Holography of the Loewner energy

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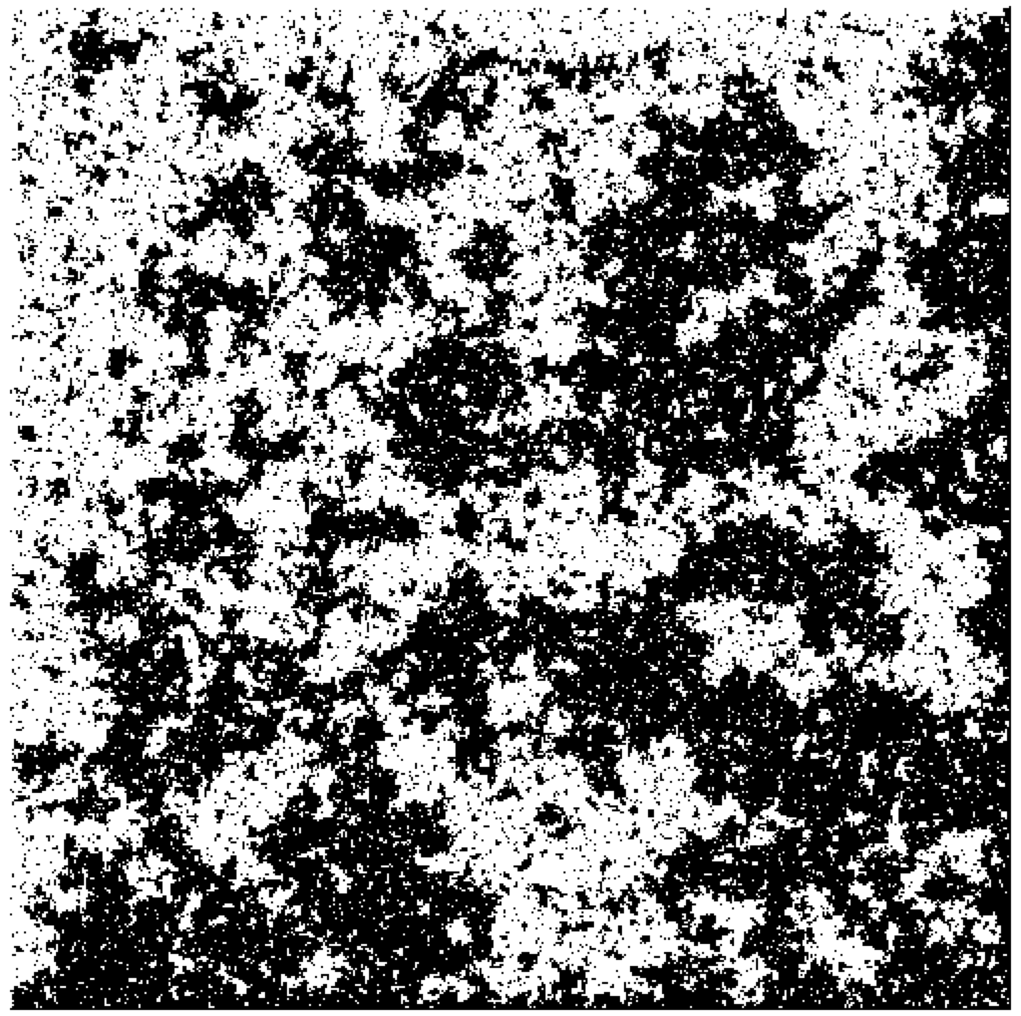
Joint with

Martin Bridgeman (Boston College), Kenneth Bromberg (Utah), Franco Vargas-Pallete (Yale)

OIST. July 31, 2023

- Loewner energy (Weil-Petersson Teichmuller space) $<\!-\!-\!>$ SLE
- Riemann sphere $\hat{\mathbb{C}}$ $<\!-\!-\!>$ hyperbolic 3-space $\mathbb{H}^3$
- **Loewner energy <— —> renormalized volume**
- Motivation from Liouville action
- Variational formula
- Quasi-Fuchsian manifolds

Schramm Loewner evolution SLE



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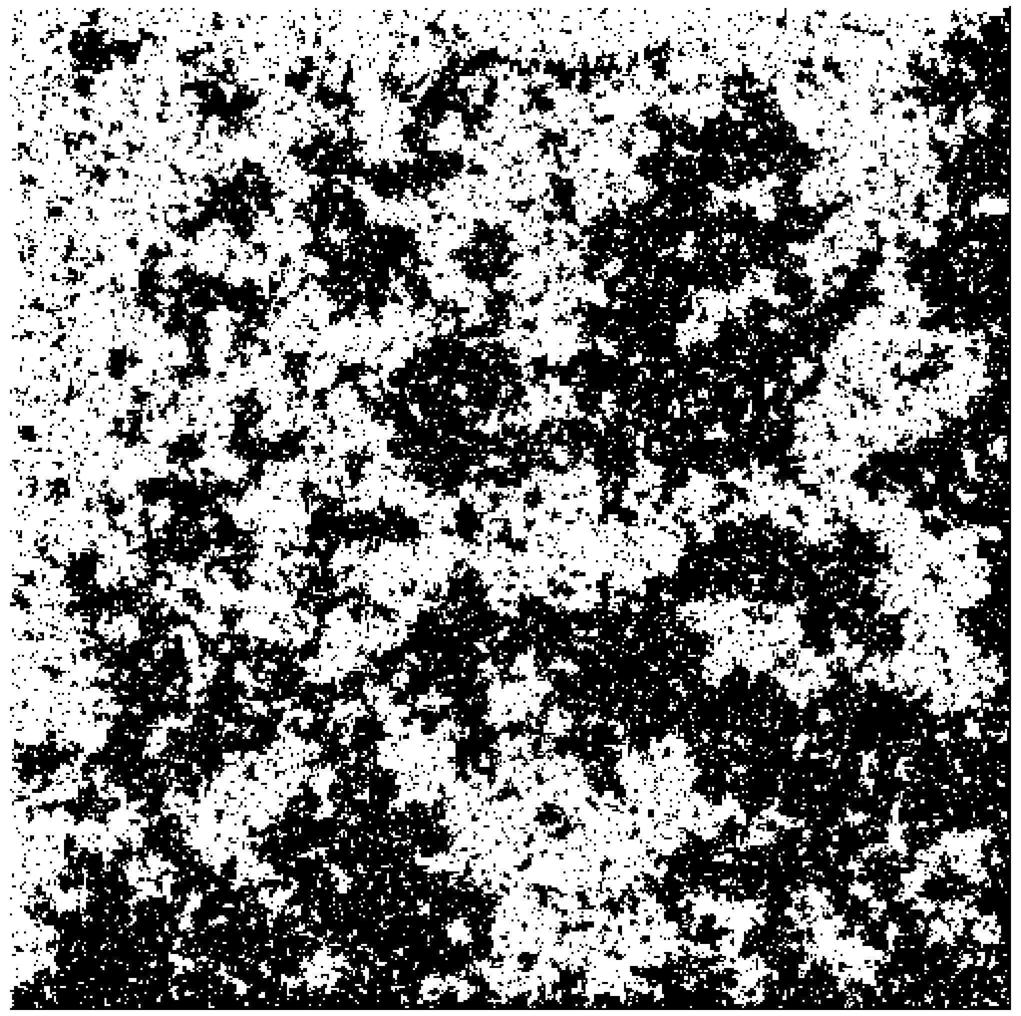

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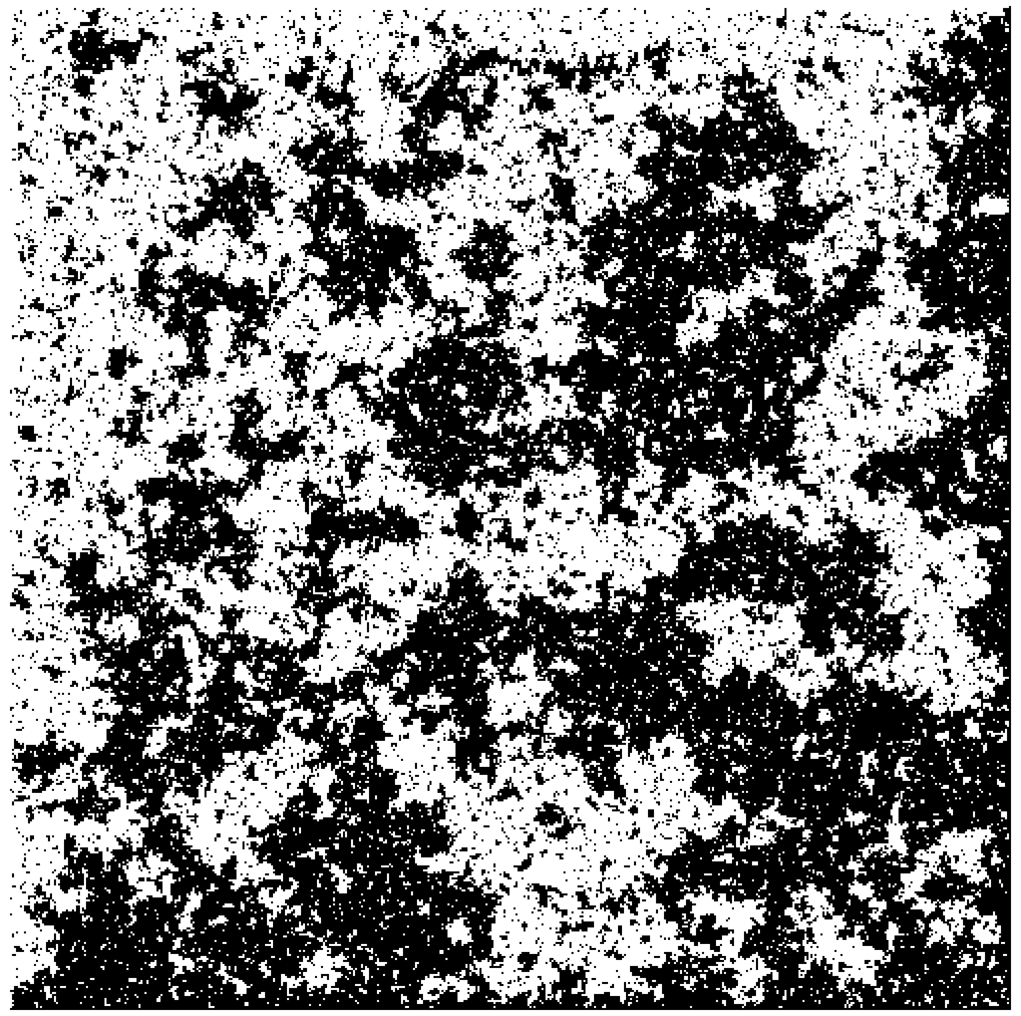

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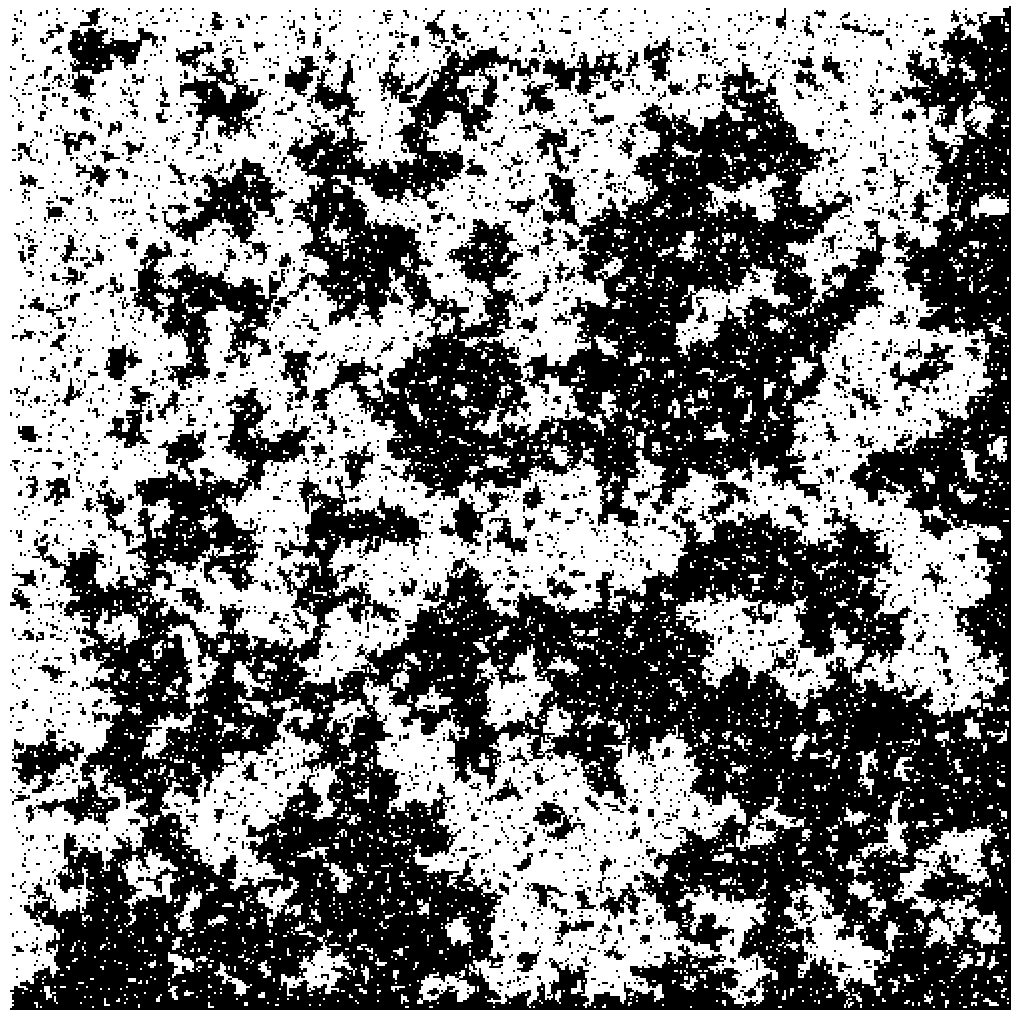

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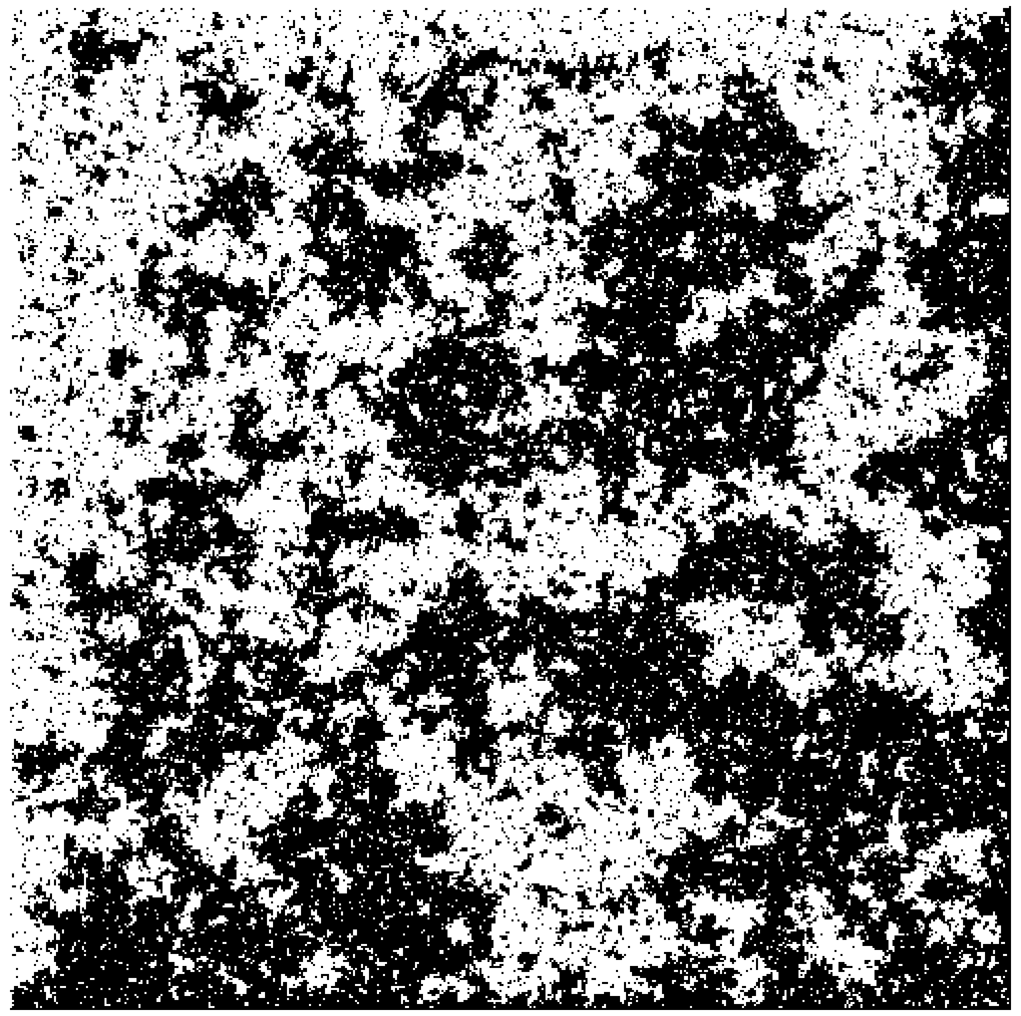

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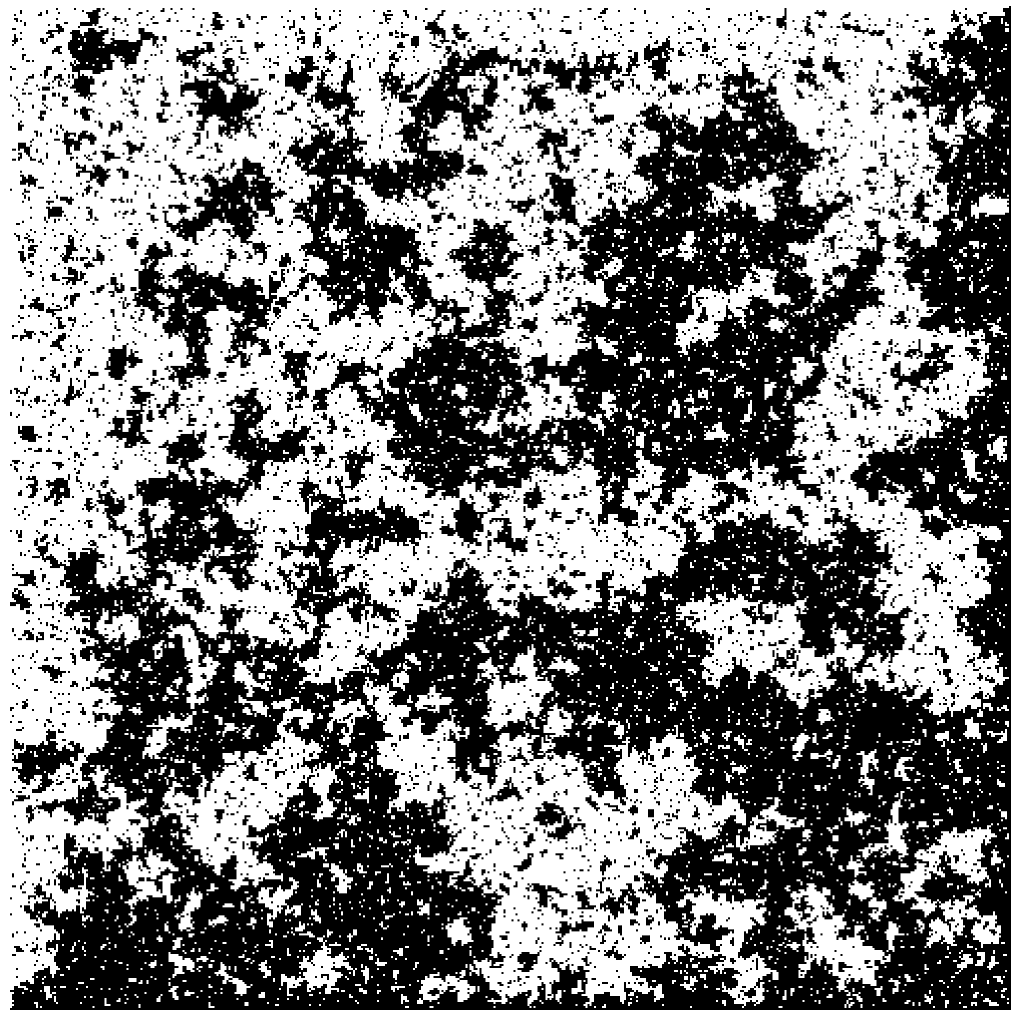

- Example: Critical Ising model —> SLE_3

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- $I^L(\eta) \in [0,\infty]$, and $I^L(\eta) = 0$ iff η is a circle.

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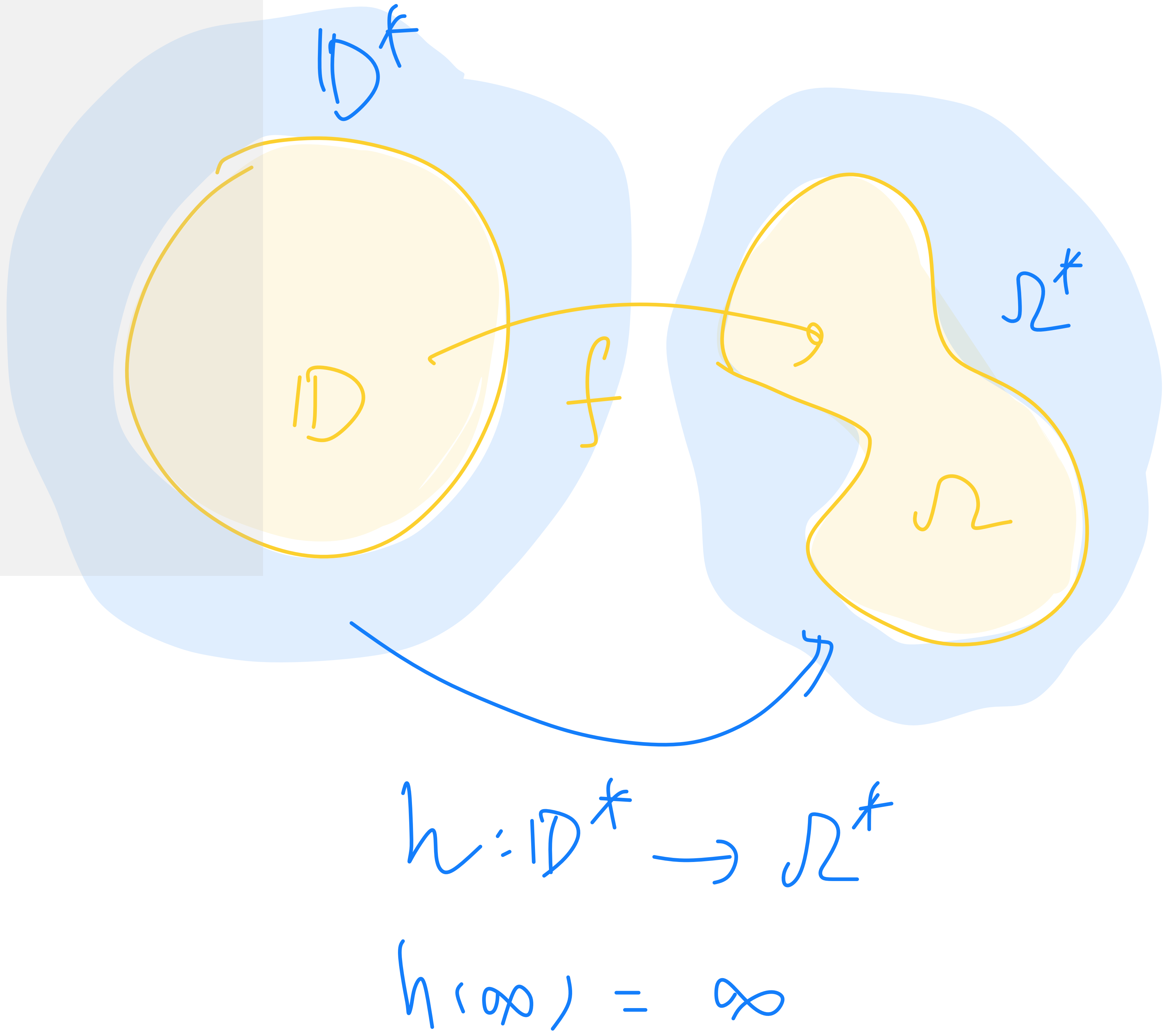

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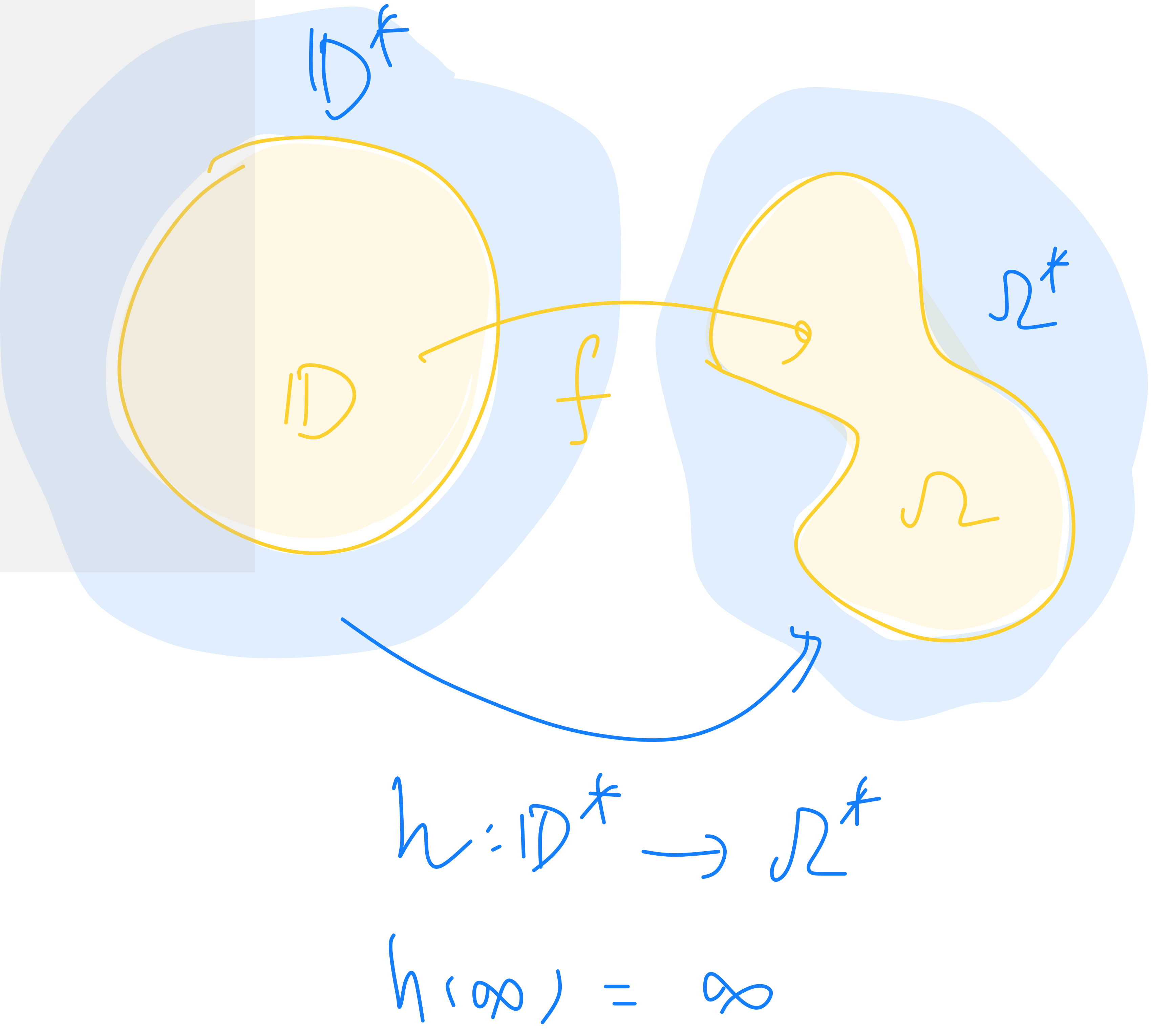

