C}^{(k)}(\theta)$ does not encounter any of the saddle points of f different from $t^{(k)}$, and that $\theta = \arg z$ is restricted to an interval

$$-\sigma_{km_1} < \theta < -\sigma_{km_2},$$

where $t^{(m_1)}$ and $t^{(m_2)}$ are adjacent to $t^{(k)}$.

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$$-\sigma_{km_1} < \theta < -\sigma_{km_2},$$

where $t^{(m_1)}$ and $t^{(m_2)}$ are adjacent to $t^{(k)}$.

Assumptions

(i) The functions $f(t)$ and $g(t)$ are analytic in a domain $\Delta^{(k)}$, whose closure is the set of all the points that can be reached by a path of steepest descent which emanates from $t^{(k)}$.

(ii) We require that $|f(t)| \to +\infty$ as $t \to \infty$ in $\Delta^{(k)}$, and $f(t)$ has several other simple saddle points in the complex t-plane situated at $t = t^{(p)}$ and indexed by $p \in \mathbb{N}$.

(iii) As $t \to \infty$ in the closure of $\Delta^{(k)}$, $\left|f^{-N-1/2}(t)g(t)\right| = o(|t|^{-1})$.

(iv) There are only finitely many saddle points that are adjacent to $t^{(k)}$, and the path of steepest descent $\mathscr{C}^{(m)}(-\sigma_{km})$ through the adjacent saddle $t^{(m)}$ does not contain any of the saddle points of f other than $t^{(m)}$.

The domain $\Delta^{(k)}$ appearing in the theory of Berry and Howls

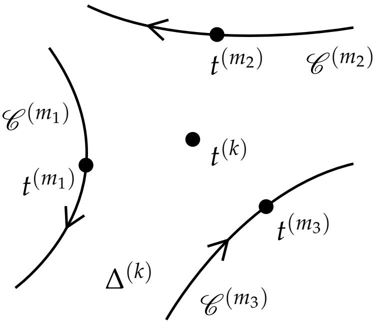

Figure 10: *Three saddle points $t^{(m)}$ adjacent to $t^{(k)}$ together with the corresponding adjacent contours $\mathscr{C}^{(m)}$, forming the boundary of the domain $\Delta^{(k)}$.*

The resurgence formula of Berry and Howls

With the above assumptions,

$$T^{(k)}(z) = \sum_{n=0}^{N-1} \frac{a_n^{(k)}}{z^n} + R_N^{(k)}(z)$$

with

$$R_N^{(k)}(z) = \frac{1}{2\pi \mathrm{i}} \frac{1}{z^N} \sum_m \frac{1}{\mathcal{F}_{km}^N} \int_0^{+\infty} \frac{t^{N-1}\mathrm{e}^{-t}}{1 - t/(\mathcal{F}_{km} z)} T^{(m)}\left(\frac{t}{\mathcal{F}_{km}}\right) \mathrm{d}t,$$

provided that $-\sigma_{km_1} < \arg z < -\sigma_{km_2}$. Here the sum runs over all the saddle points of f that are adjacent to $t^{(k)}$, and the $T^{(m)}$ is the slowly varying integral over the steepest descent contour $\mathscr{C}^{(m)}(-\sigma_{km})$ through the adjacent saddle $t^{(m)}$.

The appearance of the related integrals $T^{(m)}$ in the remainder term is called the ***resurgence property***.

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The appearance of the related integrals $T^{(m)}$ in the remainder term is called the ***resurgence property***.

Alternative representation for the remainder

If we denote $\mathscr{C}^{(m)} = \mathscr{C}^{(m)}(-\sigma_{km})$, then

$$R_N^{(k)}(z) = \frac{1}{2\pi \mathrm{i}} \frac{1}{z^N} \sum_m \int_0^{+\infty} s^{N-\frac{1}{2}} \mathrm{e}^{-s} \int_{\mathscr{C}^{(m)}} \frac{g(t)}{(f(t)-f(t^{(k)}))^{N+\frac{1}{2}}} \frac{1}{1-s/((f(t)-f(t^{(k)}))z)} \mathrm{d}t\mathrm{d}s$$

$$= \frac{1}{2\pi \mathrm{i}} \frac{1}{z^N} \sum_m \int_{\mathscr{C}^{(m)}} \frac{g(t)}{(f(t)-f(t^{(k)}))^{N+\frac{1}{2}}} \int_0^{+\infty} \frac{s^{N-\frac{1}{2}} \mathrm{e}^{-s}}{1-s/((f(t)-f(t^{(k)}))z)} \mathrm{d}s\mathrm{d}t$$

provided that $-\sigma_{km_1} < \arg z < -\sigma_{km_2}$.

