Supersymmetric Quantum Mechanics and Morse Theory

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Homework 2: Differential forms

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1 Differential forms

Let us consider Euclidean space $\mathbb{R}^3$ with coordinates x^1, x^2, x^3 endowed with Riemann metric metric $g(\partial_i, \partial_j) = g_{ij} = \delta_{ij}$. For a vector field

$$\vec{v} = v^i(x)\partial_i \in Vect(\mathbb{R}^3) \tag{1.1}$$

we can associate two differential forms

$$\omega^2_{\vec{v}} = \frac{1}{2}\sum_{ijk} \epsilon_{ijk} v^i dx^j \wedge dx^k \in \Omega^2(\mathbb{R}^3), \quad \omega^1_{\vec{v}} = \sum_{ij} g_{ij} v^j dx^i \in \Omega^1(\mathbb{R}^3). \tag{1.2}$$

Show that the following relations are identities:

1. (*5 points*) Hodge dual
$$\omega^2_{\vec{v}} = *\omega^1_{\vec{v}}, \;\; \omega^1_{\vec{v}} = *\omega^2_{\vec{v}}. \tag{1.3}$$

2. (*5 points*) Gradient formula
$$\omega^1_{\vec{\nabla} f} = df. \tag{1.4}$$

3. (*5 points*) Curl formula
$$d\omega^1_{\vec{v}} = \omega^2_{\vec{\nabla}\times\vec{v}}. \tag{1.5}$$

4. (*5 points*) Divergence
$$* \, d\omega^2_{\vec{v}} = \vec{\nabla}\cdot\vec{v}. \tag{1.6}$$

5. (*10 points*) Maxwell equations: Let us consider 4d Minkowski space $\mathbb{R}^{3,1} = \mathbb{R}^3 \times \mathbb{R}$ with coordinates $t = x^0, x^1, x^2, x^3$ and metric
$$g(\partial_i, \partial_j) = g_{ij} = \delta_{ij}, \;\; g(\partial_i, \partial_0) = 0, \;\; g(\partial_0, \partial_0) = -1 \tag{1.7}$$
For electromagnetic field in terms of electric field 3-vector $E^i(x)$ and magnetic field 3-vector $B^i(x)$ we can associate a 2-form
$$F = g_{ij} E^i dx^0 \wedge dx^j + \epsilon_{ijk} B^i dx^j \wedge dx^k = dx^0 \wedge \omega^1_{\vec{E}} + \omega^2_{\vec{B}} \tag{1.8}$$
Show that Maxwell equations in vacuum can be written in the form
$$dF = d * F = 0. \tag{1.9}$$

with $*$ being Hodge dual in four-dimensional space. We assumed the choice of units where the speed of light $c = 1$.

2 Differential forms as functions on supermanifold

In class we established a relation between differential forms and functions on supergaminold

$$F : \Omega^*(\mathbb{R}^n) \to C^\infty(\Pi T\mathbb{R}^n) : \omega \to F_\omega. \tag{2.1}$$

Let us choose $x^1, ..., x^n$ and $\psi^1, .., \psi^n$ as even and odd coordinates on $\Pi T\mathbb{R}^n$. The differential p-form

$$\omega = \sum_{i_1...i_p} \frac{1}{p!} \omega_{i_1...i_p} dx^{i_1}...dx^{i_p} \in \Omega^p(\mathbb{R}^n) \tag{2.2}$$

maps to a function

$$F_\omega(x, \psi) = \sum_{i_1...i_p} \frac{1}{p!} \omega_{i_1...i_p} \psi^{i_1}...\psi^{i_p} \in C^\infty(\Pi T\mathbb{R}^n). \tag{2.3}$$

1. (*5 points*) Show that the differential d and vector field substitution ι_v are first order differential operators on $C^\infty(\Pi T\mathbb{R}^n)$

$$F_{d\omega} = D_d F_\omega, \quad F_{\iota_v \omega} = D_{\iota_v} F_\omega \tag{2.4}$$

   and evaluate D_d and D_{ι_v} in (x, ψ) coordinates.