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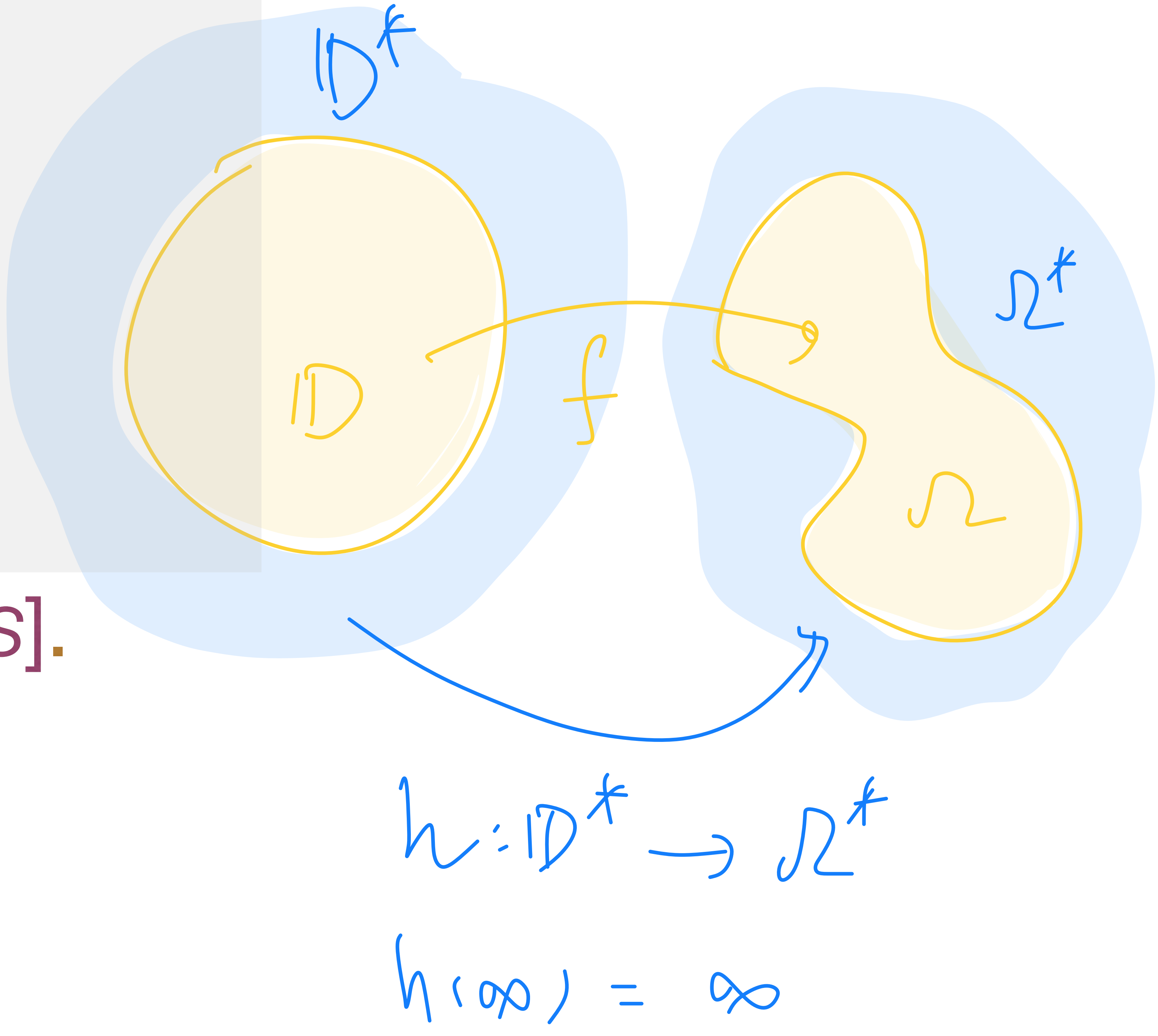

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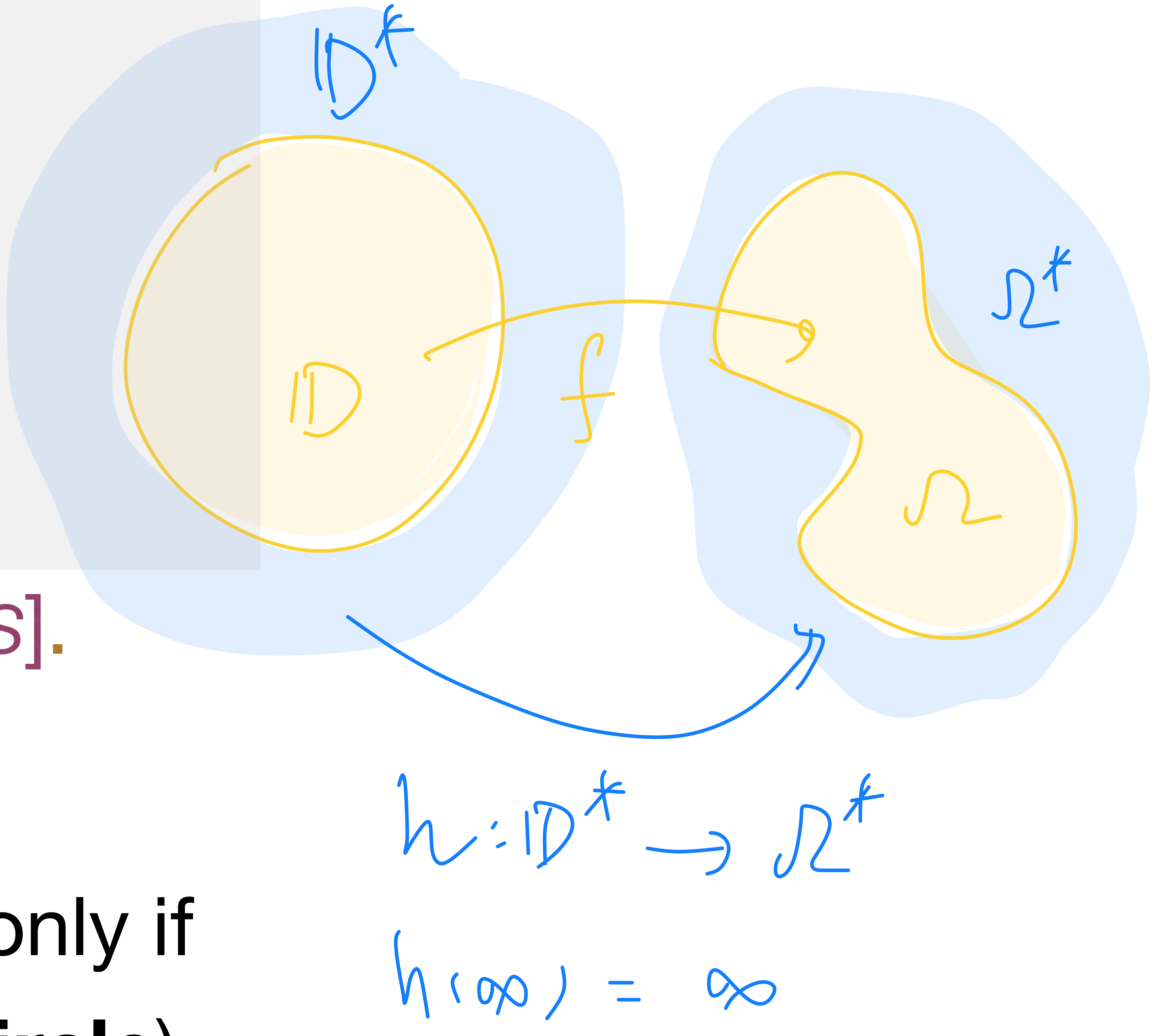

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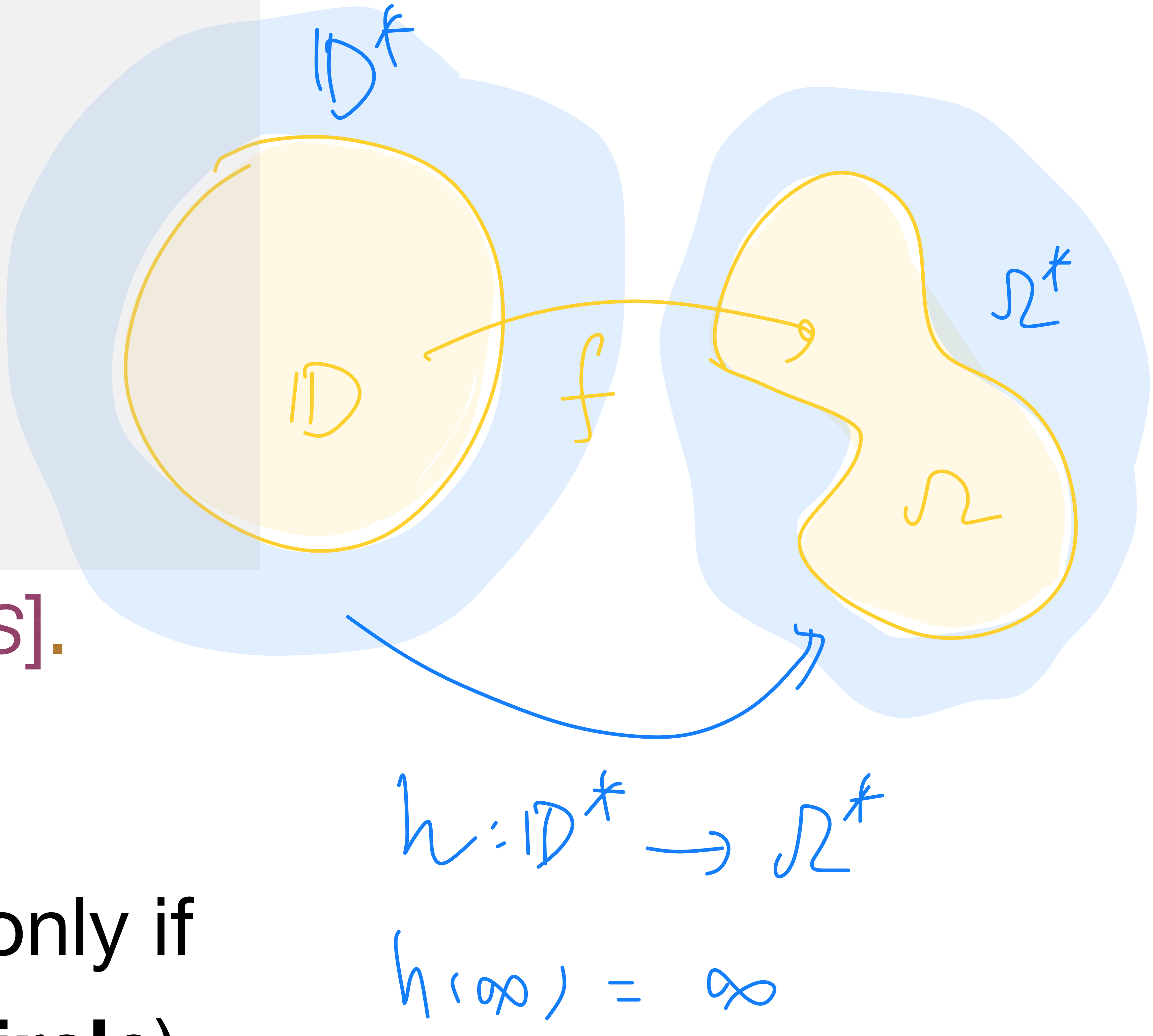

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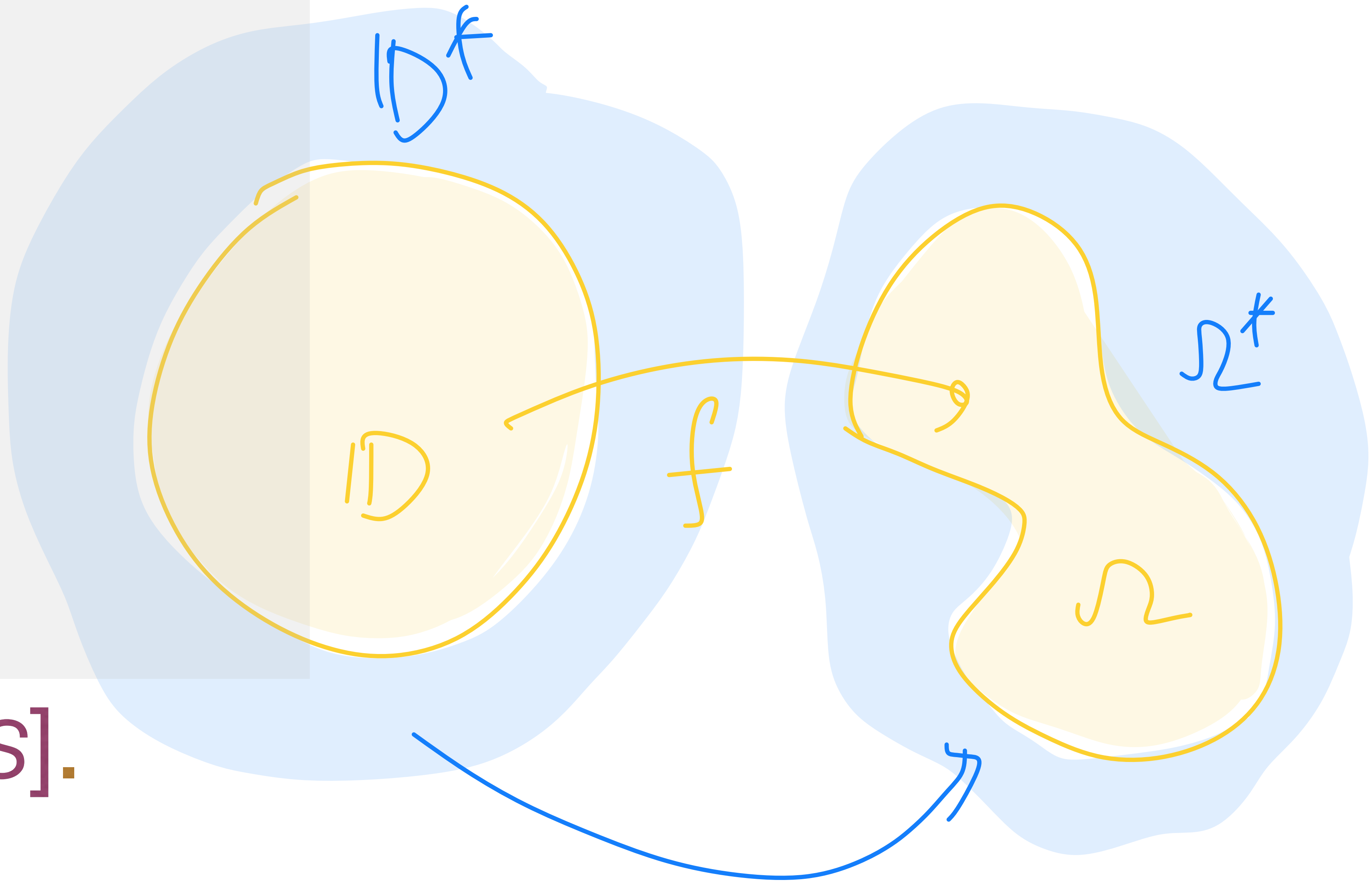

$h : \mathbb{D}^* \to \Omega^*$

$h(\infty) = \infty$

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[Takhtajan-Teo, *MAMS*]: $I^L : WP(S^1) \to \mathbb{R}_{\geq 0}$ is the **Kahler potential** of the unique homogeneous Kahler metric on Weil-Petersson universal Teichmuller space $\mathscr{T}_0(1) = Mob(S^1)\backslash WP(S^1)$.

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preserves $\mathbb{H}$ and $\mathbb{H}^*$.

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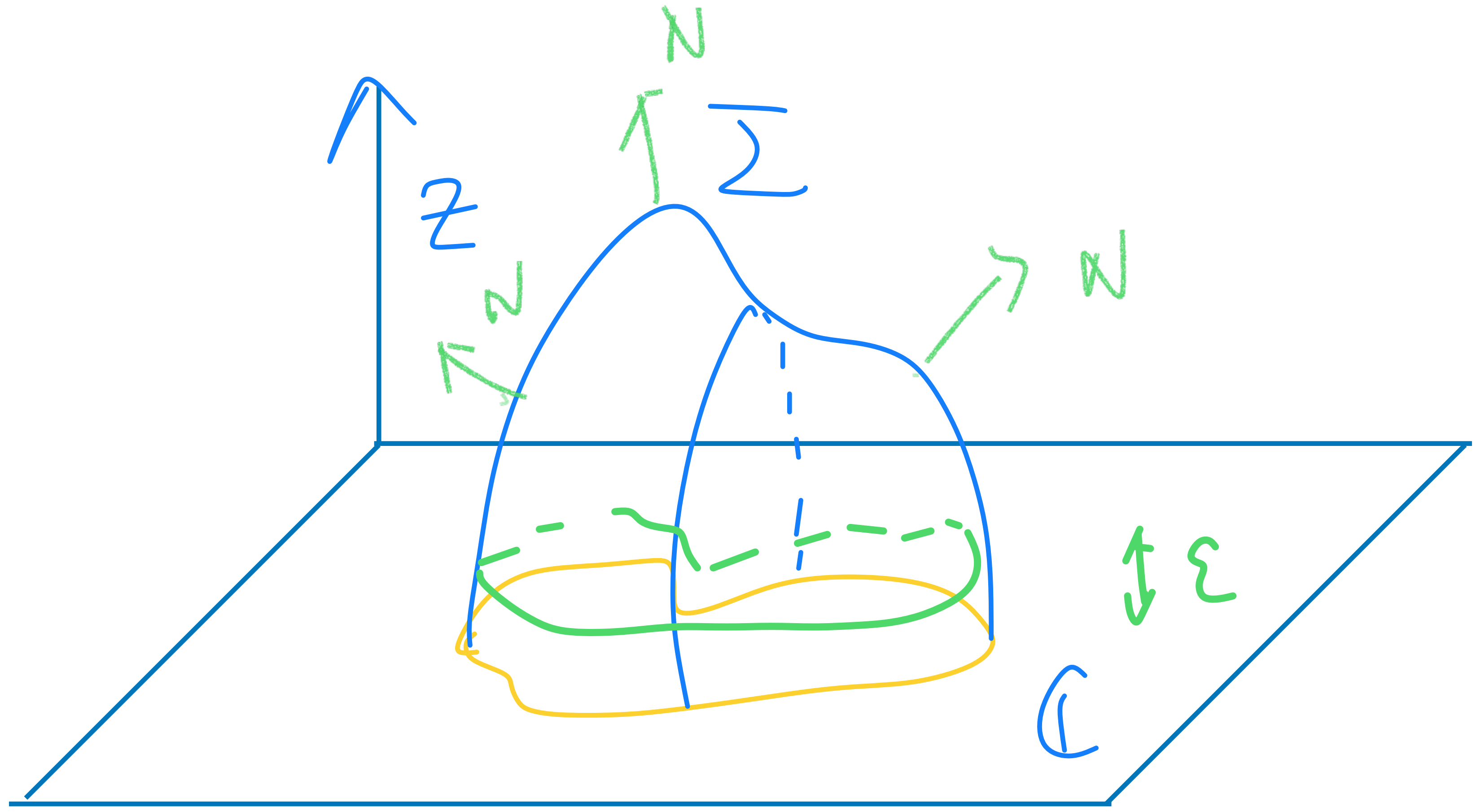
Starting point of AdS/CFT correspondence.

Holography of the Loewner energy in $\mathbb{H}^3$?

Theorem (Bishop, *preprint***)**

$I^L(\eta) < \infty$ iff η bounds a minimal surface Σ in $\mathbb{H}^3$ with finite total curvature $\int_\Sigma k^2 dA_{hyp} < \infty$ which is also equivalent to finite renormalized area.

$\lim_{\varepsilon \to 0} A_\varepsilon - L_\varepsilon$



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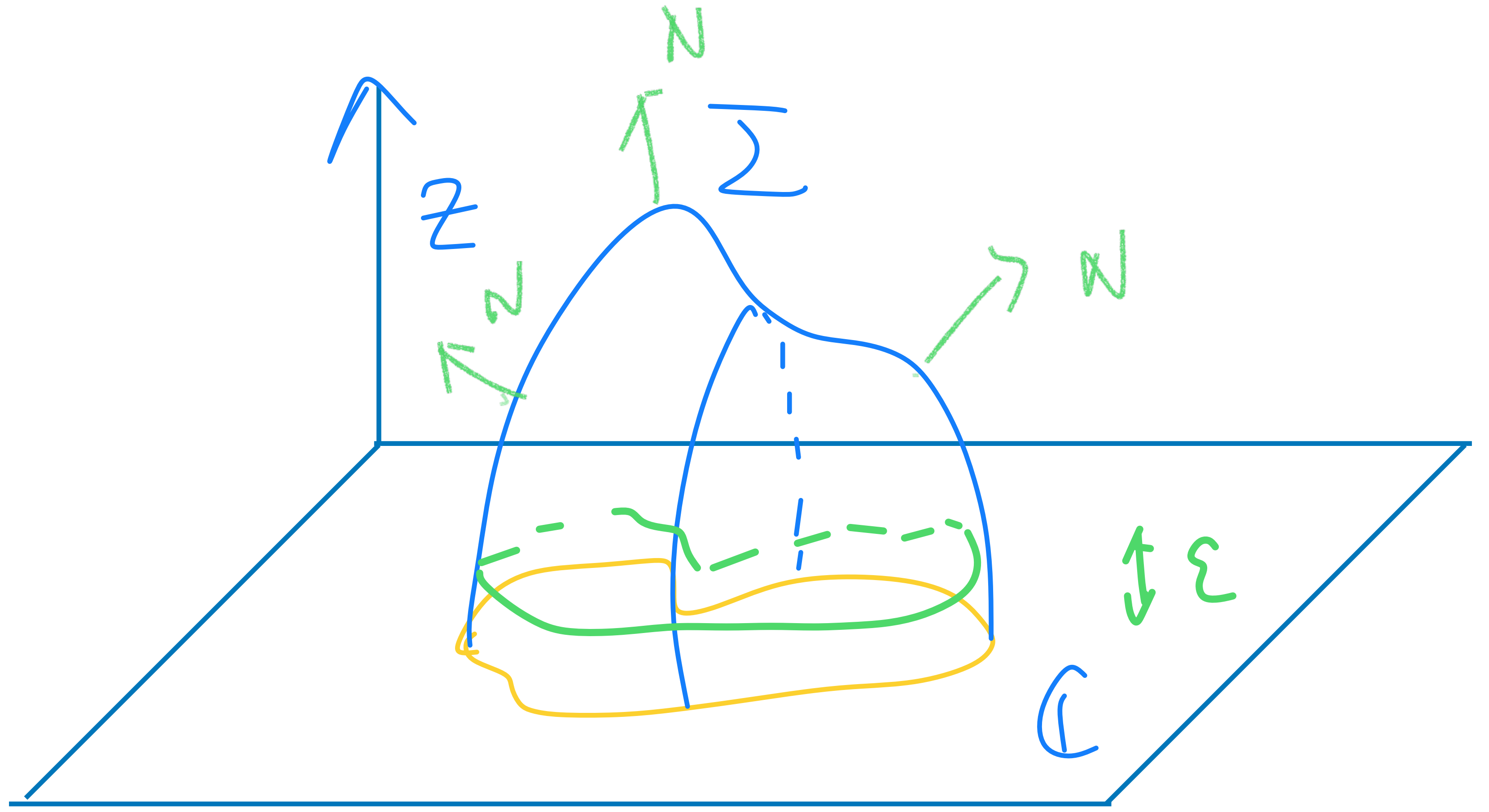

Notation: On $T\Sigma$

- $B : T\Sigma \to T\Sigma$ is the shape operator $Bu = -\nabla_u N$
- Principal curvatures are the eigenvalues $\{k_1, k_2\}$ of B.
- $H = (k_1 + k_2)/2$ is the mean curvature of Σ.
- Σ is minimal iff $H \equiv 0$, i.e $k := k_1 = -k_2$.