Proposition (G. N., 2018)

With the above assumptions, the remainder term $R_N^{(k)}(z)$ has the integral representation

$$R_N^{(k)}(z) = \frac{\Gamma\left(N+\frac{1}{2}\right)}{2\pi \mathrm{i}} \frac{1}{z^N} \sum_m \int_{\mathscr{C}^{(m)}} \frac{g(t)}{(f(t)-f(t^{(k)}))^{N+\frac{1}{2}}} \Lambda_{N+\frac{1}{2}}(\mathrm{e}^{\mp \pi \mathrm{i}}(f(t)-f(t^{(k)}))z)\mathrm{d}t,$$

for $-\sigma_{km_1} - \frac{\pi}{2} < \arg z < -\sigma_{km_2} + \frac{\pi}{2}$ and with $\pm = \mathrm{sgn}(\arg z + \sigma_{km})$.

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Error bound

Proposition (G. N., 2018)

With the above assumptions, the remainder term $R_N^{(k)}(z)$ can be bounded as

$$\left|R_N^{(k)}(z)\right| \le \frac{\Gamma\left(N+\frac{1}{2}\right)}{2\pi}\frac{1}{|z|^N}\sum_m \int_{\mathscr{C}^{(m)}}\left|\frac{g(t)}{\left(f(t)-f(t^{(k)})\right)^{N+\frac{1}{2}}}\mathrm{d}t\right| \sup_{r\ge 1}\left|\Lambda_{N+\frac{1}{2}}\left(\mathrm{e}^{\mp\pi\mathrm{i}}\mathcal{F}_{km} z r\right)\right|,$$

for $-\sigma_{km_1}-\frac{\pi}{2} < \arg z < -\sigma_{km_2}+\frac{\pi}{2}$ and with $\pm = \operatorname{sgn}(\arg z + \sigma_{km})$.

The absolute value of the first omitted term can be written

$$\frac{\left|a_N^{(k)}\right|}{|z|^N} = \frac{\Gamma\left(N+\frac{1}{2}\right)}{2\pi}\frac{1}{|z|^N}\left|\oint_{(t^{(k)}+)}\frac{g(t)}{\left(f(t)-f(t^{(k)})\right)^{N+\frac{1}{2}}}\mathrm{d}t\right|$$

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Example: parabolic cylinder function with large arguments

Nico M. Temme showed that the parabolic cylinder function admits the asymptotic expansion

$$U(-\mu, 2\mu^{\frac{1}{2}}\cosh\alpha) \sim \frac{\mu^{\frac{\mu}{2}-\frac{1}{4}}\mathrm{e}^{-\frac{\mu}{2}(\sinh(2\alpha)-2\alpha+1)}}{\sqrt{2\sinh\alpha}} \sum_{n=0}^{\infty} \frac{\mathsf{A}_n(\coth\alpha)}{\mu^n},$$

as $\mu \to +\infty$, with $\alpha > 0$.

The coefficients $\mathsf{A}_n(\coth\alpha)$ are polynomials in $\coth\alpha$ of degree $3n$ and can be computed using the recurrence relation

$$\mathsf{A}_{n+1}(x) = -\frac{(x^2-1)^2}{4}\mathsf{A}_n'(x) - \frac{1}{16}\int_1^x (5t^2-2)\mathsf{A}_n(t)\mathrm{d}t,$$

for $n \geq 0$ with $\mathsf{A}_0(x) = 1$.

One can derive this expansion with an exact remainder by applying the Berry–Howls method to the integral representation

$$U(a,z) = \frac{\mathrm{e}^{\frac{1}{4}z^2}}{\mathrm{i}\sqrt{2\pi}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \mathrm{e}^{-zt+\frac{1}{2}t^2} t^{-a-\frac{1}{2}}\mathrm{d}t, \quad c > 0.$$

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Example: parabolic cylinder function with large arguments

For any non-negative integer N and for $|\arg \mu| < \frac{3\pi}{2}$, we have

$$U(-\mu, 2\mu^{\frac{1}{2}} \cosh \alpha) = \frac{\mu^{\frac{\mu}{2}-\frac{1}{4}} \mathrm{e}^{-\frac{\mu}{2}(\sinh(2\alpha)-2\alpha+1)}}{\sqrt{2\sinh \alpha}} \left(\sum_{n=0}^{N-1} \frac{\mathsf{A}_n(\coth \alpha)}{\mu^n} + R_N(\mu, \alpha) \right),$$

where the remainder $R_N(\mu, \alpha)$ satisfies

$$|R_N(\mu, \alpha)| \le \frac{|\mathsf{A}_N(\coth \alpha)|}{|\mu|^N} \sup_{r \ge 1} \left| \Lambda_{N+\frac{1}{2}}((\sinh(2\alpha) - 2\alpha)\mu r) \right|$$

$$\le \frac{|\mathsf{A}_N(\coth \alpha)|}{|\mu|^N} \times \begin{cases} 1 & \text{if } \ |\arg \mu| \le \frac{\pi}{2}, \\ \min\left(|\csc(\arg \mu)|, \chi(N+\frac{1}{2}) + 1 \right) & \text{if } \ \frac{\pi}{2} < |\arg \mu| \le \pi, \\ \dfrac{\sqrt{2\pi(N+\frac{1}{2})}}{|\cos(\arg \mu)|^{N+\frac{1}{2}}} + \chi(N+\frac{1}{2}) + 1 & \text{if } \ \pi < |\arg \mu| < \frac{3\pi}{2}. \end{cases}$$