2. (*5 points*) Using the differential operator representation verify the properties of external derivative d and ι_v

   - Differential

$$d^2 = 0. \tag{2.5}$$

   - Antysymmetry

$$\iota_v \iota_w \omega = -\iota_w \iota_v \omega. \tag{2.6}$$

   - Derivation: Let $\omega \in \Omega^k(\mathbb{R}^n)$ and $\mu \in \Omega^m(\mathbb{R}^n)$

$$d(\omega \wedge \mu) = d\omega \wedge \mu + (-1)^k \omega \wedge d\mu, \tag{2.7}$$

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$$\iota_v(\omega \wedge \mu) = \iota_v\omega \wedge \mu + (-1)^k \omega \wedge \iota_v \mu. \tag{2.8}$$

3. (*5 points*) We can use Cartan formula to describe Lie derivative via

$$\mathcal{L}_v = \{d, \iota_v\} \tag{2.9}$$

In terms of the functions on $C^\infty(\Pi T\mathbb{R}^n)$ the D_{L_v} is the graded commutator of two first order differential operators, so it is also a first order differential operator

$$D_{L_v} = \{D_d, D_{\iota_v}\} \tag{2.10}$$

Evaluate the D_{L_v} in coordinates x, ψ.

4. (*5 points*) Using differential operator representation verify the properties of Lie derivative

- Derivation property: Let $\omega \in \Omega^k(\mathbb{R}^n)$ and $\mu \in \Omega^m(\mathbb{R}^n)$

$$\mathcal{L}_v(\omega \wedge \mu) = \mathcal{L}_v\omega \wedge \mu + \omega \wedge \mathcal{L}_v\mu.$$

- Lie algebra :

$$(\mathcal{L}_v\mathcal{L}_w - \mathcal{L}_w\mathcal{L}_v)\omega = \mathcal{L}_{[v,w]}\omega.$$

- Lie algebra representation:

$$(\mathcal{L}_v\iota_w - \iota_w\mathcal{L}_v)\omega = \iota_{[v,w]}\omega.$$

5. (*10 points*) Let us restrict our attention to the differential forms μ, ω that go to zero at infinity sufficiently fast to the integrals

$$\langle \omega, \mu \rangle = \int_{\mathbb{R}^n} \omega \wedge *\mu \tag{2.11}$$

are finite and we can define the Hodge-dual operators. Verify the Hodge dual of a differential p-form ω as a function on supermanifold $\Pi T\mathbb{R}^n$

$$(-1)^{\frac{1}{2}(n-p)(n+p-1)} F_{*\omega}(\psi, x) = \int_{\mathbb{R}^{0|n}} d^n\eta \;\; e^{\sum \delta_{ij}\psi^i\eta^j} F_\omega(x, \eta) \tag{2.12}$$

with integration over $d^n\eta$ being Grassmann integration.

Hint: Let α be a monomial expression of Grassmann variables, not depending on η^i then

$$\int_{\mathbb{R}^{0|n}} d^n\eta \;\; \alpha\eta^1...\eta^n = \alpha. \tag{2.13}$$

6. (*10 points*) The hodge dual d^* of external derivative d is the second order differential operator D_{d^*} on $C^\infty(\Pi T\mathbb{R}^n)$. Use the Hodge star formula (2.12) to evaluate this operator in x^i, ψ^i coordinates

$$F_{d^*\omega} = D_{d^*}F_\omega. \tag{2.14}$$

3 Laplacian with superpotential

Given a smooth function h on closed smooth manifold M with Riemann metric g we can construct a differential operator

$$d_h = e^{-h}de^h = d + dh\wedge : \Omega^p(M) \to \Omega^{p+1}(M). \tag{3.1}$$

1. (*5 points*) Show that d_h is the differential.

2. (*5 points*) Show that

$$H^k(\Omega^*(M), d_h) = H^k_{DR}(M). \tag{3.2}$$

3. (*10 points*) Let us for simplicity assume that $M = \mathbb{R}^n$ with Euclidean metric $g_{ij} = \delta_{ij}$ and we restrict our attention to the forms, that vanish sufficiently fast at infinity, so the integrals over $\mathbb{R}^n$ are finite. Describe the dual operator d_h^* defined via

$$\langle d_h\omega, \mu\rangle = \langle\omega, d_h^*\mu\rangle, \;\; \mu \in \Omega^k(M) \tag{3.3}$$

in terms of d^* and ι_v.

Hint: It might be helpful to represent the differential forms as a functions on supermanifold $\Pi T\mathbb{R}^n$.

4. (*10 points*) Let us for simplicity assume that $M = \mathbb{R}^n$ with Euclidean metric $g_{ij} = \delta_{ij}$ and we restrict our attention to the forms, that vanish sufficiently fast at infinity, so

the integrals over $\mathbb{R}^n$ are finite. Evaluate the Laplacian

$$\Delta_h = d_h d_h^* + d_h^* d_h \tag{3.4}$$

in the form of differential operator acting on $C^\infty(\Pi T\mathbb{R}^n)$.