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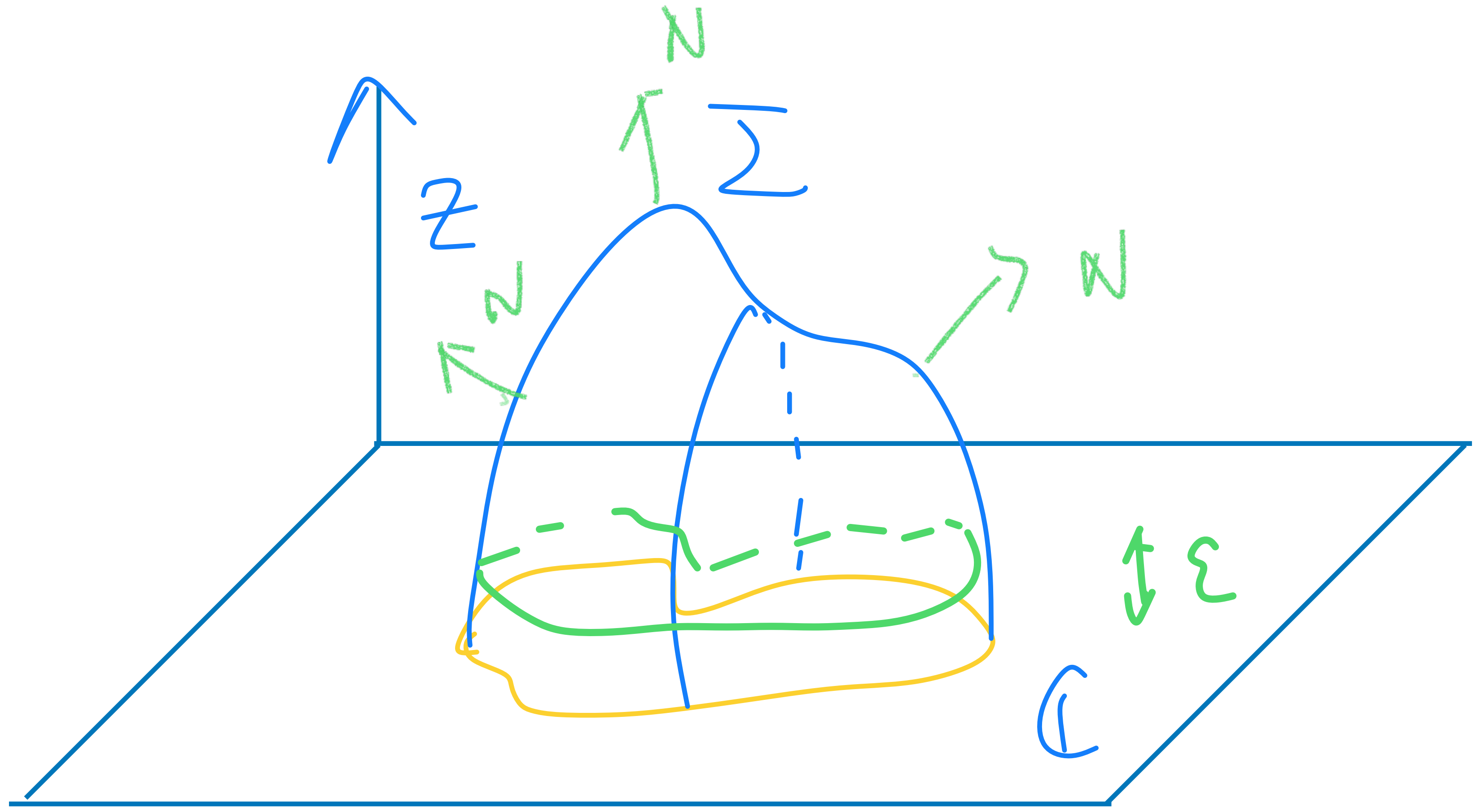

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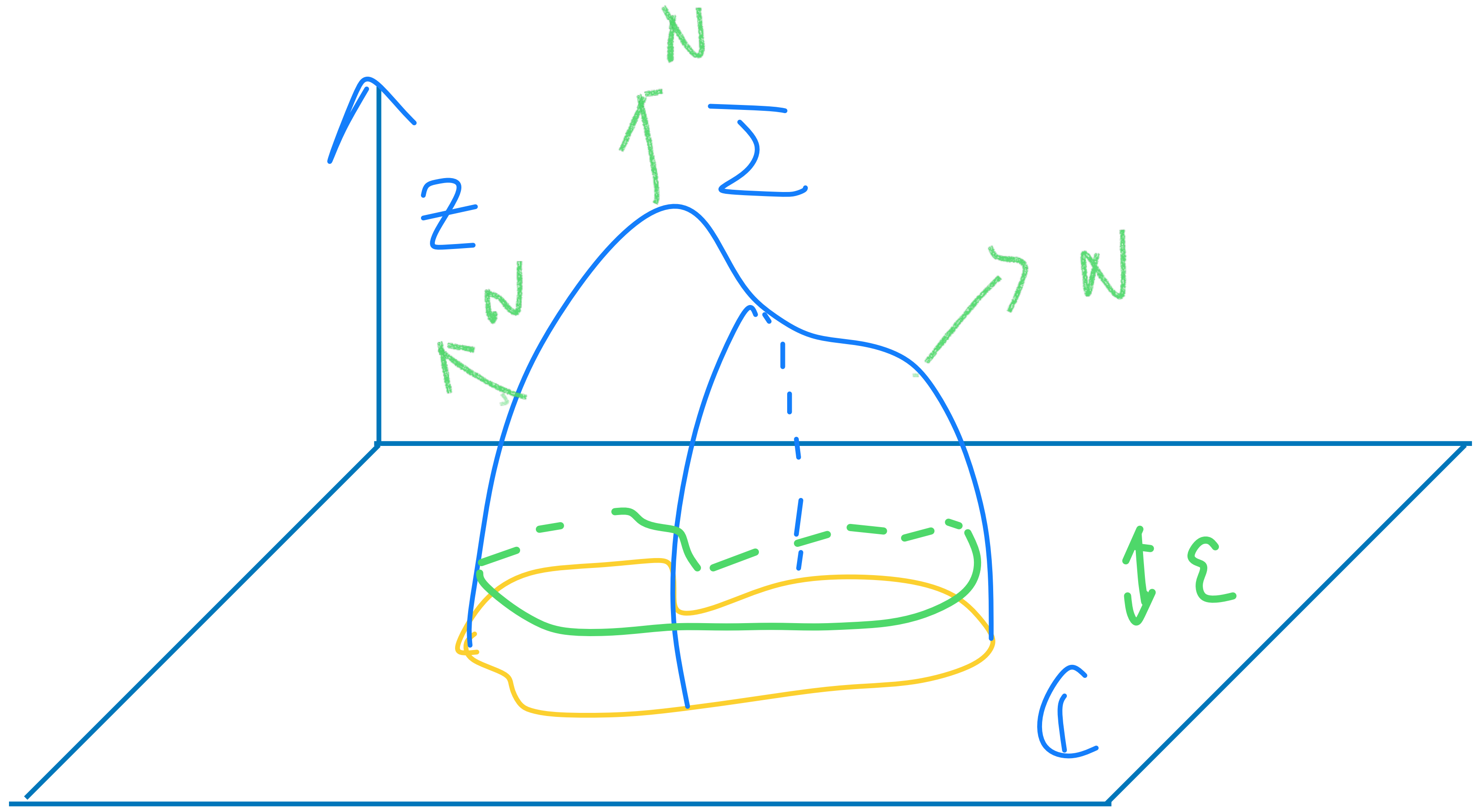

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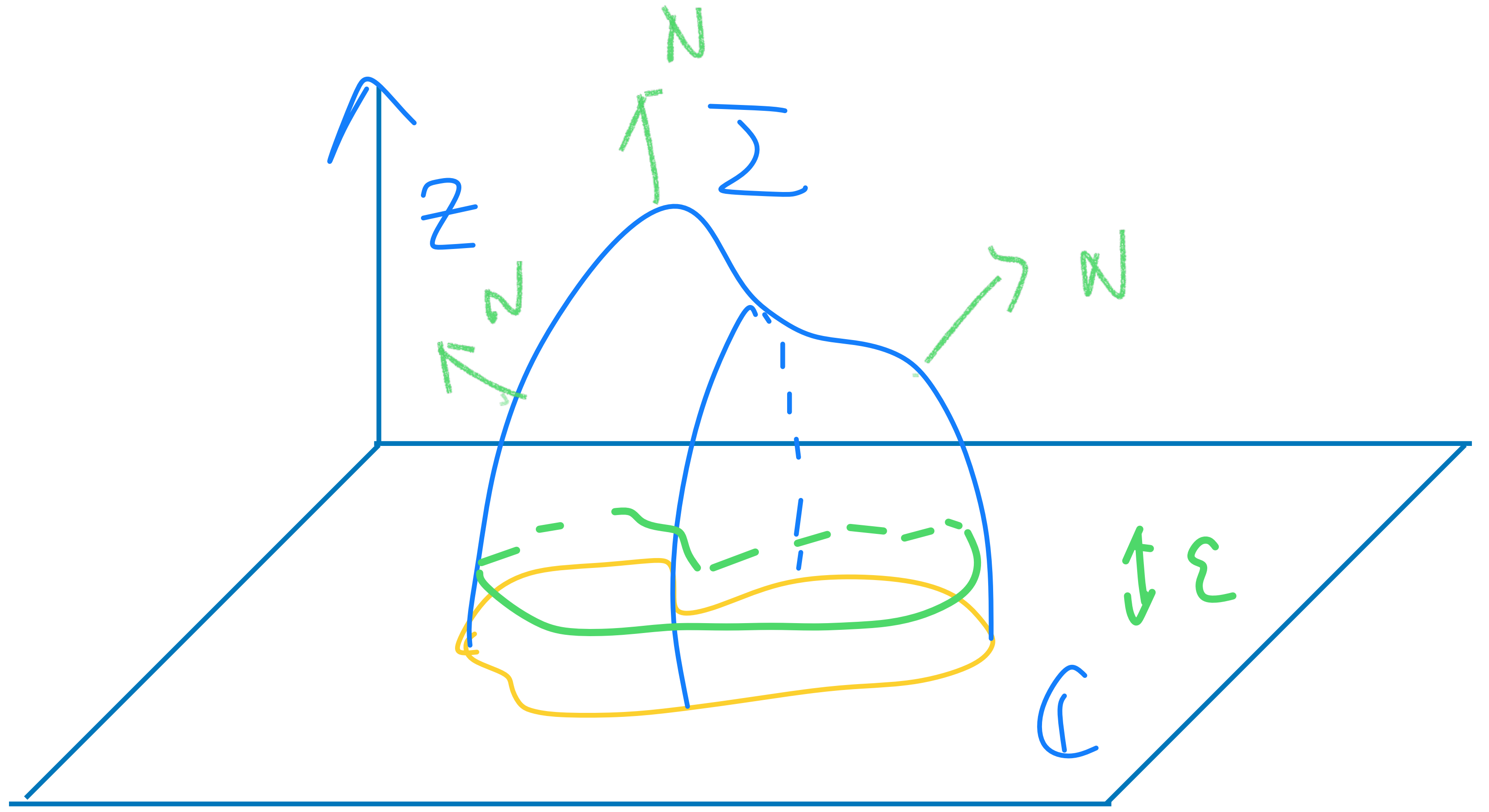

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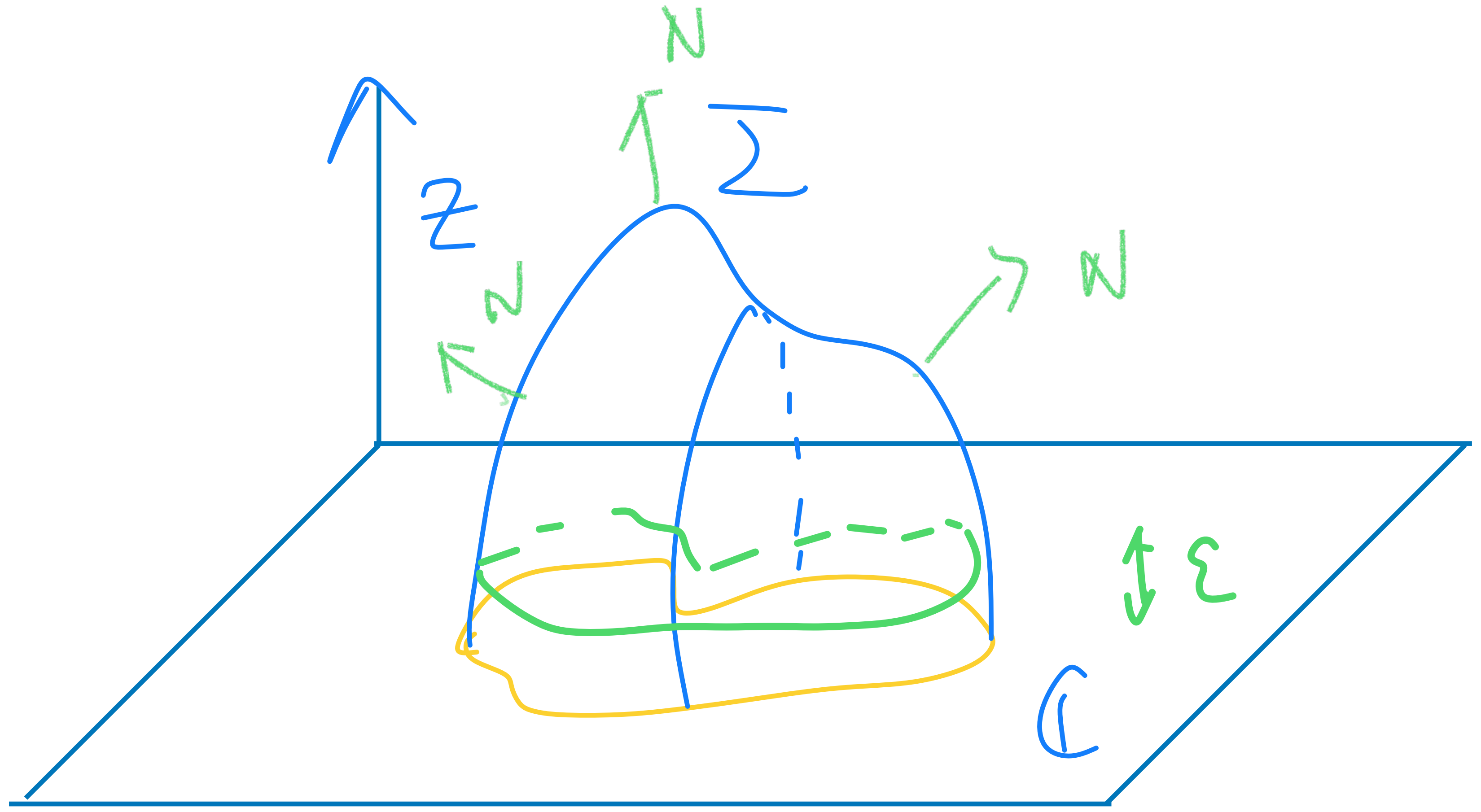

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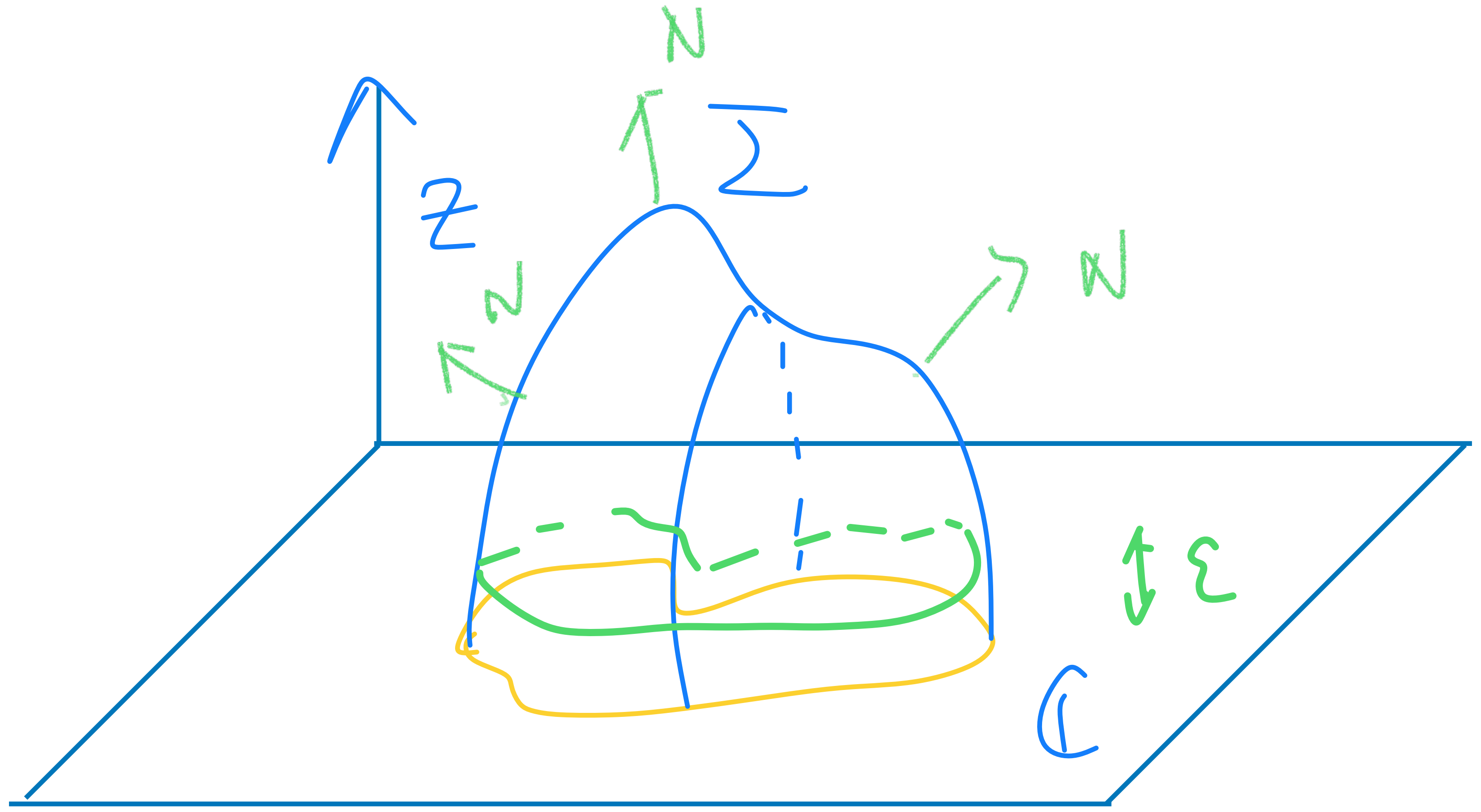

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- The total curvature of different surfaces are also different.

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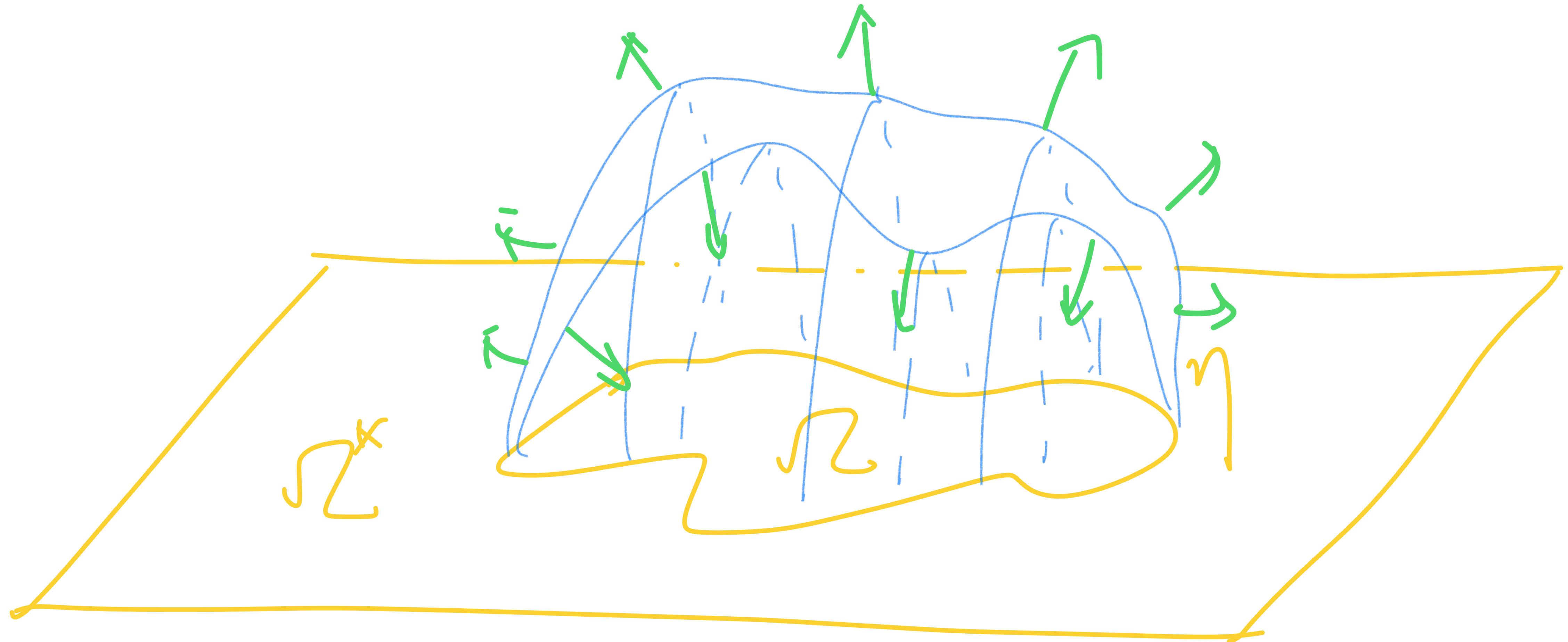

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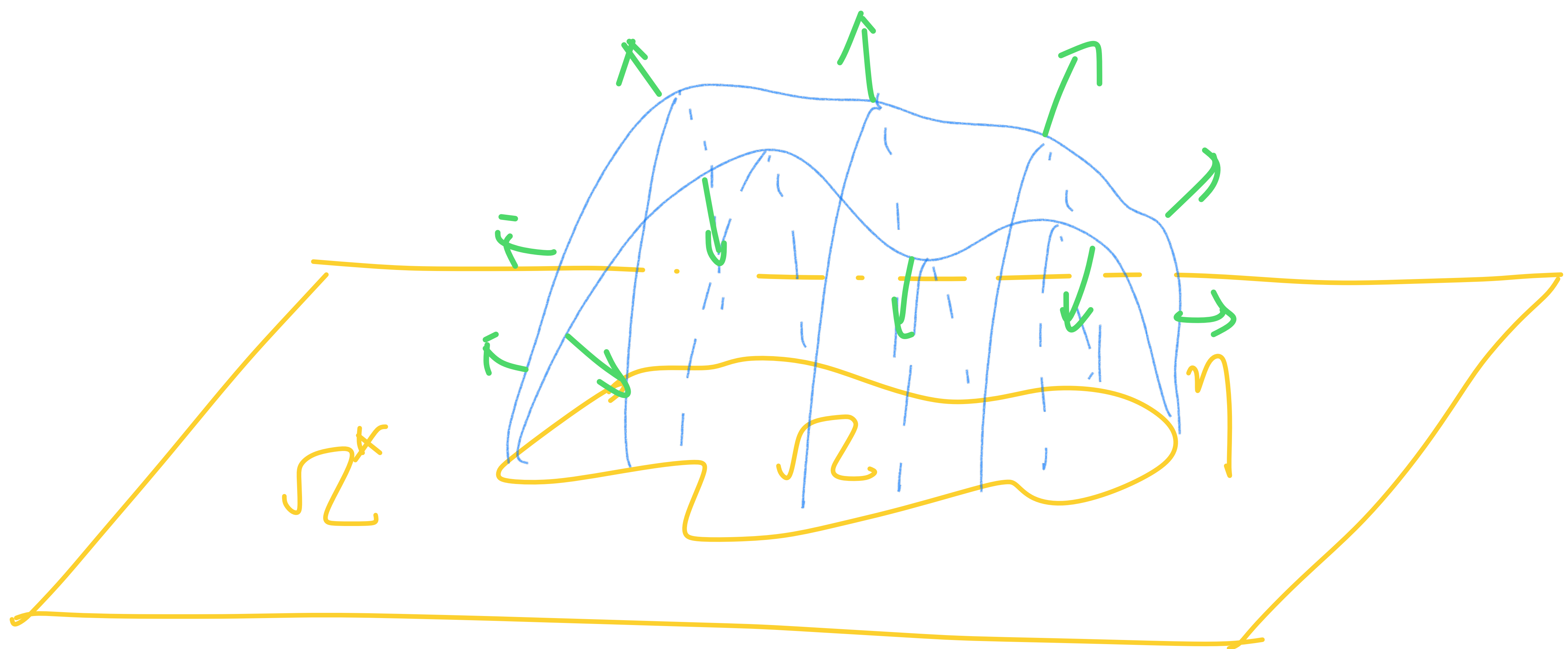

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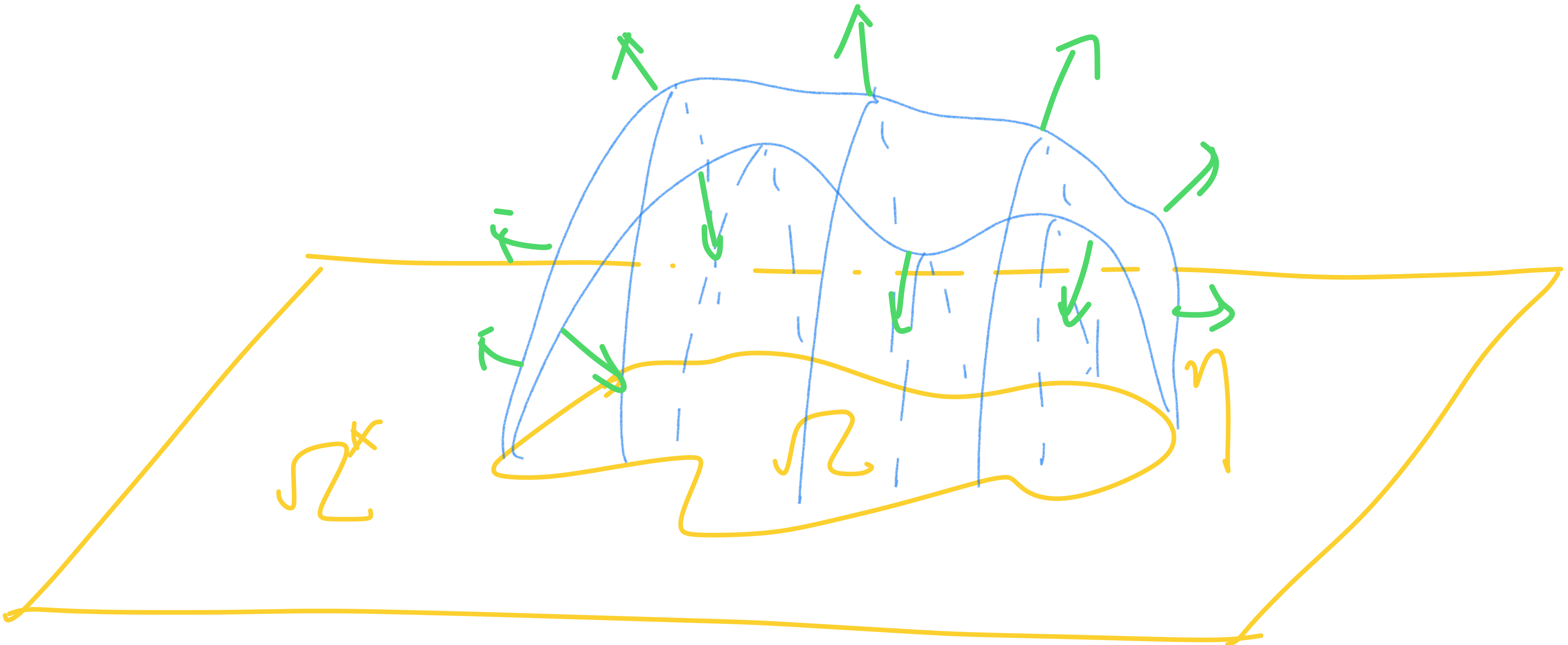