This result may be applied to the Hermite polynomials outside the oscillatory regime, since

$$H_n(\sqrt{2n+1} \cosh \alpha) = 2^{\frac{2\mu-1}{4}} \mathrm{e}^{\mu \cosh^2 \alpha} U(-\mu, 2\mu^{\frac{1}{2}} \cosh \alpha), \quad \mu = n + \tfrac{1}{2}.$$

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Olver's conjecture

It is well known that the Airy function $\mathrm{Ai}(z)$ has an infinite number of negative zeros. We denote them by a_k, arranged in ascending order of absolute value with k a positive integer. The large negative zeros of $\mathrm{Ai}(z)$ are known to posses the divergent asymptotic expansion

$$a_k \sim -\gamma_k^{2/3}\left(1+\frac{5}{48\gamma_k^2}-\frac{5}{36\gamma_k^4}+\frac{77125}{82944\gamma_k^6}-\frac{108056875}{6967296\gamma_k^8}+\ldots\right),$$

where $\gamma_k = \frac{3}{8}\pi(4k-1)$ (Jeffrey C. P. Miller, 1946).

Conjecture (Frank W. J. Olver, 1999)

In the expansion of a_k, the Nth error term is bounded by the first neglected term and has the same sign for all values of $N \geq 1$. In addition, starting from the second term, the terms alternate in sign.

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A function that returns the zeros

Theorem (G. N., 2021)

There exists a function $T(w)$, which is analytic in the closed sector $|\arg w| \leq \frac{\pi}{2}$ and has the following properties.

(i) For each $k \geq 1$, $T(\gamma_k) = -a_k$.

(ii) $T(w)$ remains bounded as $w \to 0$ in the sector $|\arg w| \leq \frac{\pi}{2}$.

(iii) For any $s > 0$, $\operatorname{Im}\left(\mathrm{e}^{-\frac{\pi}{3}\mathrm{i}} T(\mathrm{i}s)\right) < 0$.

(iv) $w^{-2/3} T(w) = 1 + \mathcal{O}(w^{-2})$ as $w \to \infty$ in the sector $|\arg w| \leq \frac{\pi}{2}$.

(v) $\operatorname{Im}\left(\mathrm{e}^{-\frac{\pi}{3}\mathrm{i}} T(\mathrm{i}s)\right) = o(s^{-r})$ as $s \to +\infty$, with any fixed $r > 0$.

Confirming Olver's conjecture

The above theorem combined with a Cauchy–Heine-type argument, shows that for any $k \geq 1$ and $N \geq 1$,

$$a_k = -T(\gamma_k) = -\gamma_k^{2/3}\left(1 + \sum_{n=1}^{N-1} \frac{T_n}{\gamma_k^{2n}} + R_N(\gamma_k)\right)$$

with

$$T_n = (-1)^n \frac{2}{\pi} \int_0^{+\infty} s^{2n-\frac{5}{3}} \operatorname{Im}\left(\mathrm{e}^{-\frac{\pi}{3}\mathrm{i}} T(\mathrm{i}s)\right) \mathrm{d}s$$

and

$$R_N(\gamma_k) = \frac{1}{\gamma_k^{2N}} (-1)^N \frac{2}{\pi} \int_0^{+\infty} \frac{s^{2N-\frac{5}{3}} \operatorname{Im}\left(\mathrm{e}^{-\frac{\pi}{3}\mathrm{i}} T(\mathrm{i}s)\right)}{1 + (s/\gamma_k)^2} \mathrm{d}s.$$

By our theorem and the mean value theorem for improper integrals,

$$R_N(\gamma_k) = \theta_{k,N} \frac{1}{\gamma_k^{2N}} (-1)^N \frac{2}{\pi} \int_0^{+\infty} s^{2N-\frac{5}{3}} \operatorname{Im}\left(\mathrm{e}^{-\frac{\pi}{3}\mathrm{i}} T(\mathrm{i}s)\right) \mathrm{d}s = \theta_{k,N} \frac{T_N}{\gamma_k^{2N}}$$

with a suitable $0 < \theta_{k,N} < 1$, answering Olver's conjecture in the affirmative.

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Problems for future research

- Studying the analogous problem for multidimensional integrals of the form

$$I^{(k)}(z) = \int \cdots \int_{\mathscr{S}_k} \mathrm{e}^{-zf(t_1,\ldots,t_d)} g(t_1,\ldots,t_d) \mathrm{d}t_1 \cdots \mathrm{d}t_d$$

over a d dimensional surface $\mathscr{S}_k$ which is doubly infinite in extent in all complex variables and runs between specified valleys at infinity associated with an isolated critical point $t^{(k)}$ of f. The resurgence properties were studied by Howls (1997).

- Constructing error bounds for uniform asymptotic expansions arising from integrals (e.g., coalescing saddle points, saddle point near a pole, saddle point near and endpoint). The resurgence properties for integrals with coalescing saddles were studied by Adri B. Olde Daalhuis in 2000.

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Thank you for your attention!