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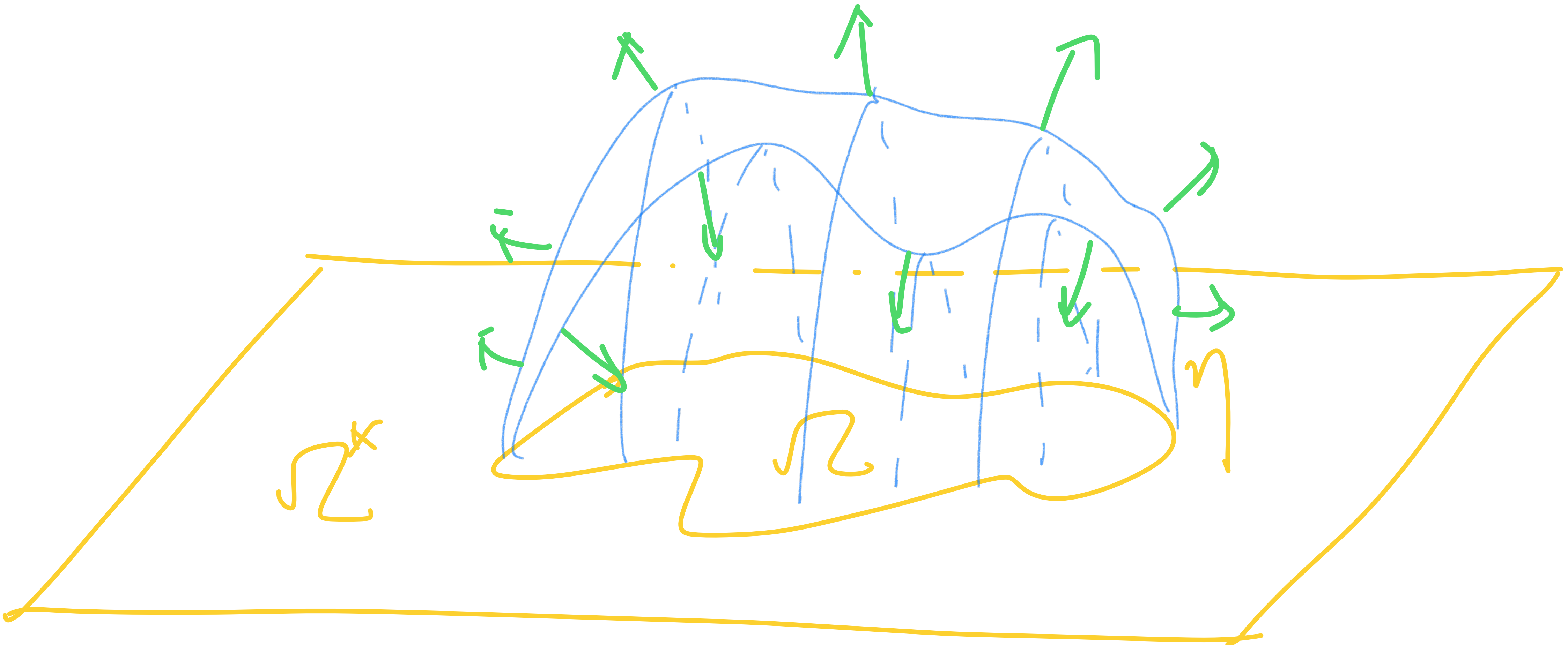

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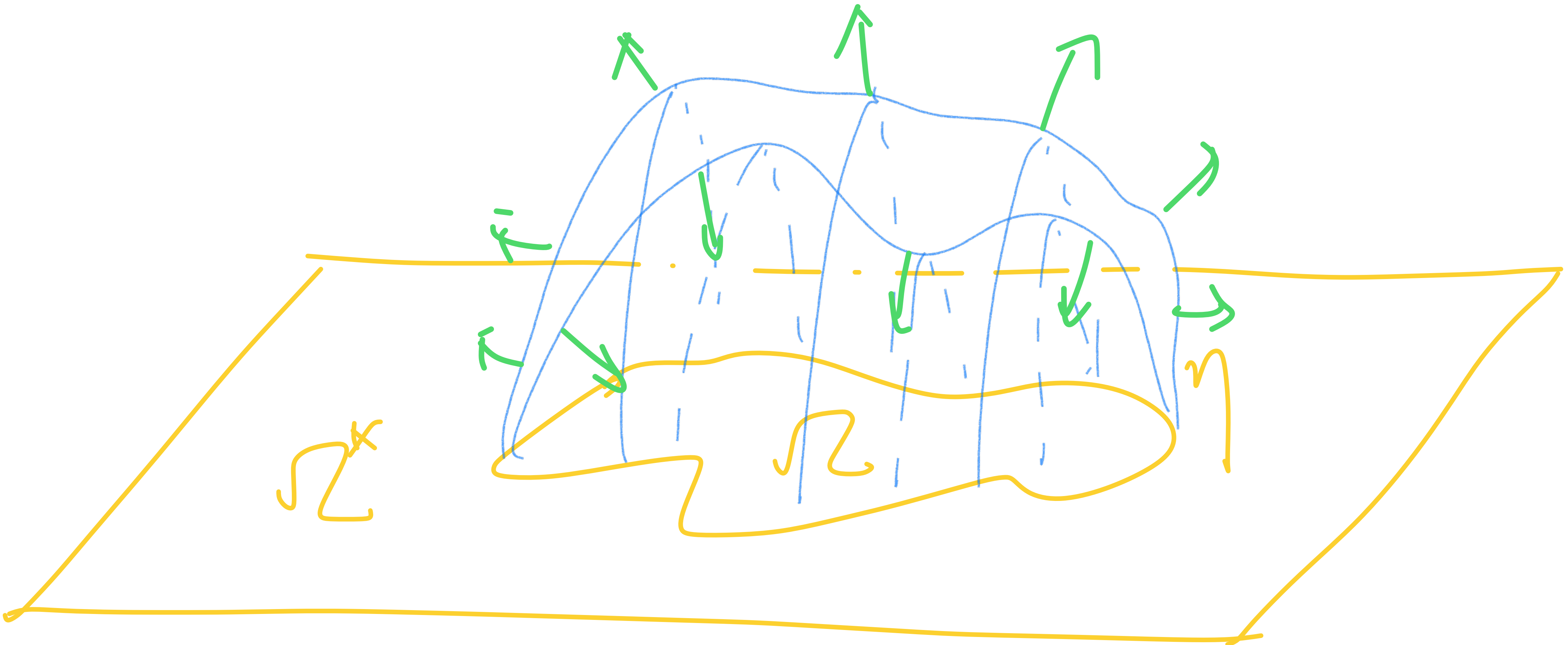

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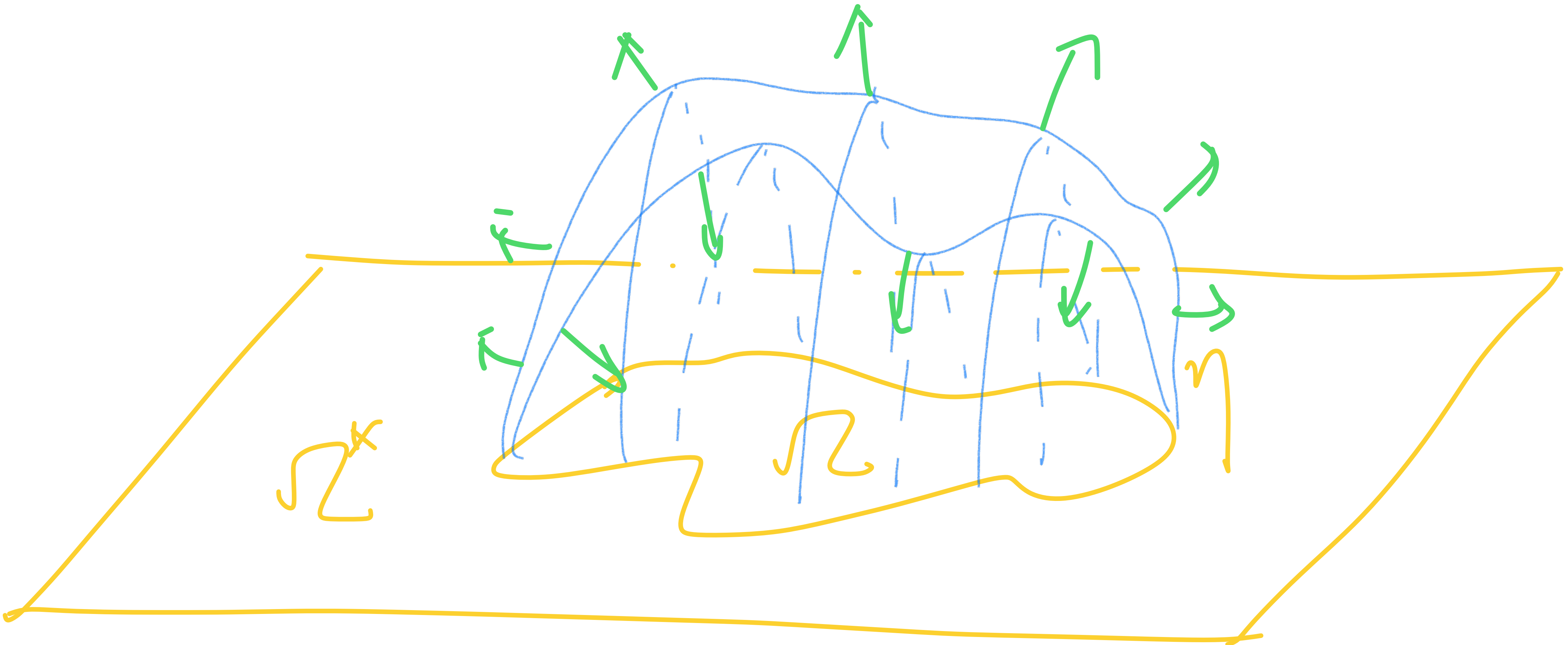
H = mean curvature



Renormalized volume: Motivation

[Hennigson-Skenderis][Graham-Witten] [Krasnov]

[Krasnov-Schlenker][Takhtajan-Zograf][Takhtajan-Teo]

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But the classical Liouville action $S[g_{hyp}, 0] = \mathrm{Area}(X) = -2\pi\chi = 4\pi(\text{genus} - 1)$: $S[g_{hyp}, 0]$ does not depend on the moduli.

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Our result is the version for the universal Teichmuller space (dim = ∞).

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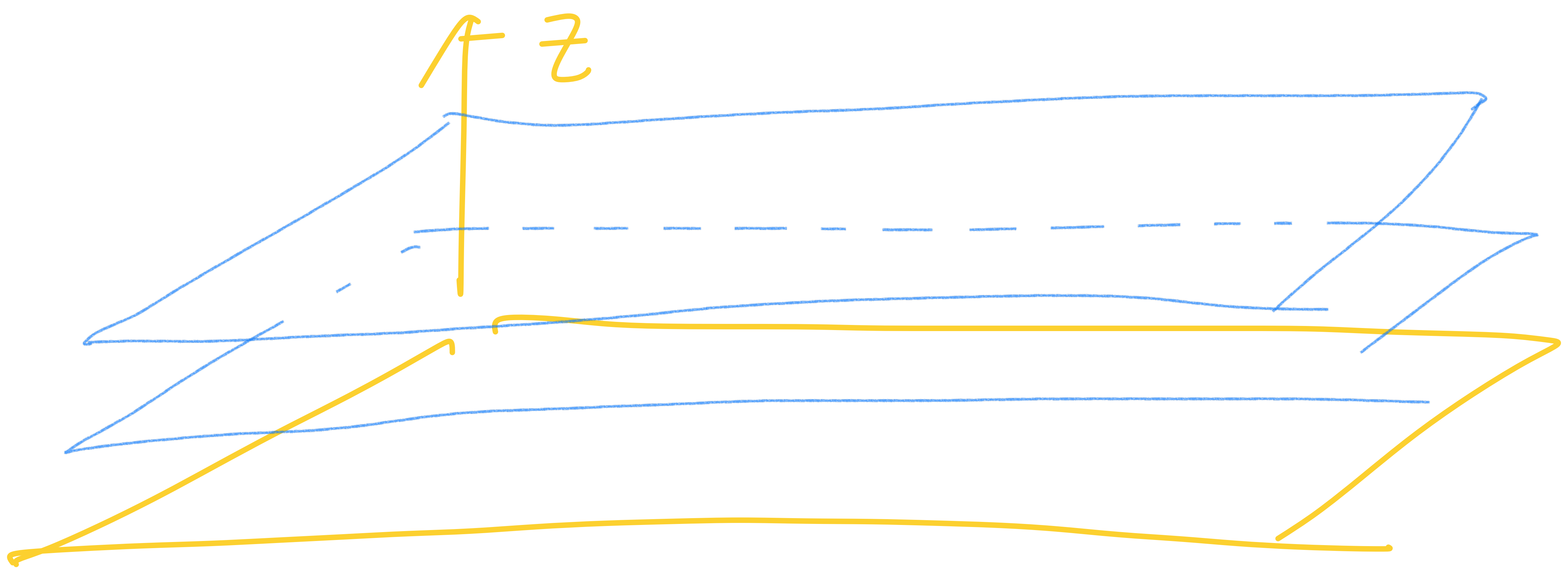

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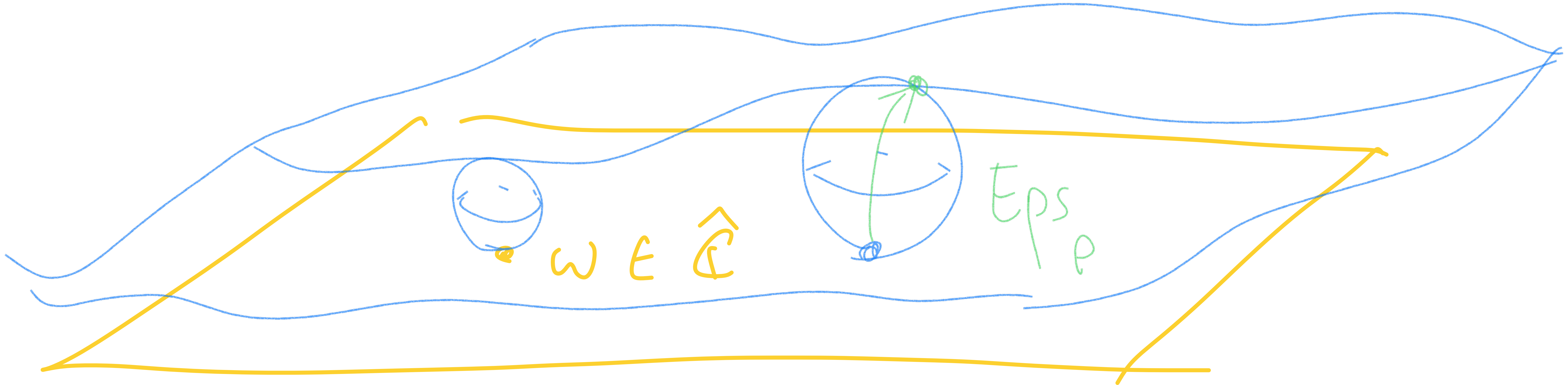

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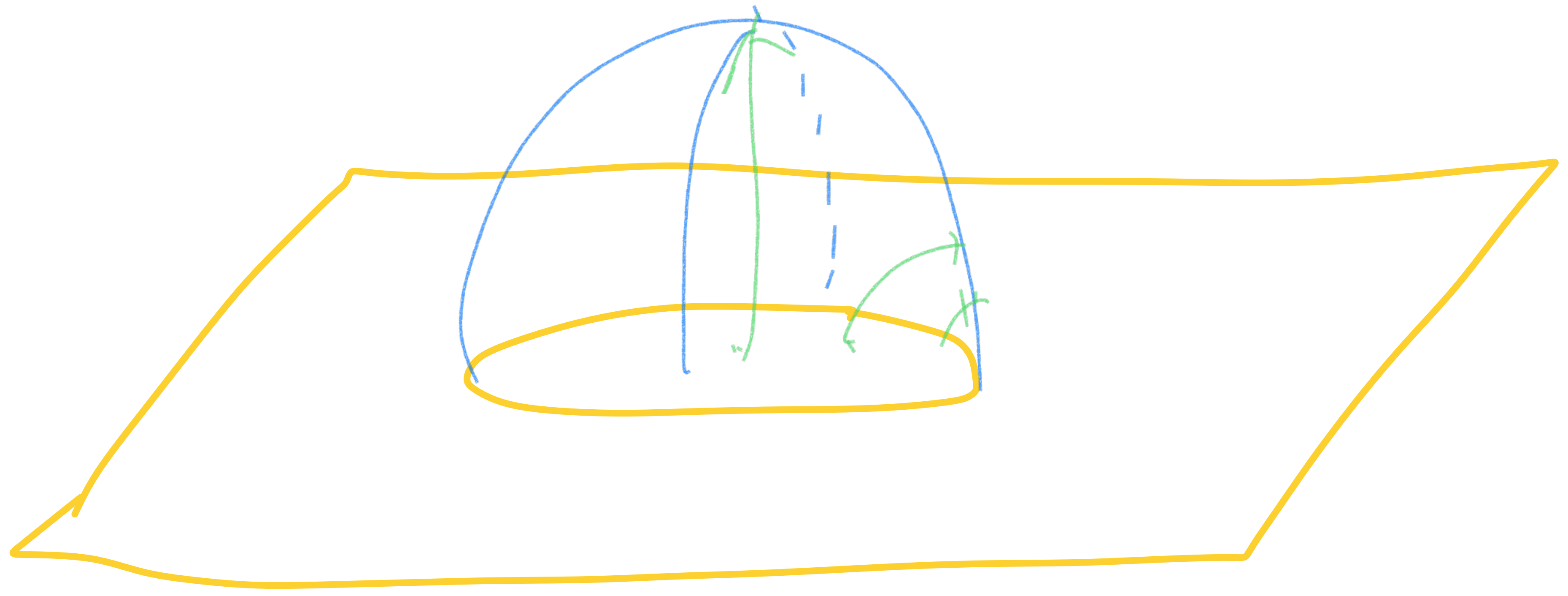

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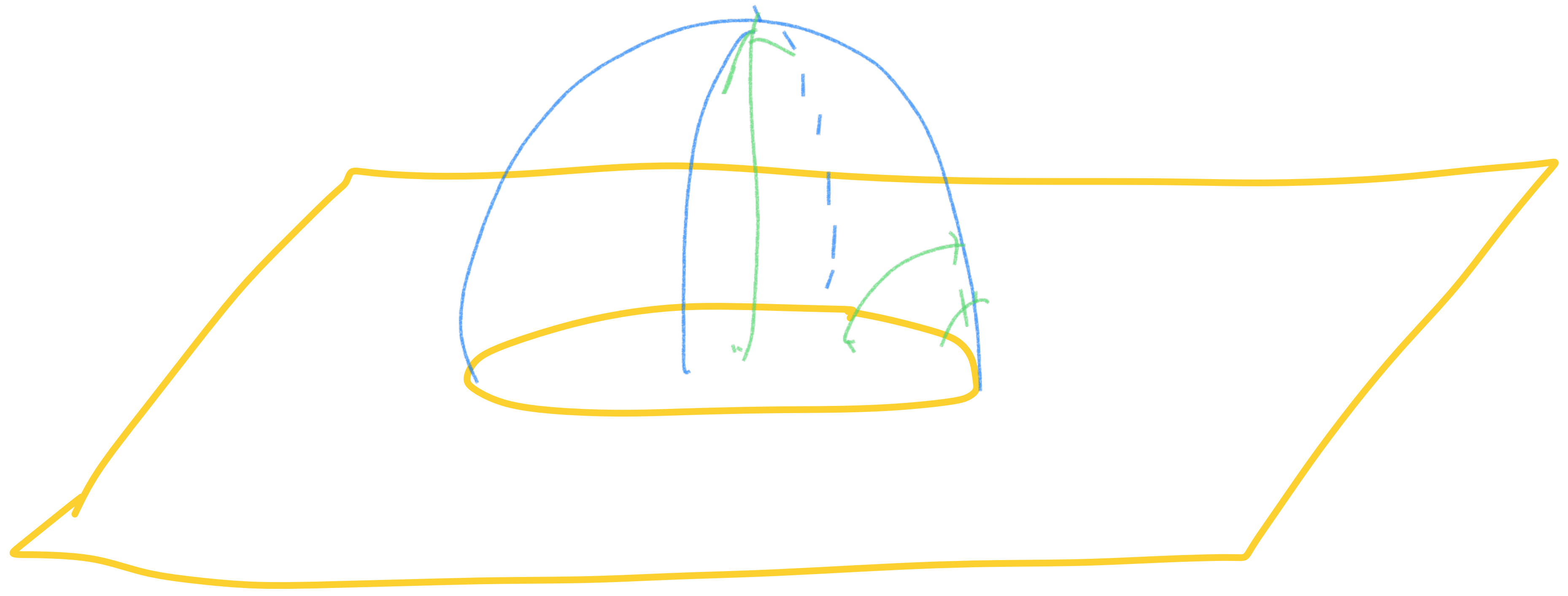

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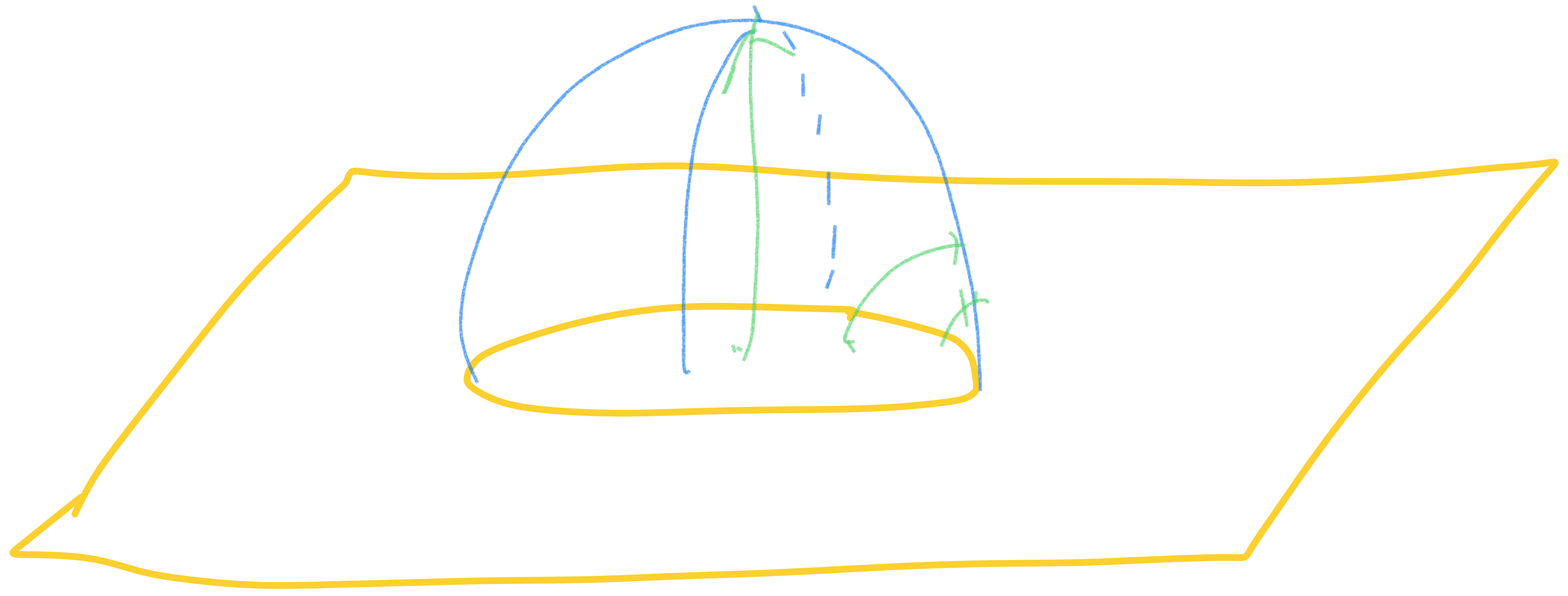

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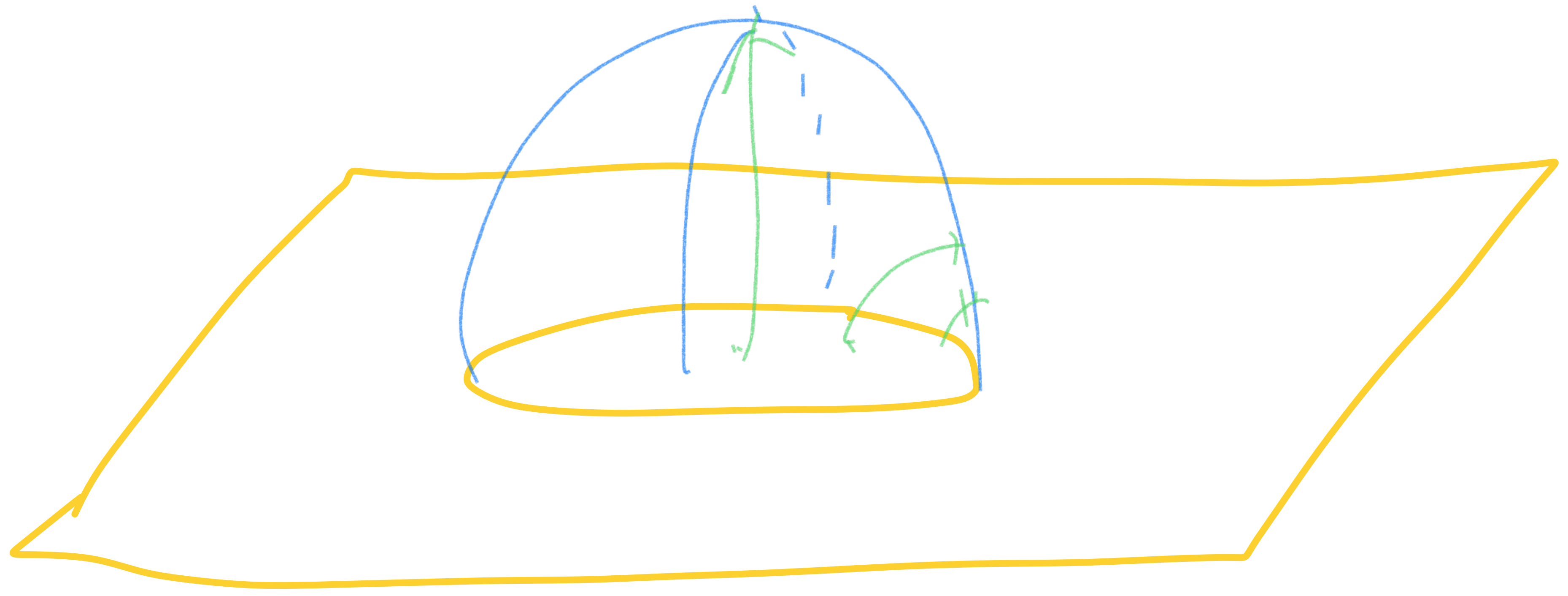

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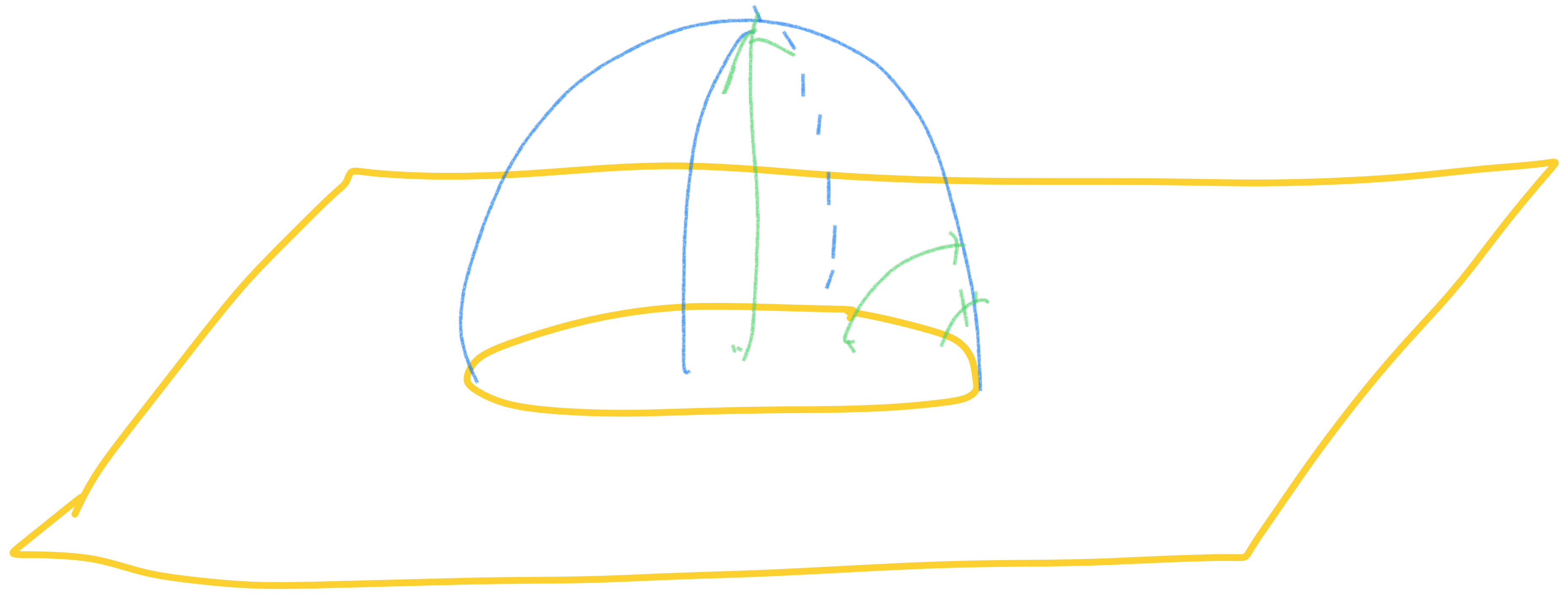

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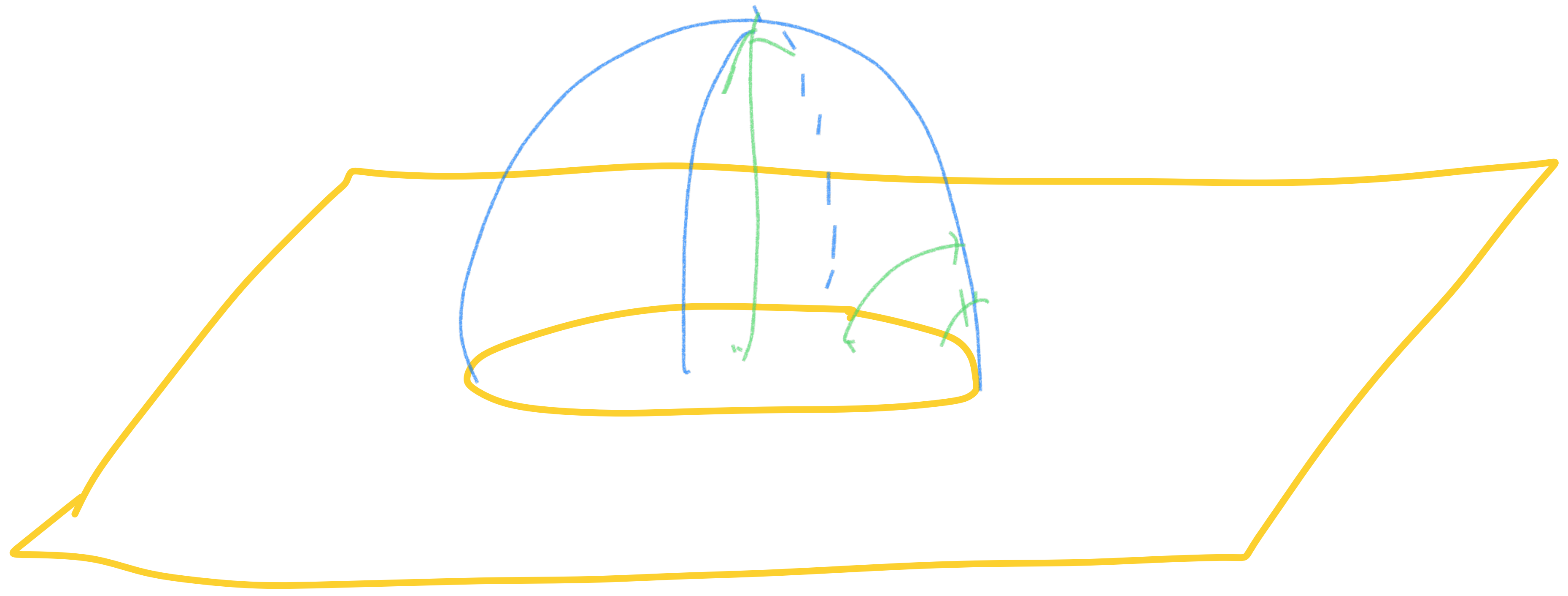

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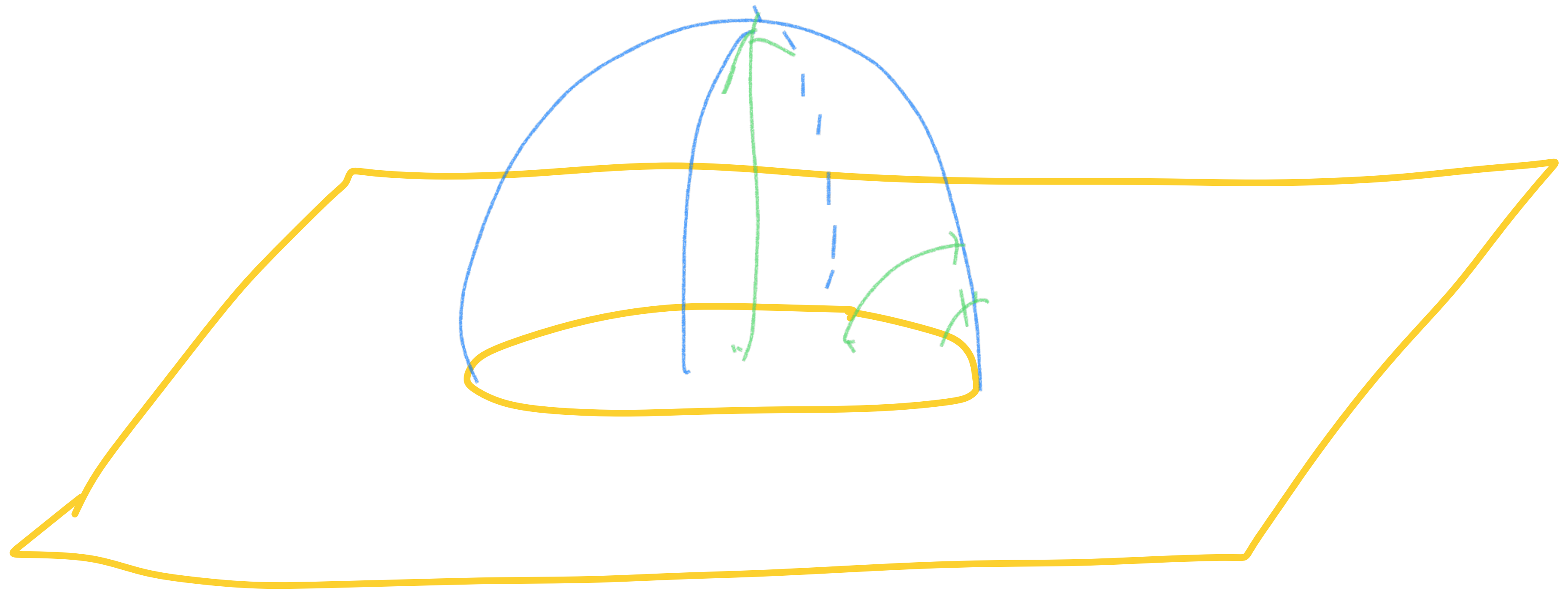
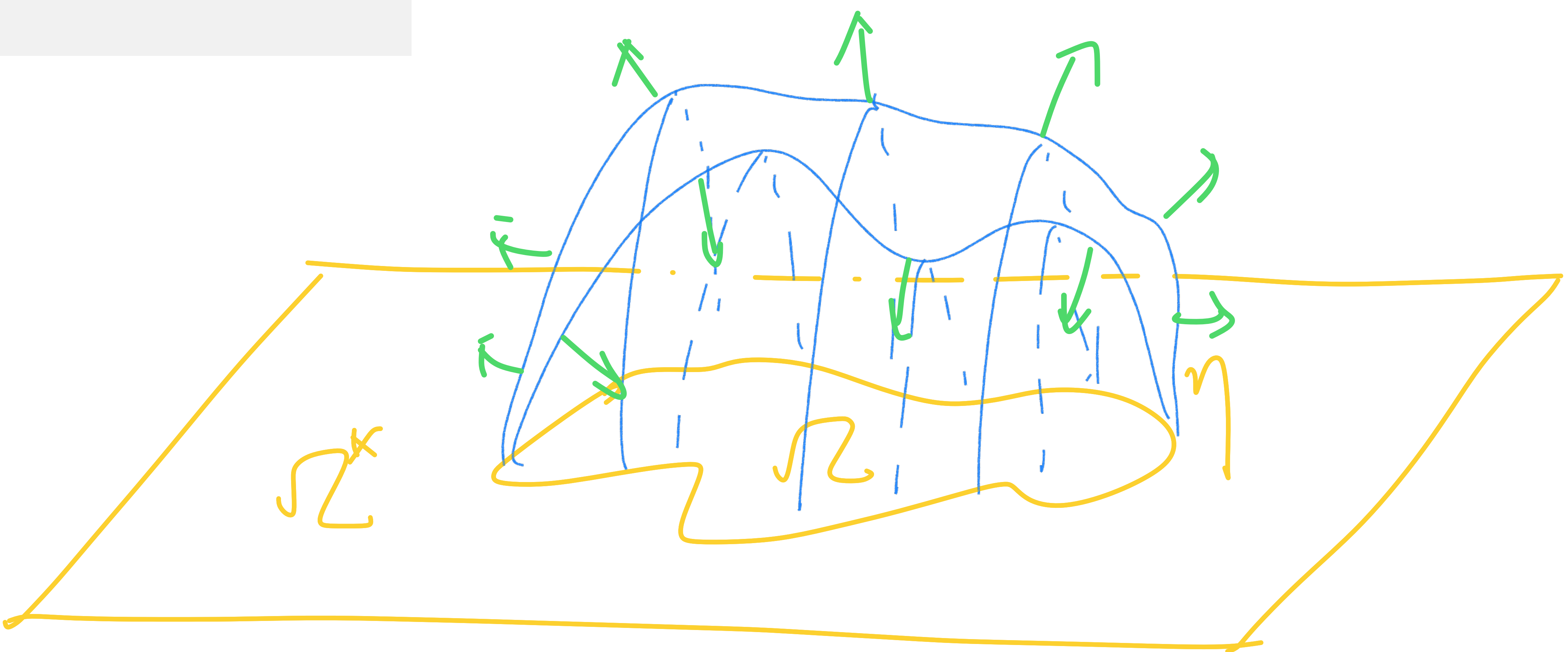


Definition of N_η and $V_R(N_\eta)$



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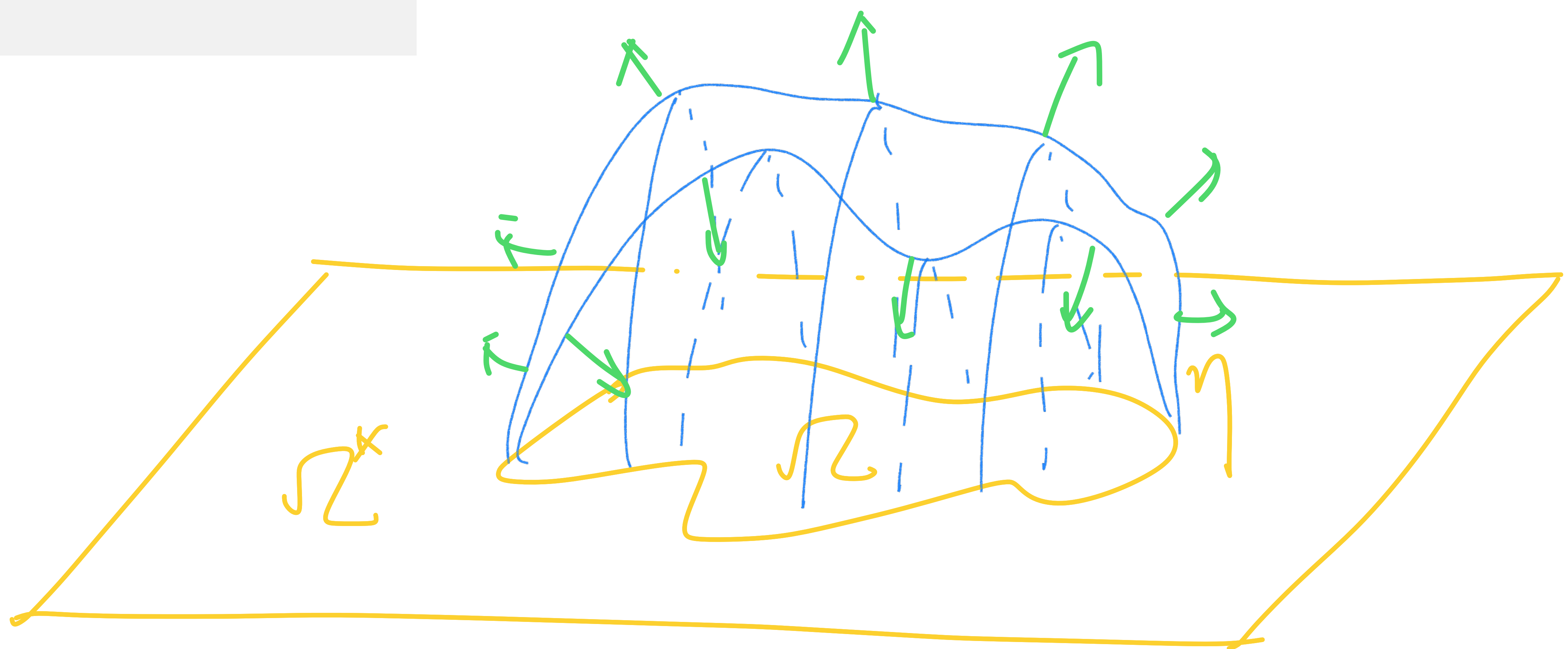

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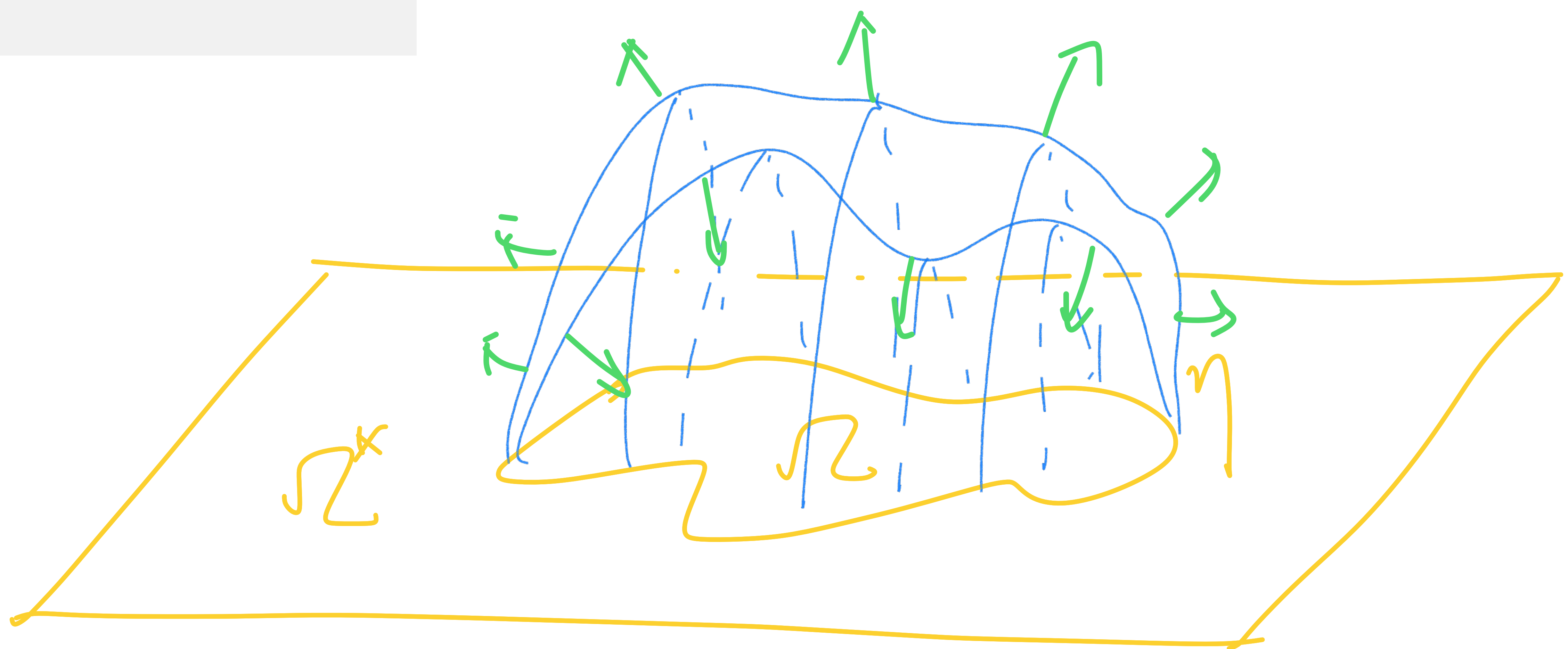

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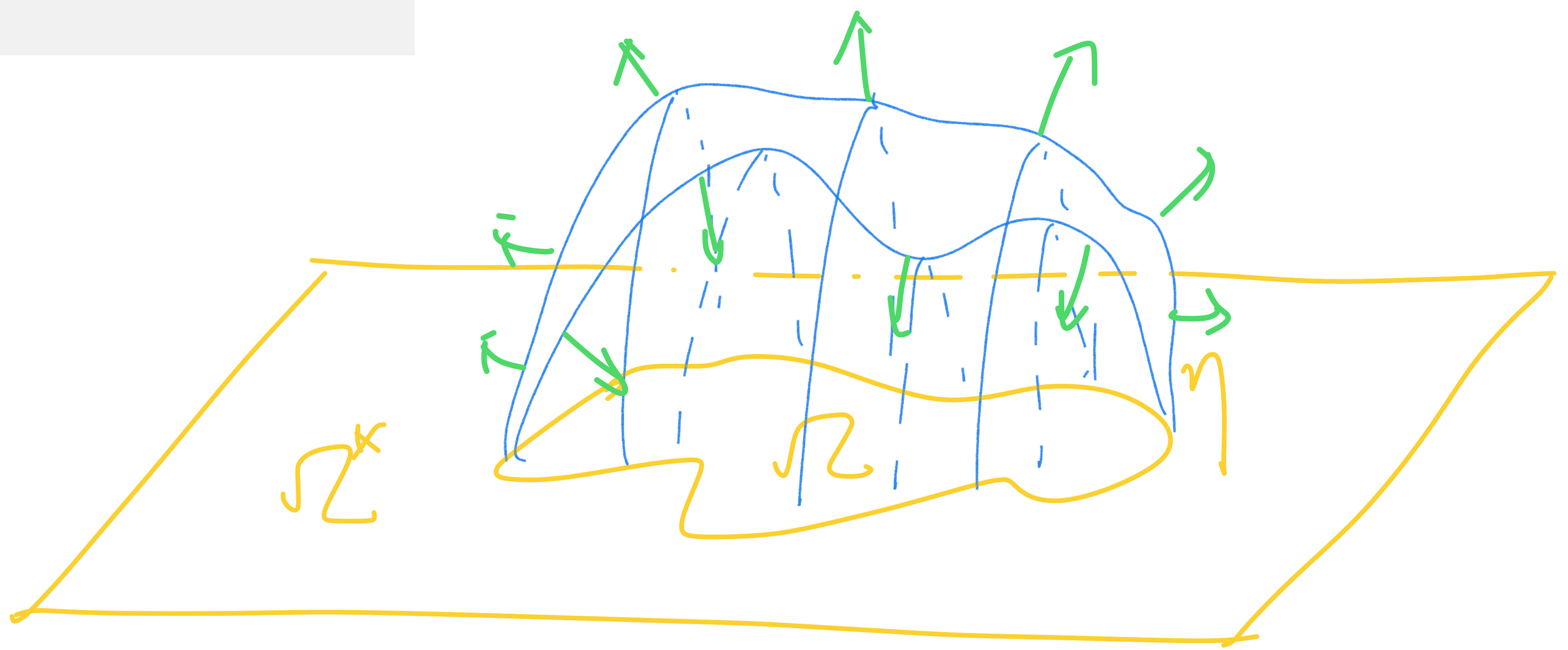

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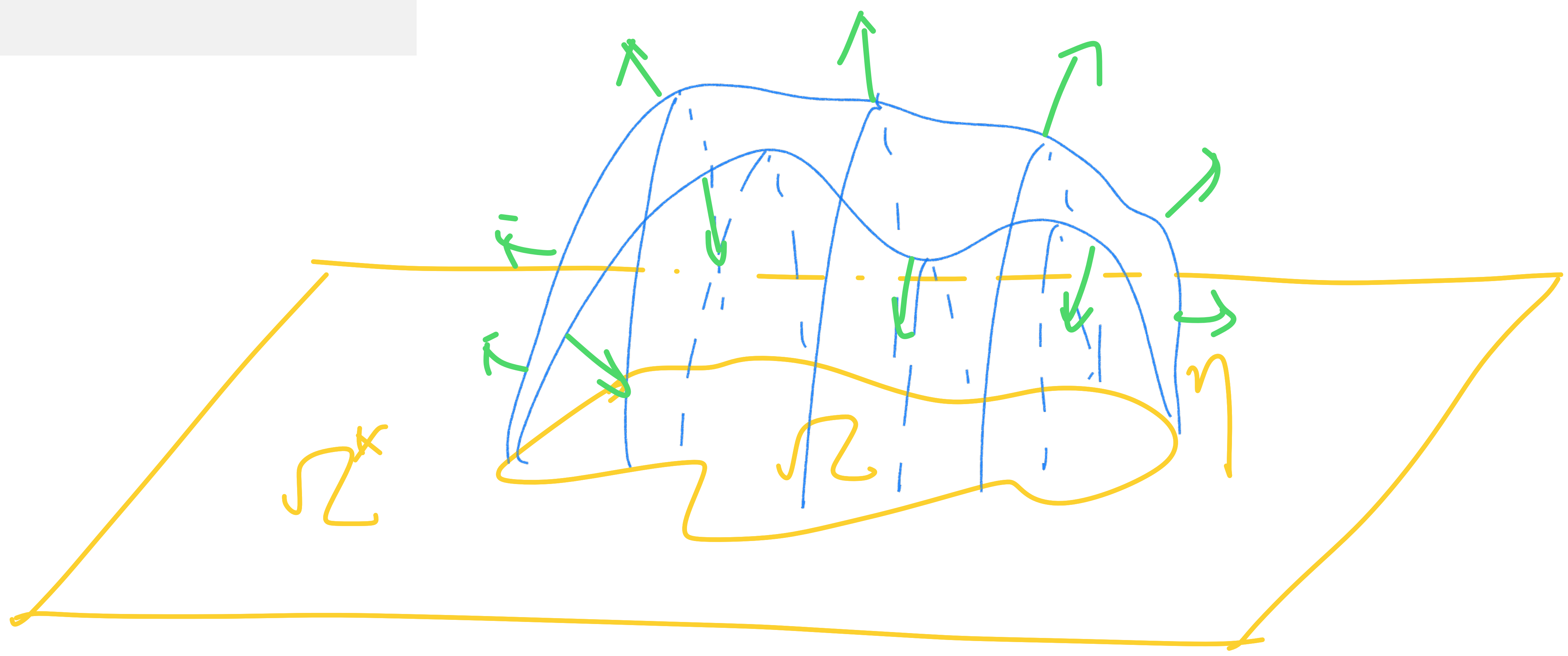

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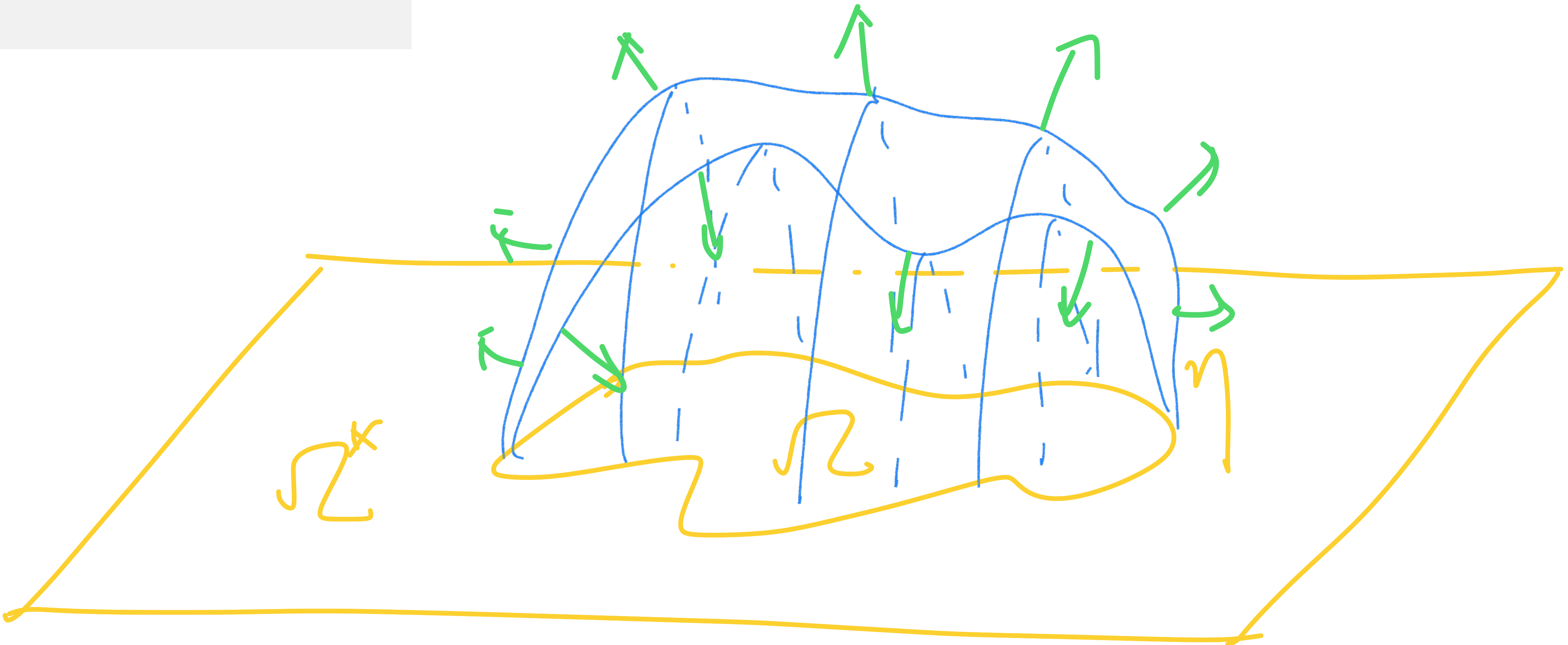

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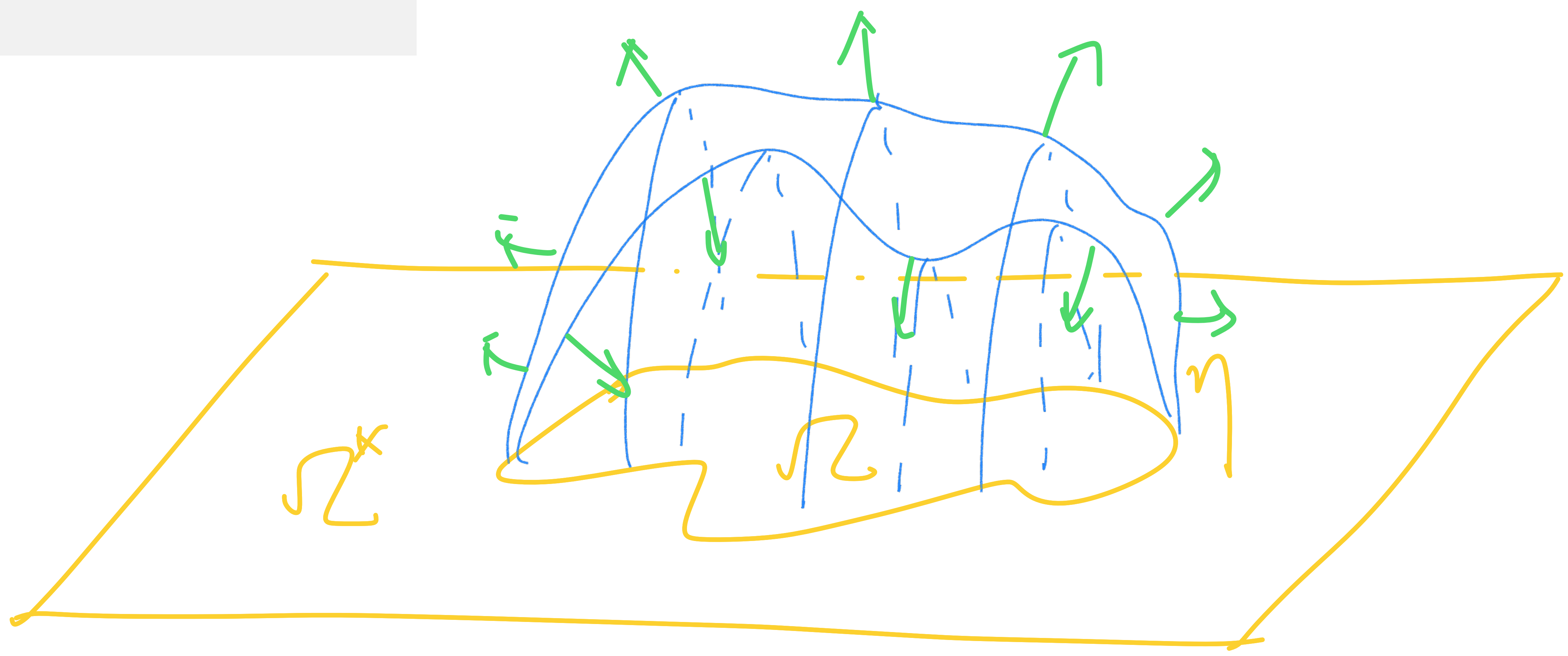

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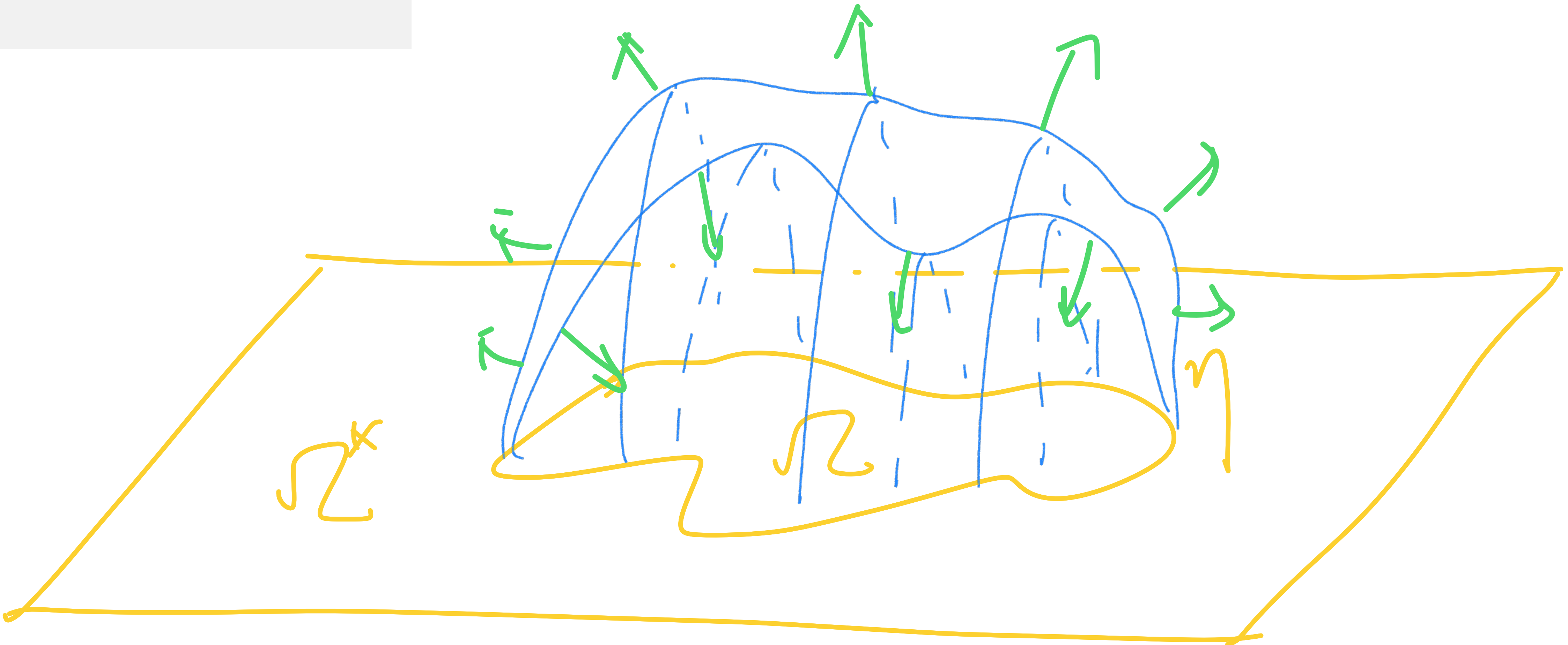

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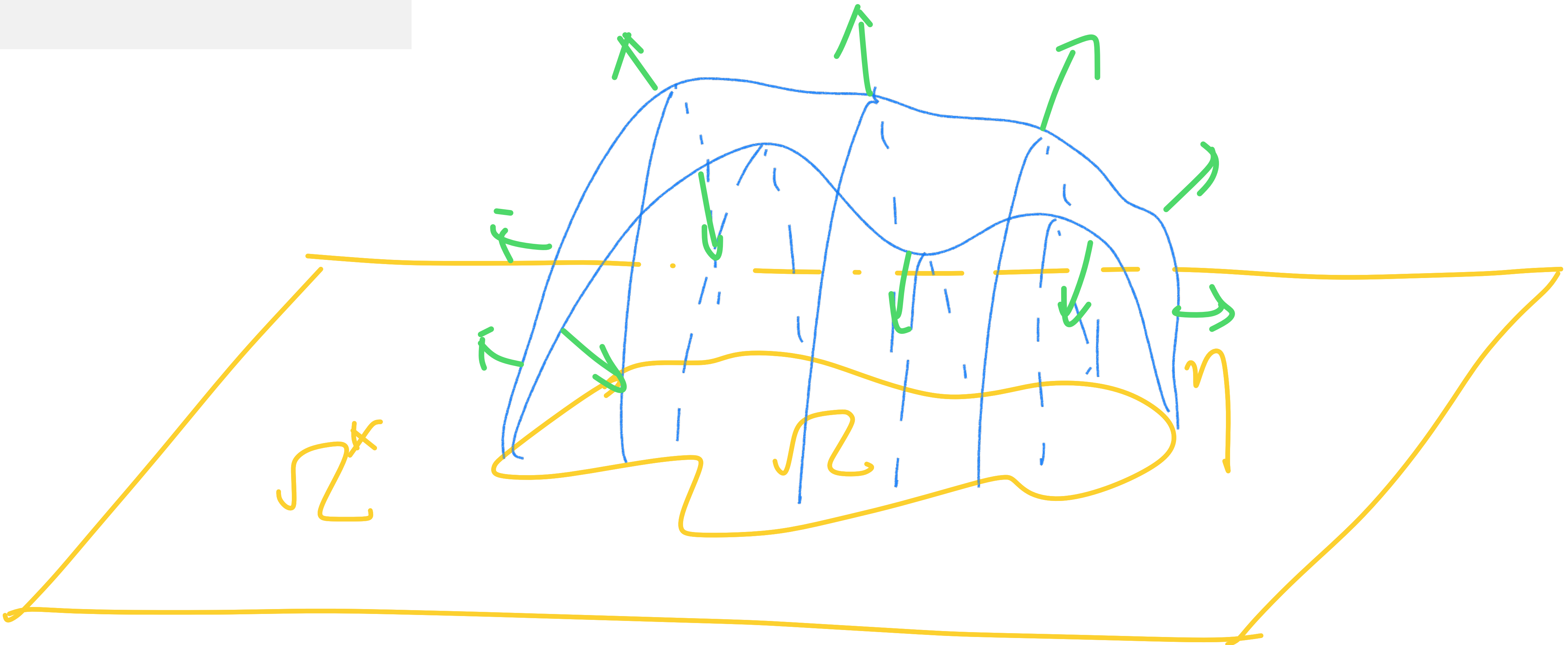

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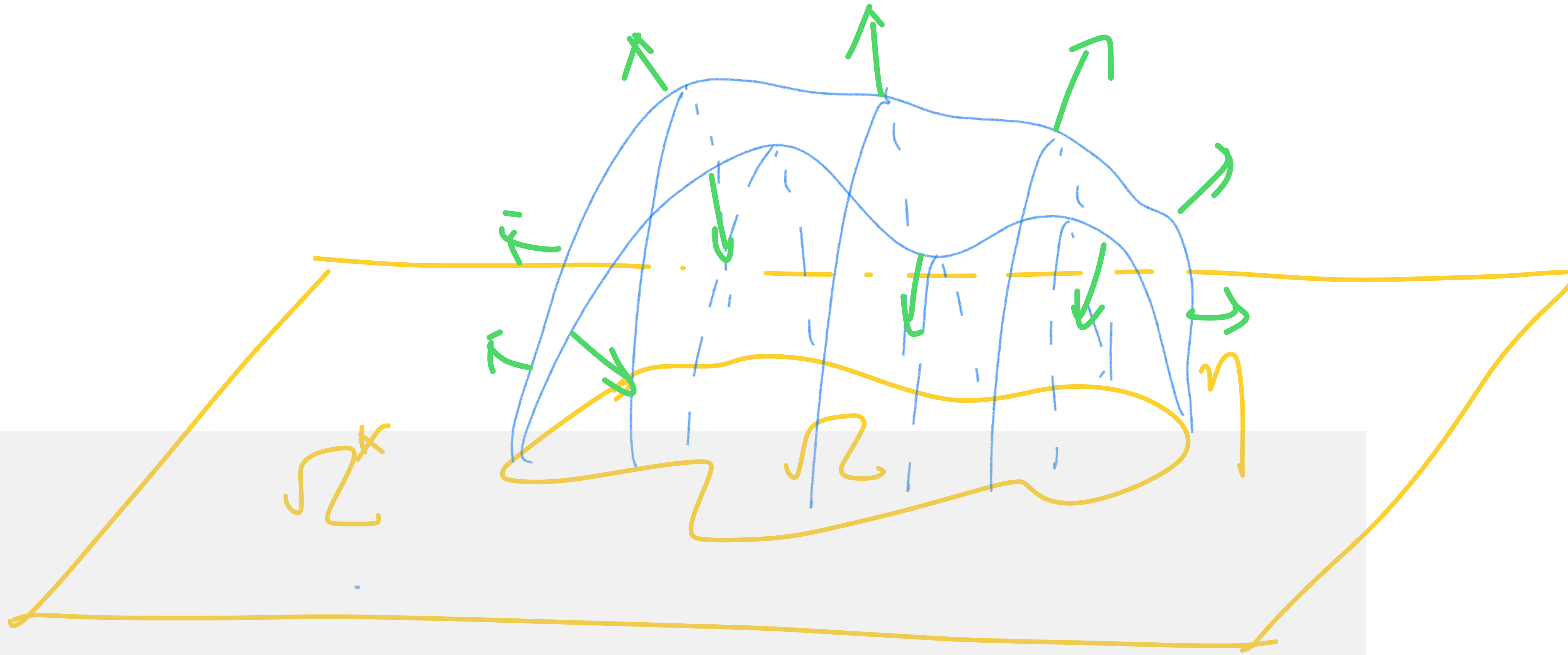
Proof: We show that $Vol(N_\eta) < \infty$, and $I^L(\eta)$ and $4V_R(N_\eta)/\pi$ satisfy the same variation formula and vanish when $\eta = S^1$.

Variation formula



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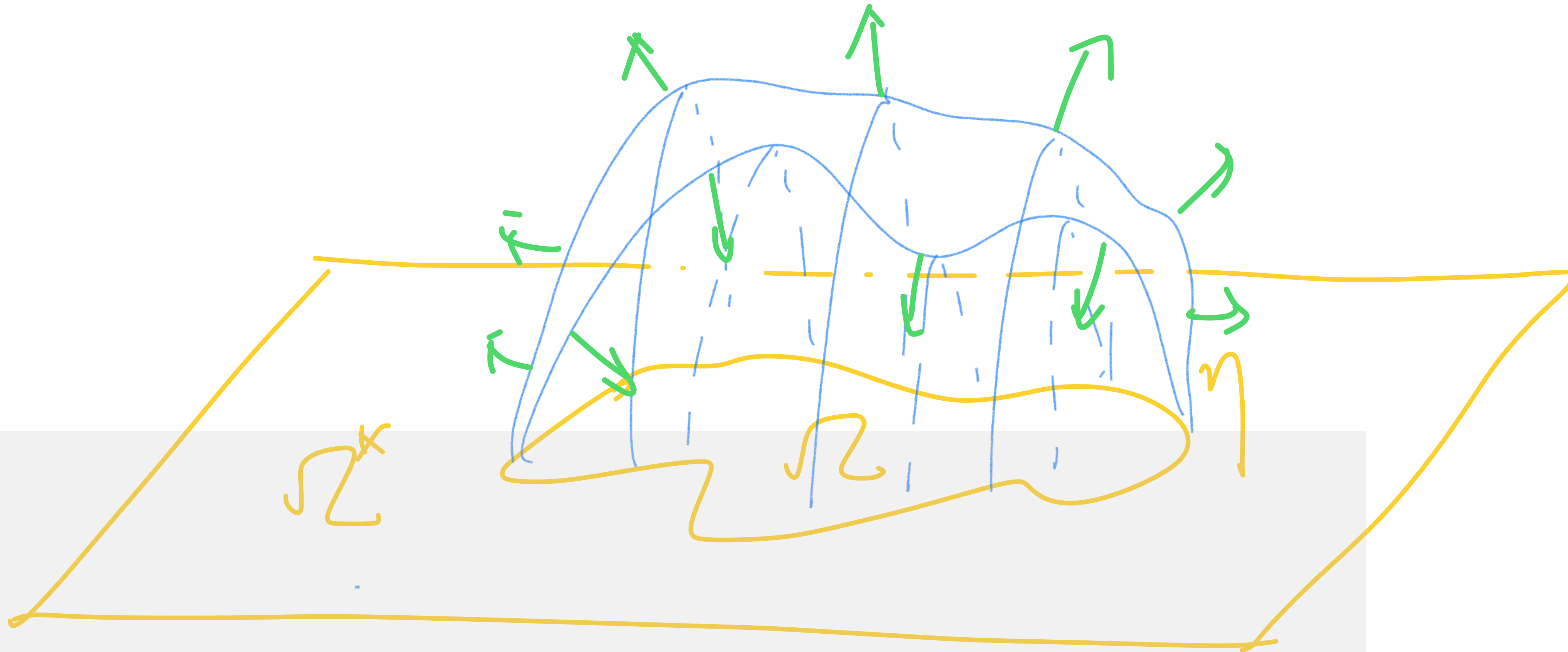

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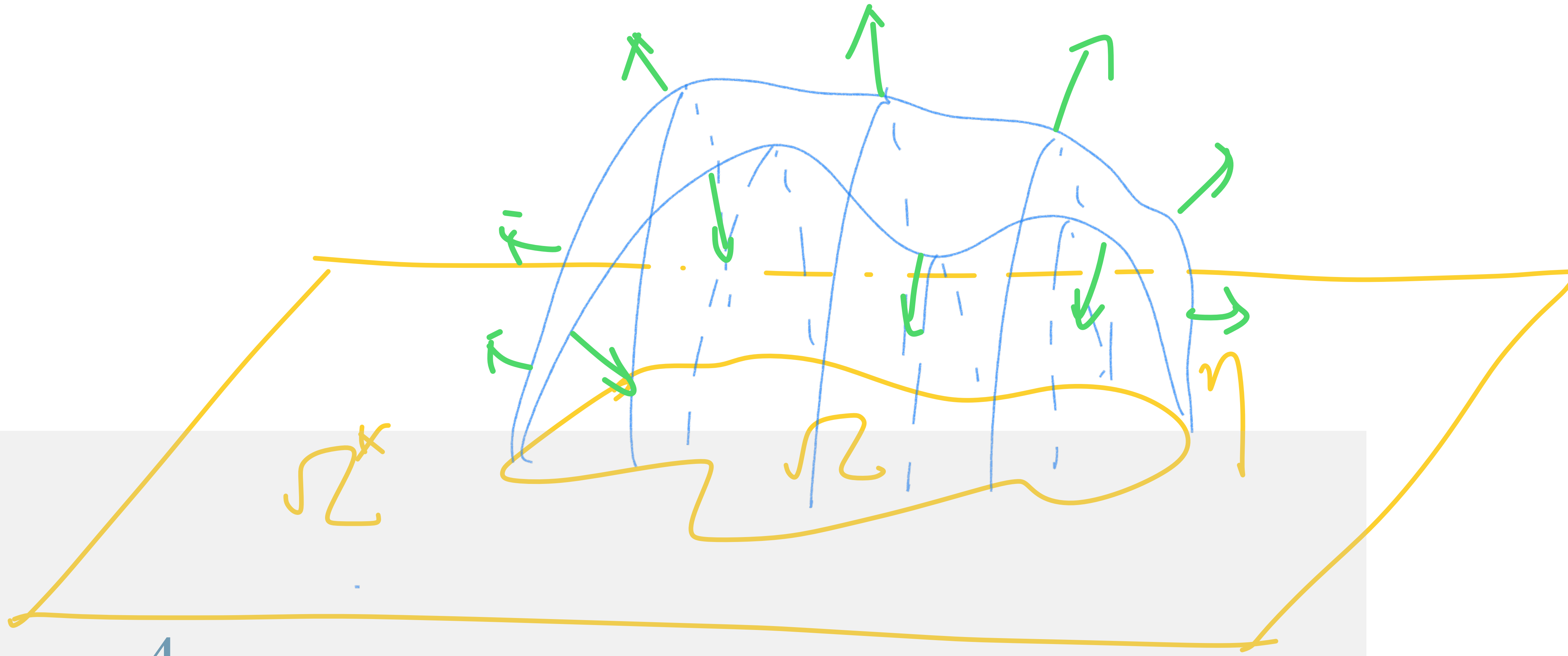

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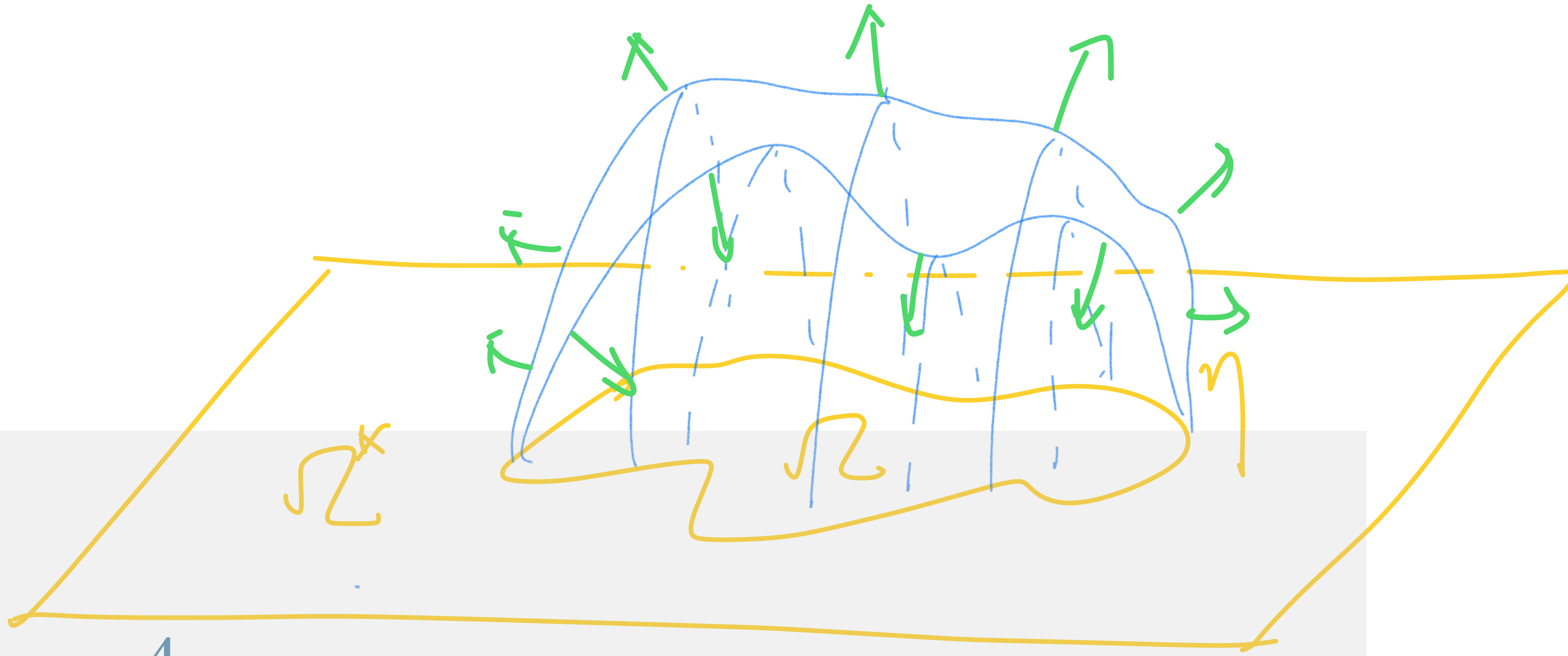

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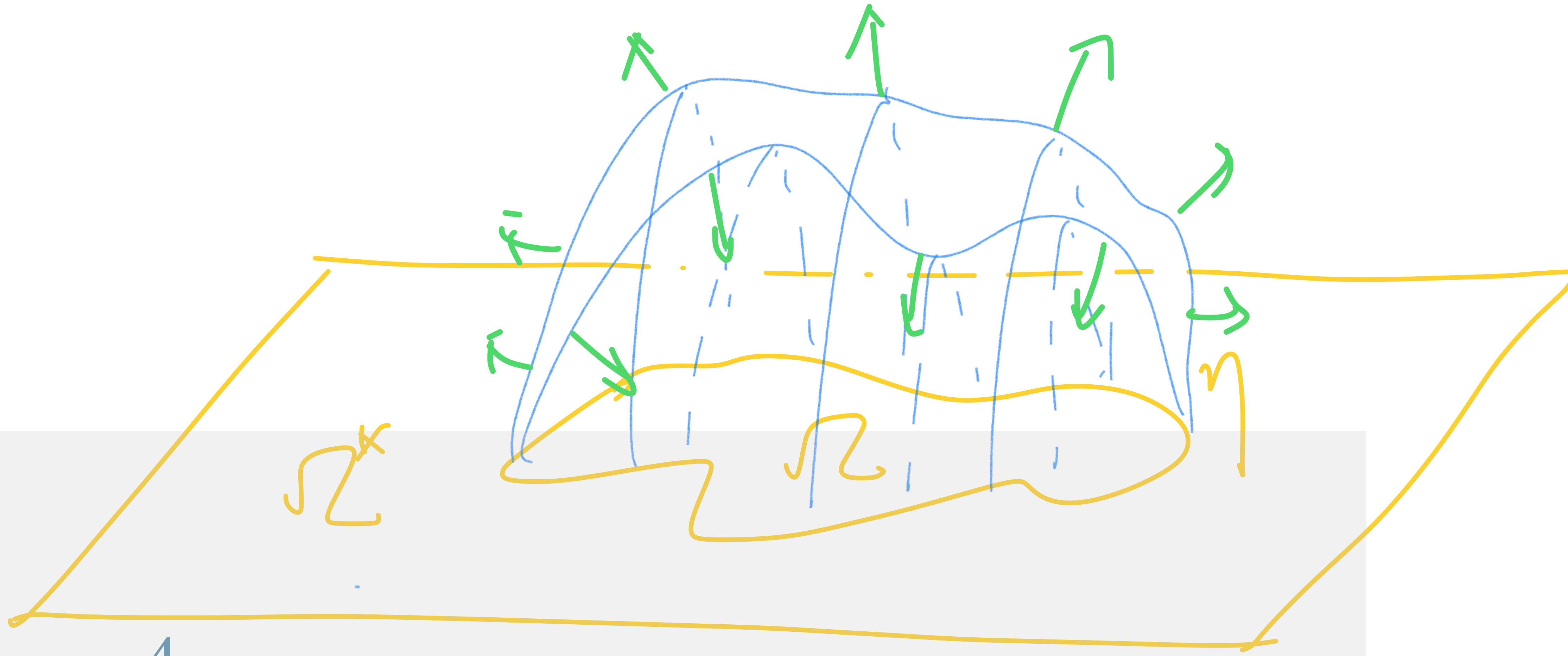

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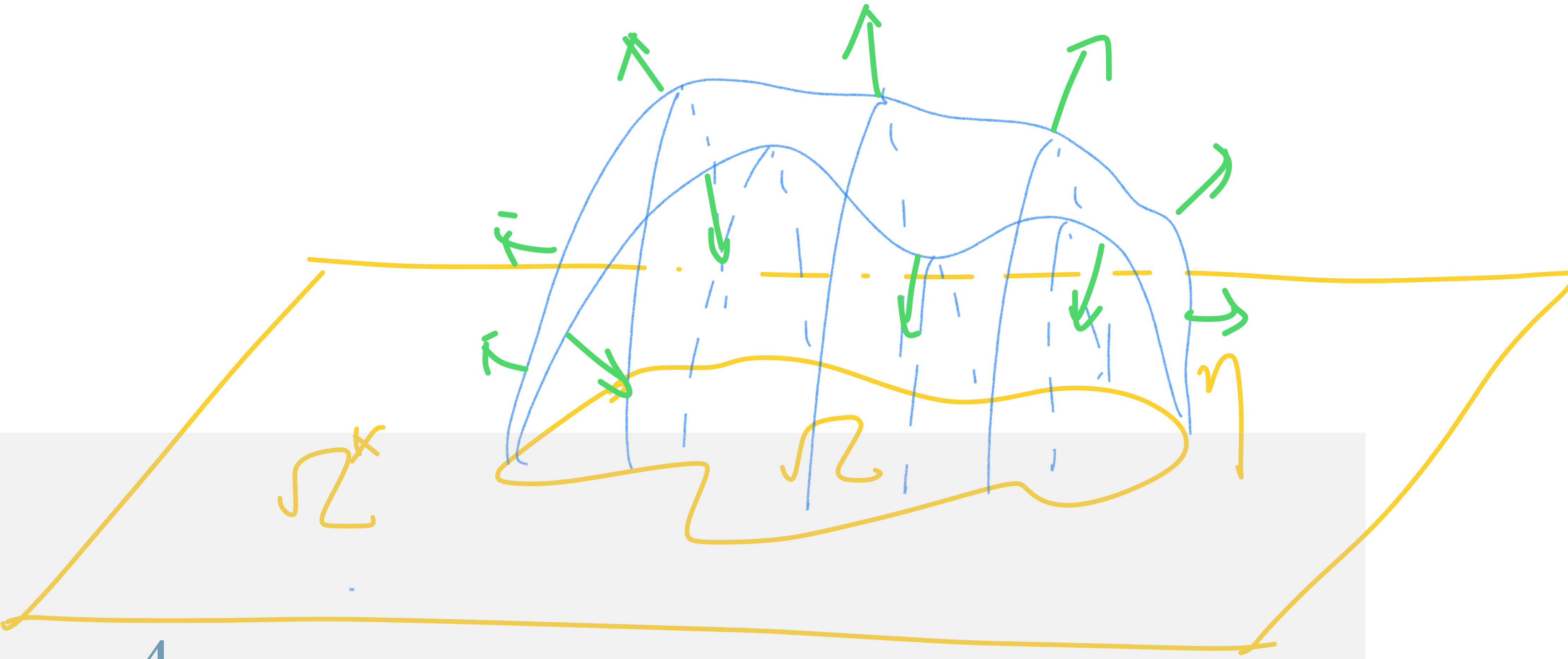

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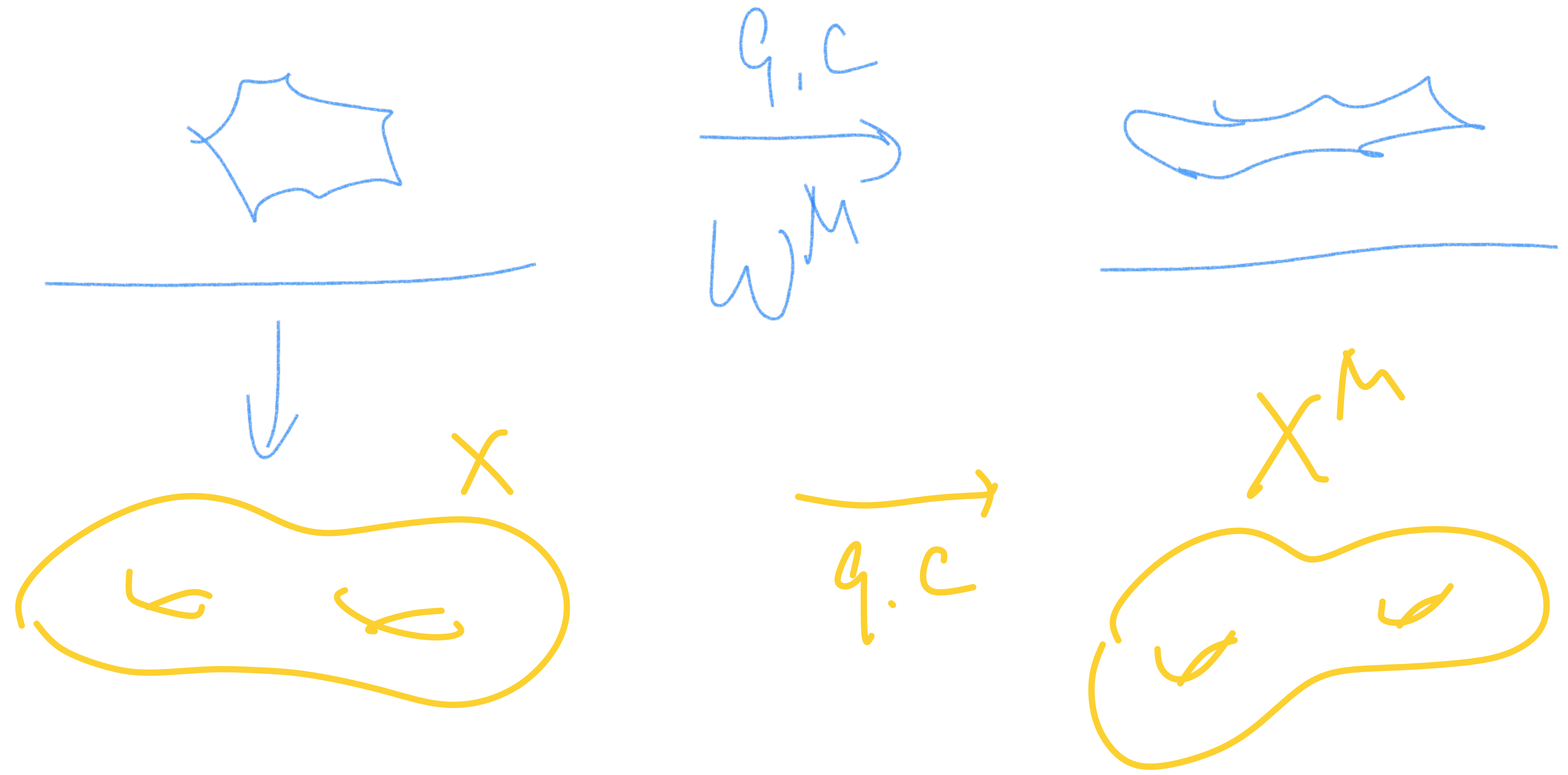
Analogous Liouville action: Quasi-Fuchsian case

[Takhtajan-Teo, *CMP*], [Krasnov-Schlenker, *CMP*]

Teichmuller spaces



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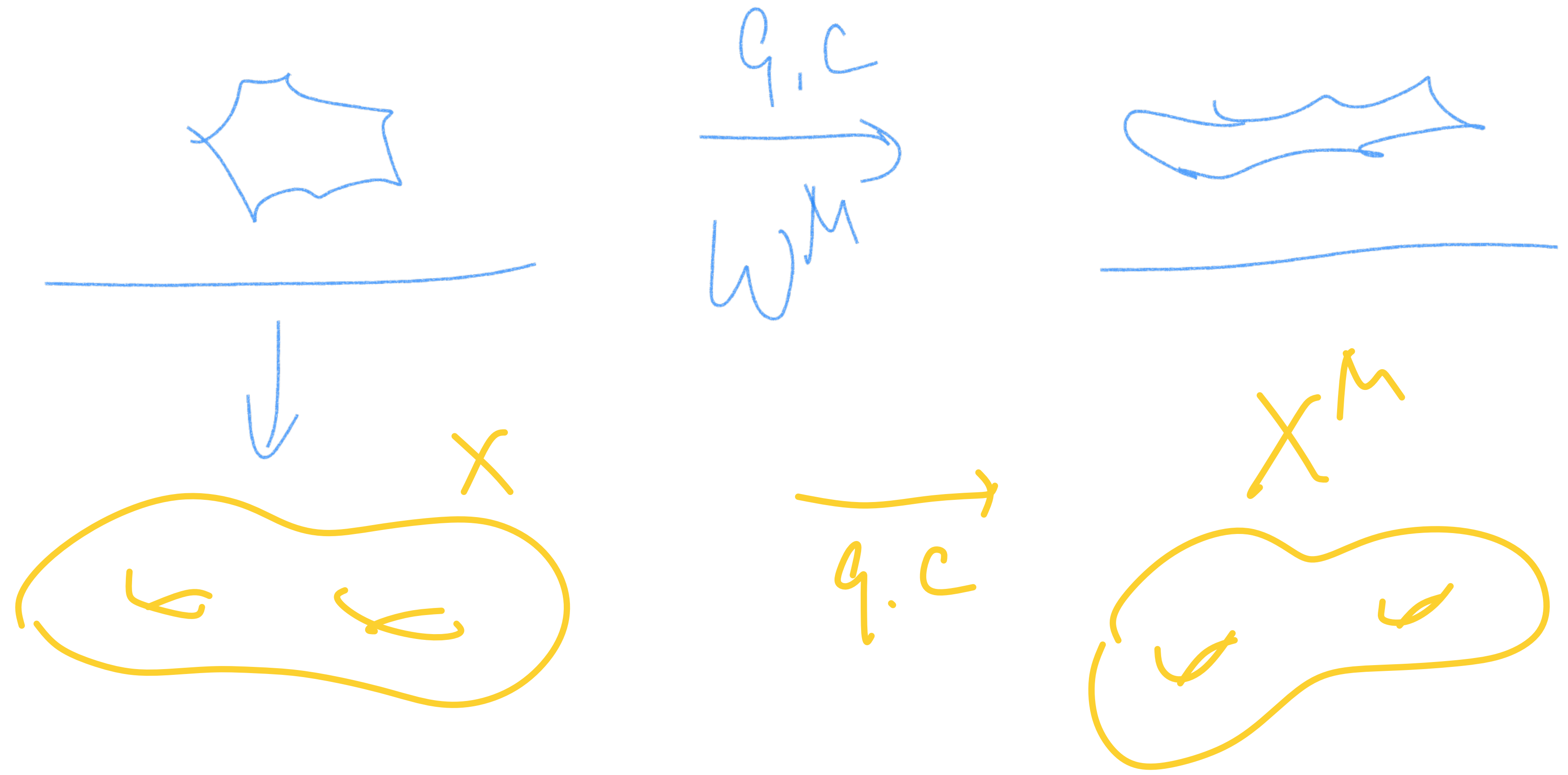

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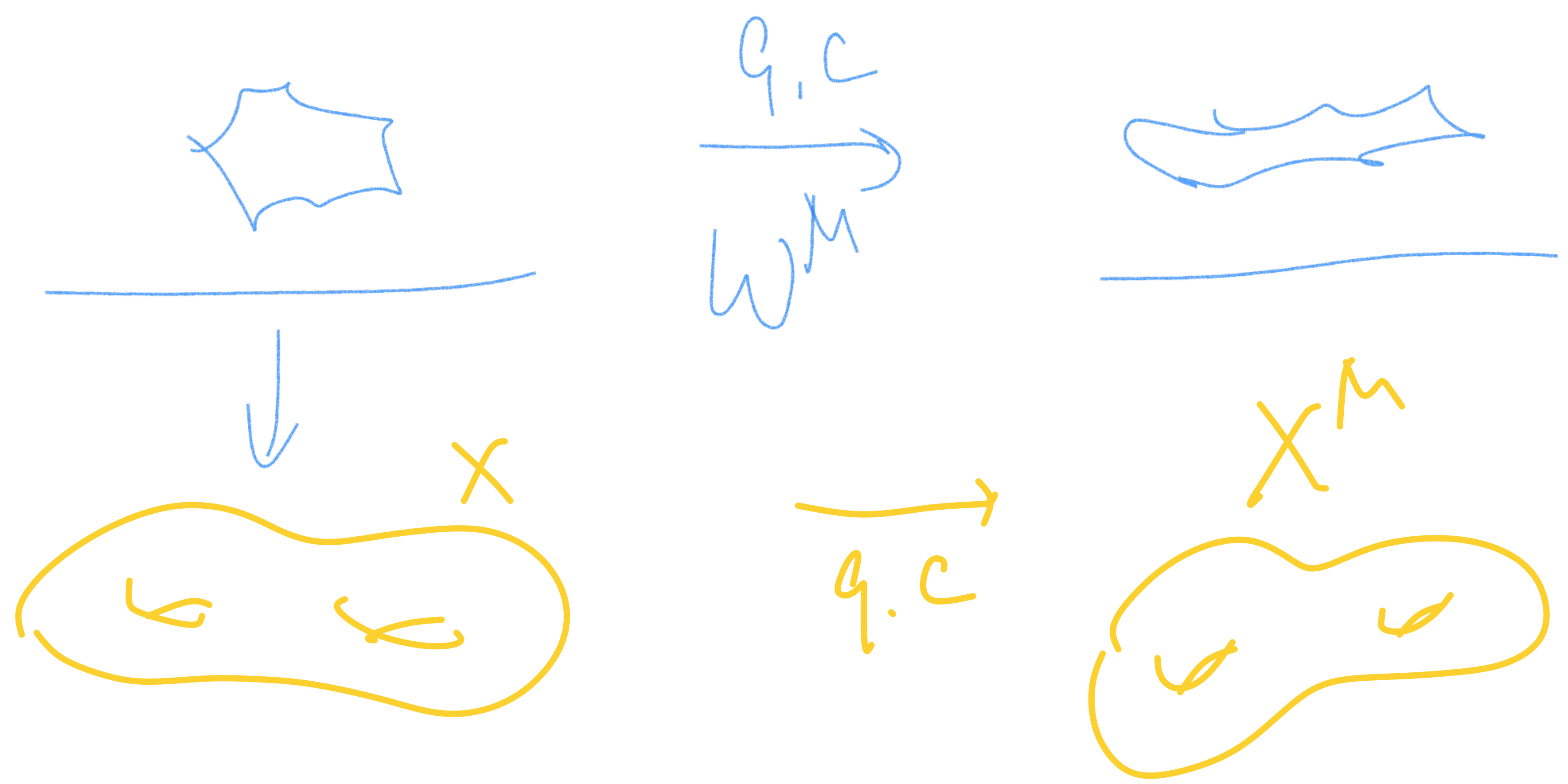

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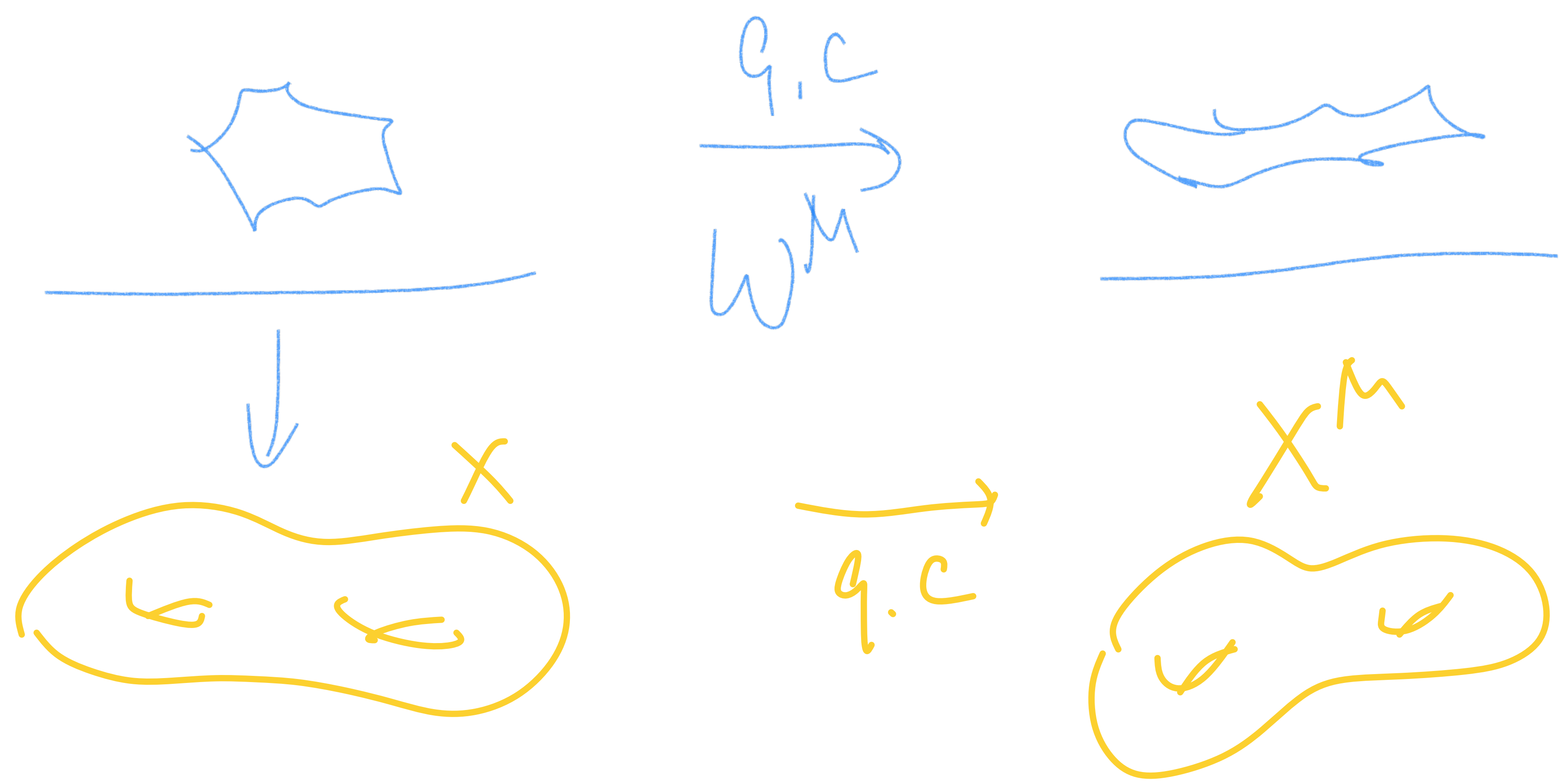

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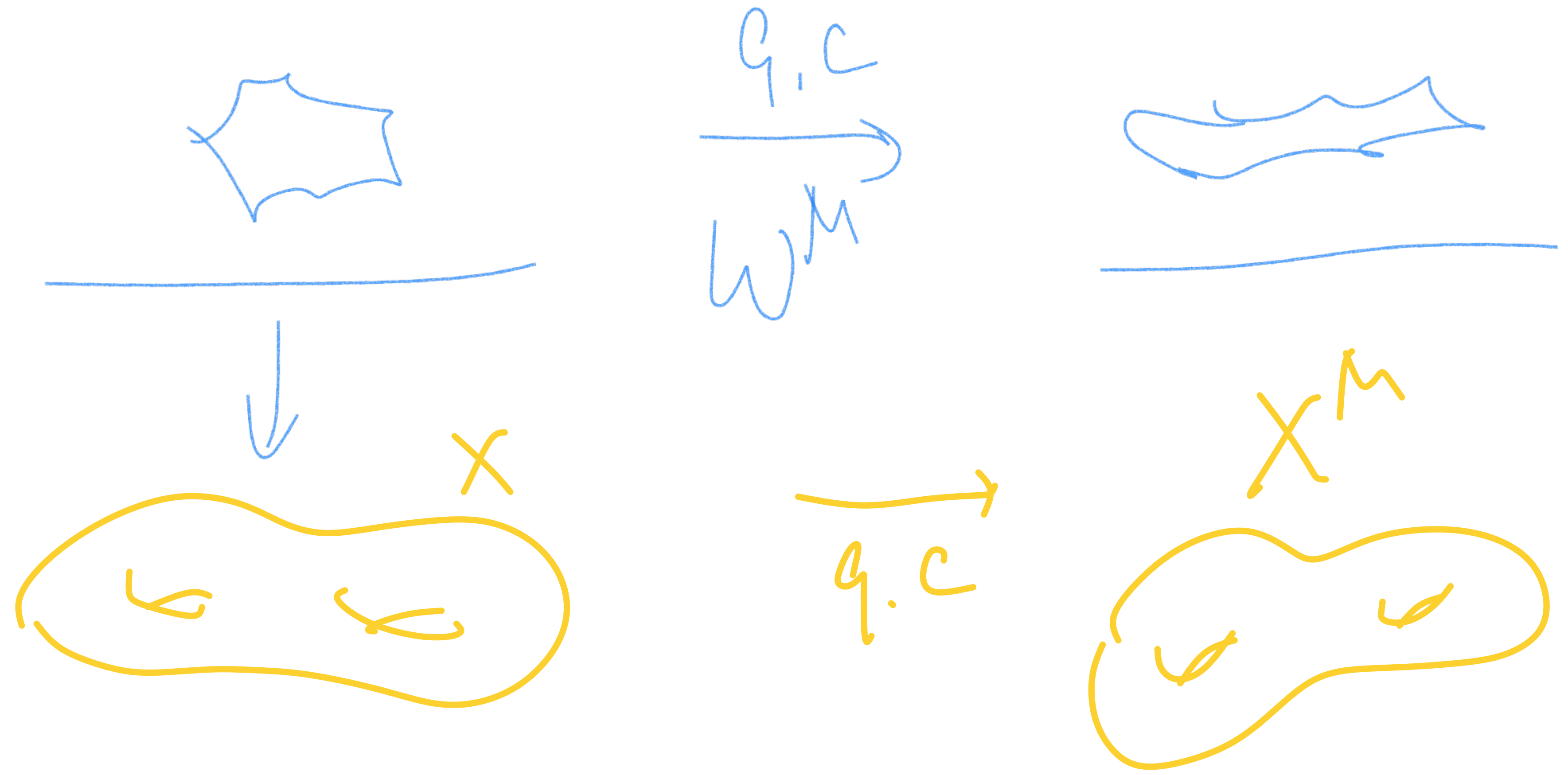

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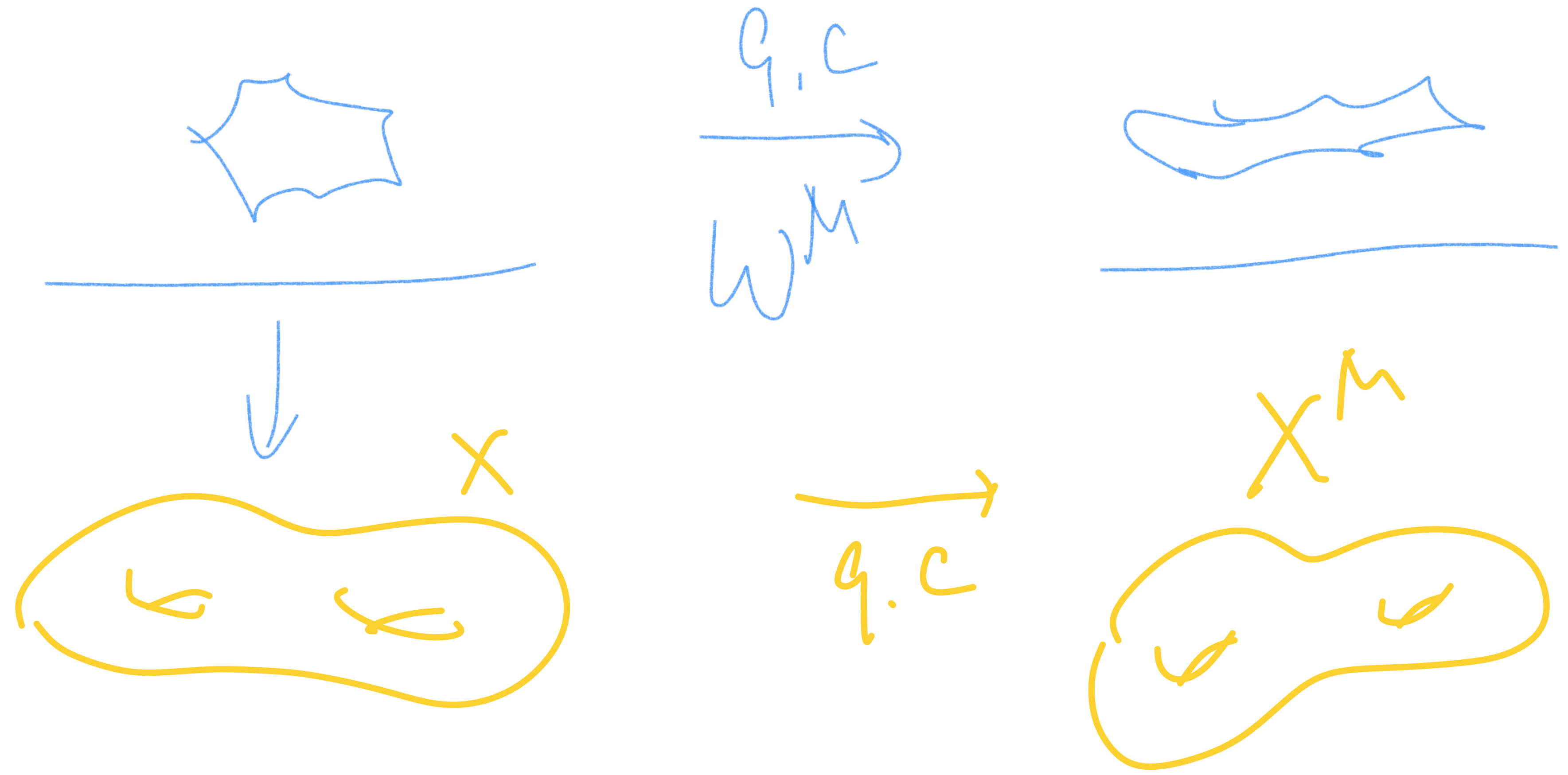

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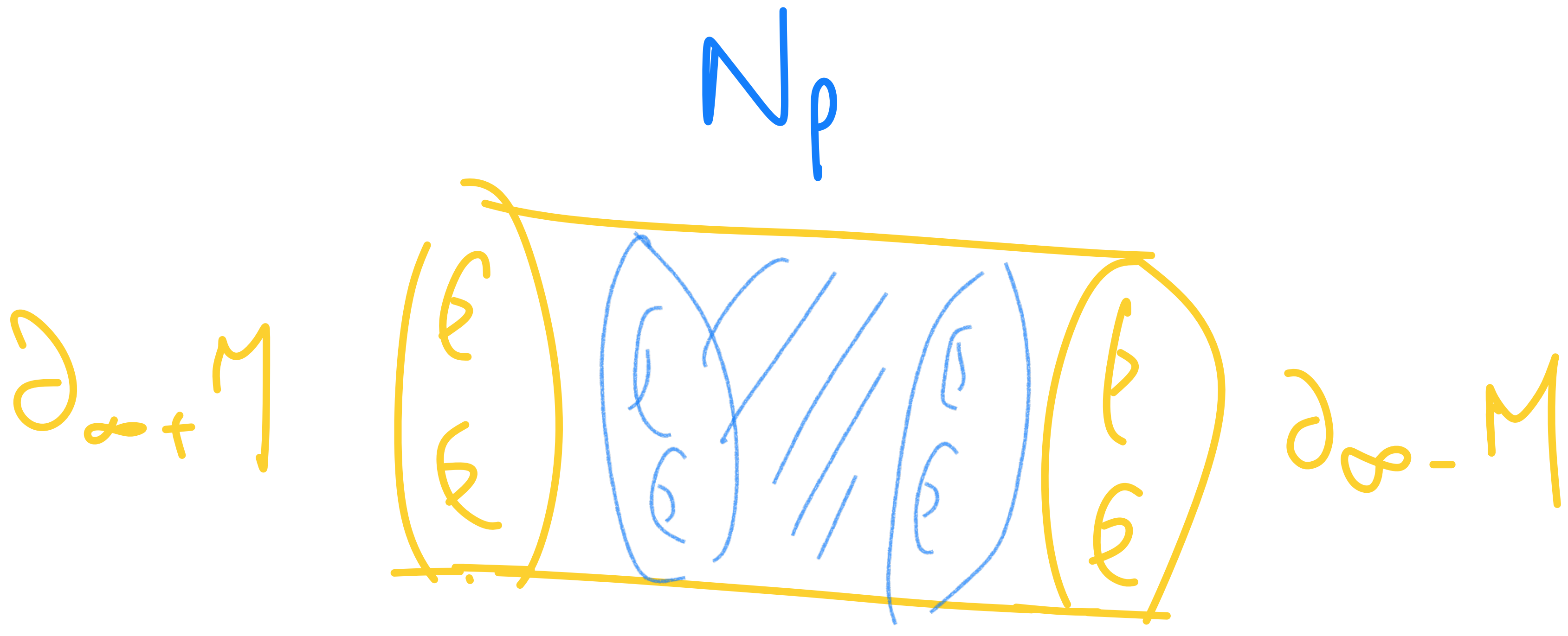

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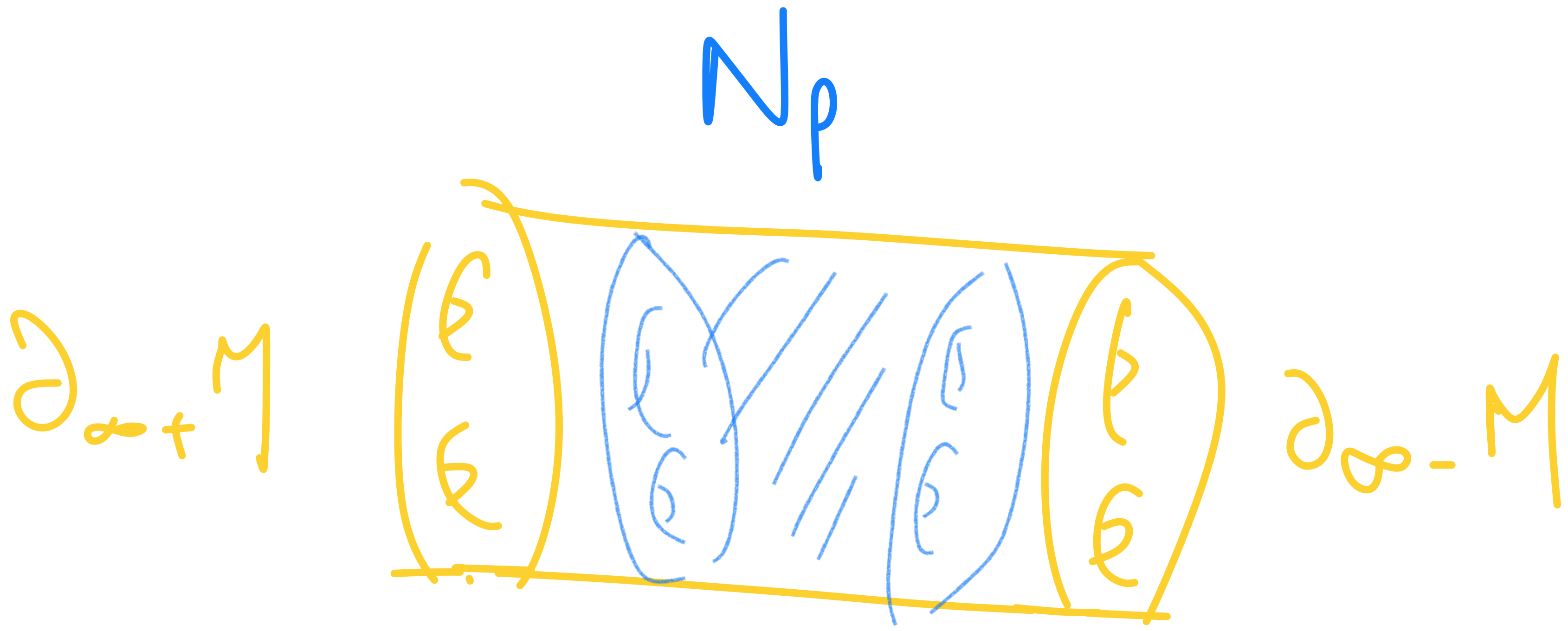

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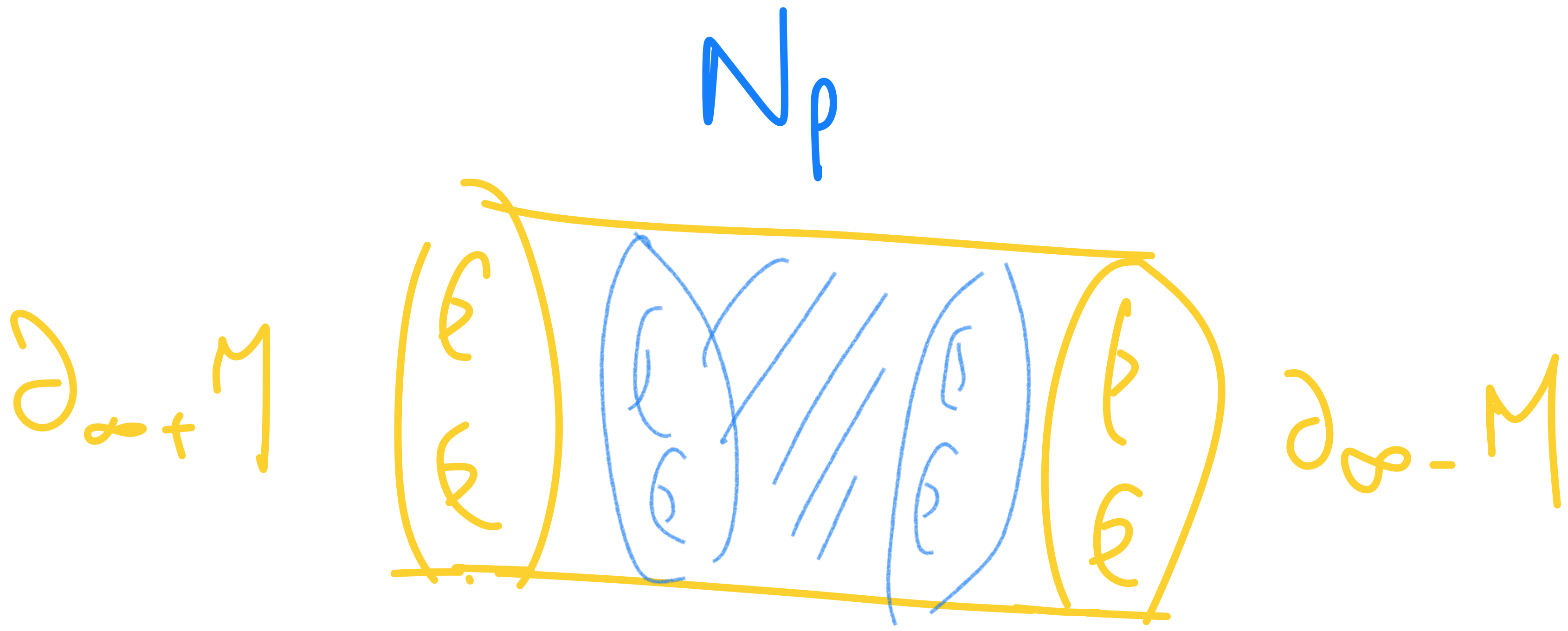

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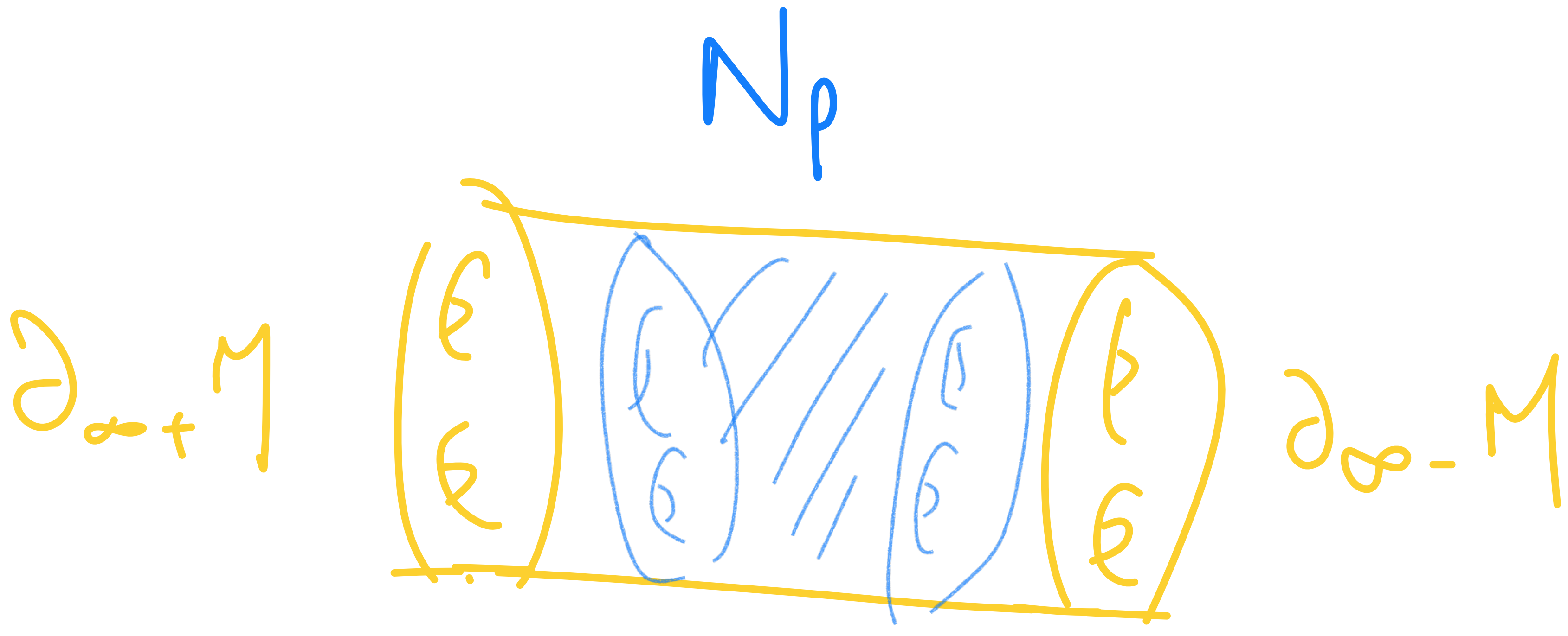

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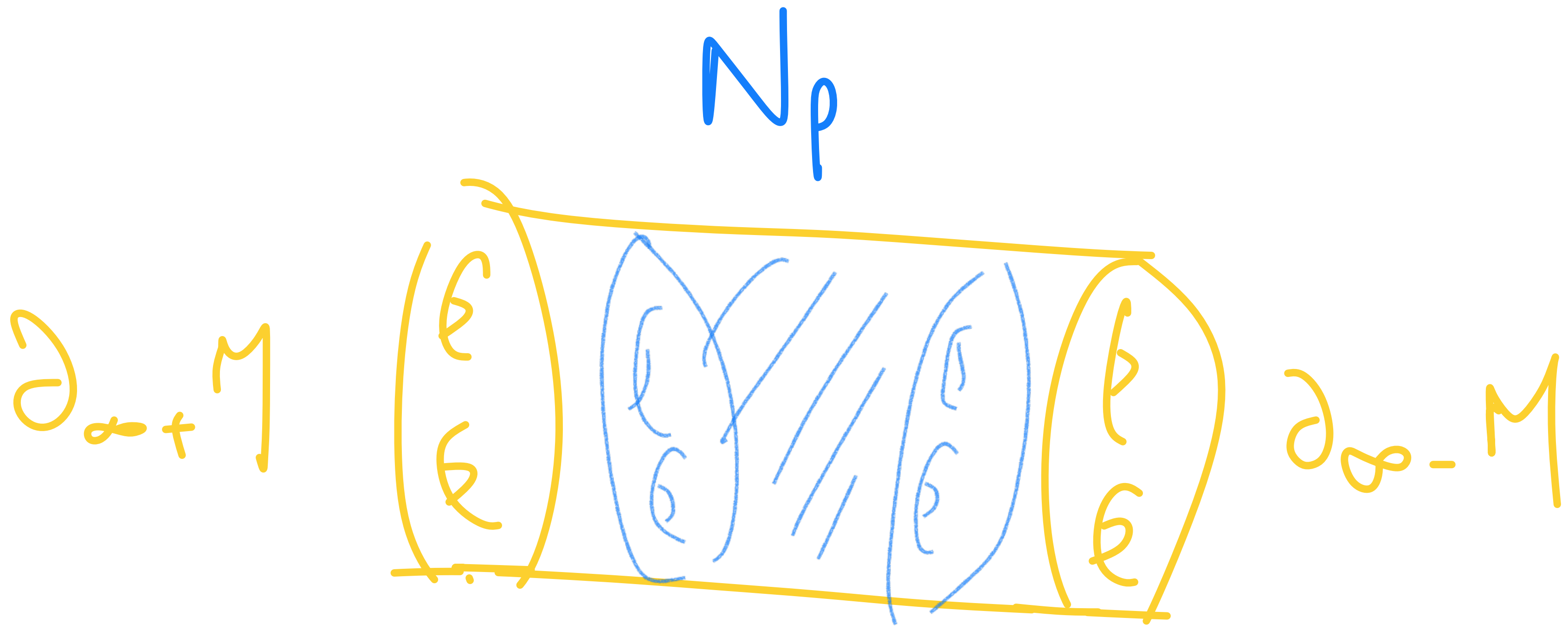

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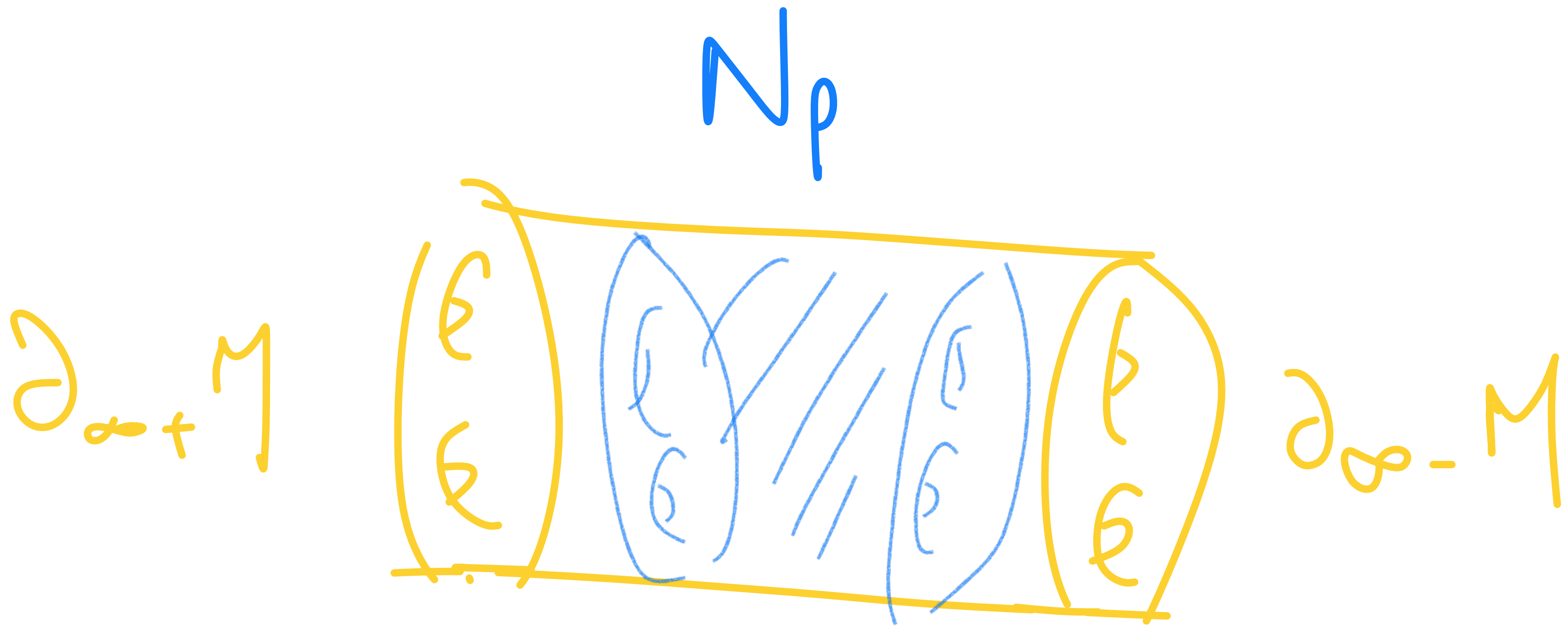

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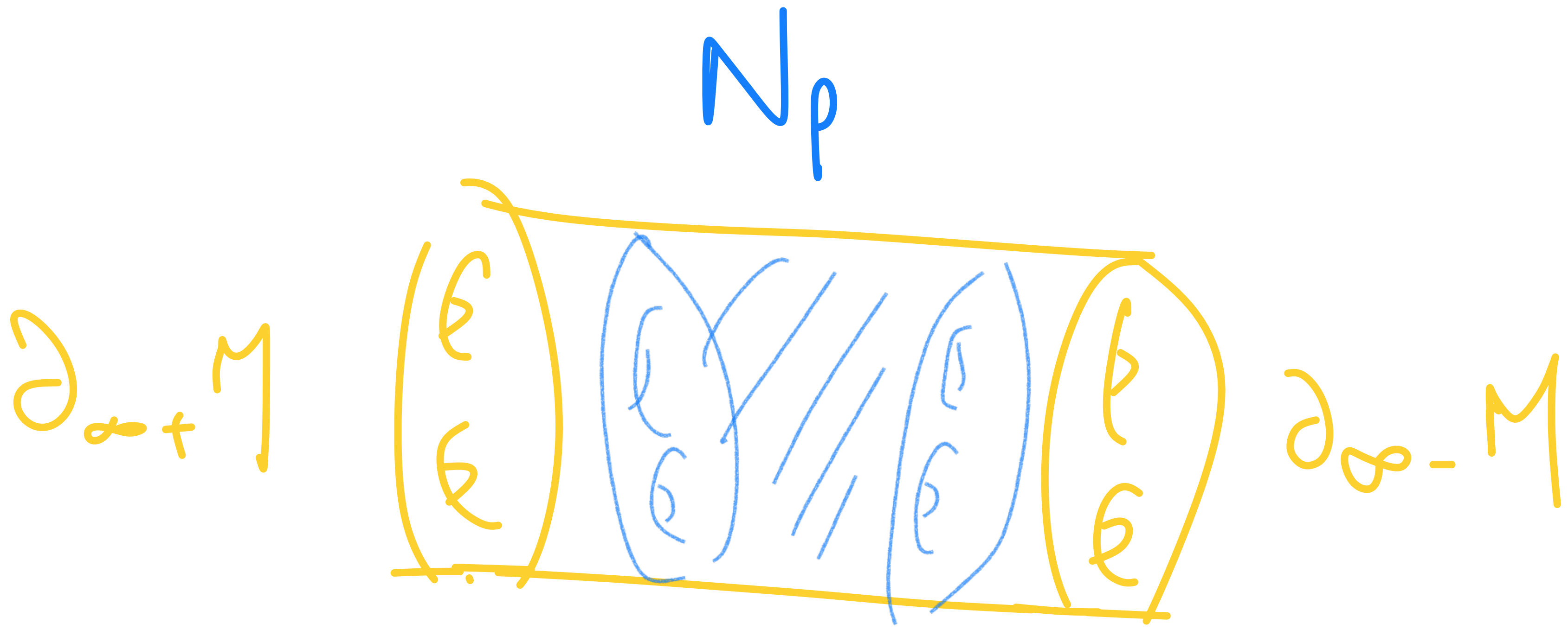

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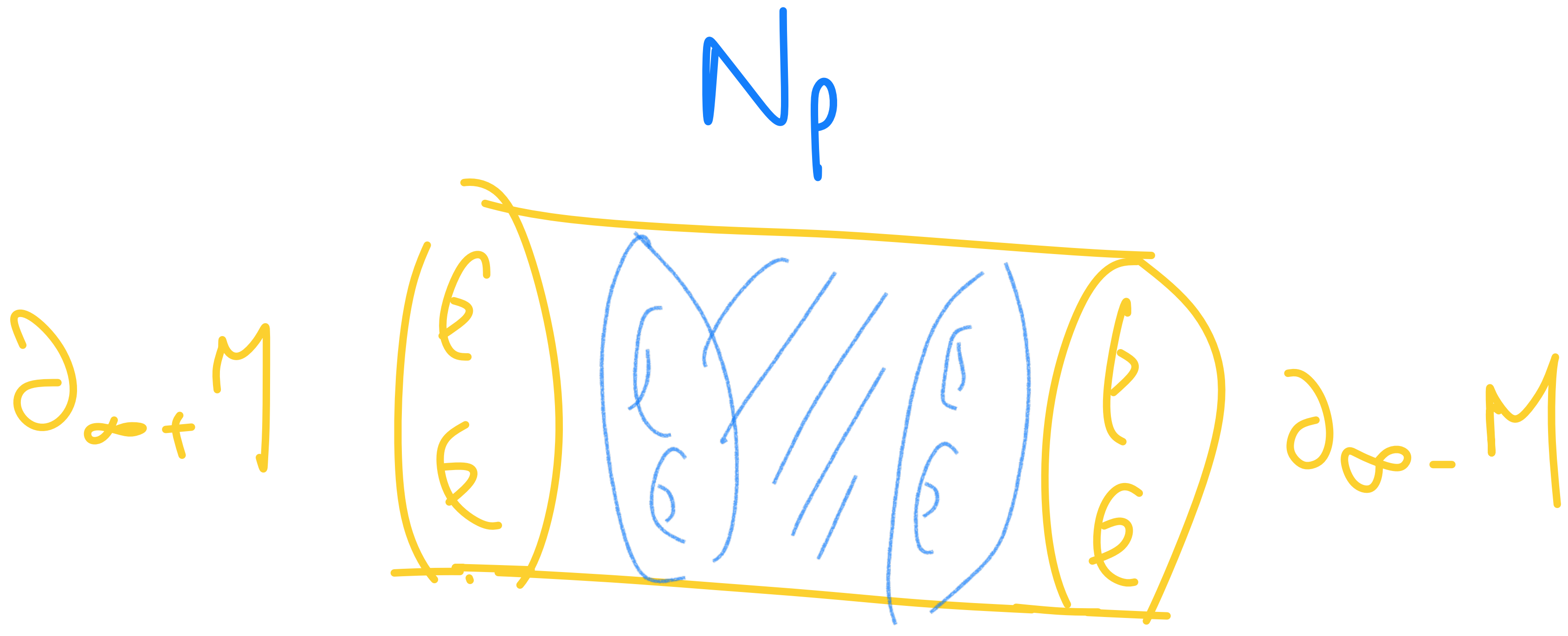

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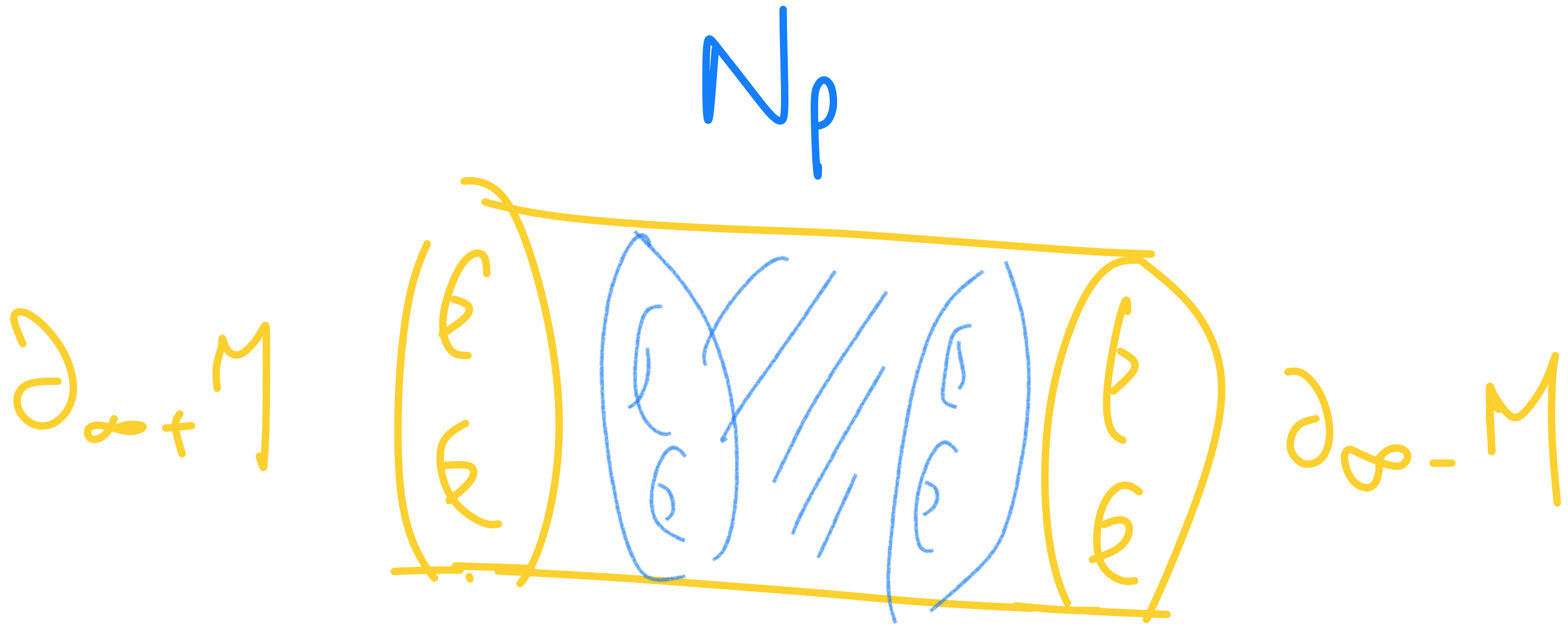

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- Proof relies on the variational formula of the Loewner energy and the renormalized volume.

Conclusion

- The Loewner energy is the *large deviation rate function* of SLE_{0+} and also the *Kahler potential* for the Weil-Petersson metric on universal Teichmuller space.

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Thanks!