Potential theory and random walks in metric spaces

May 30-June 2, 2023

Schedule

	Tuesday 5-30	Wednesday 5-31	Thursday 6-1	Friday 6-2
9:30- 10:30 am	Nageswari Shanmugalingam	Kuwae Kazuhiro	Takashi Kumagai	Qing Liu
10:30- 11:00 am	Coffee break	Coffee break	Coffee break	Coffee break
11:00-noon	Mathav Murugan	Mathav Murugan	Mathav Murugan	Free Discussion
	Lunch	Lunch	Lunch	Lunch
2:00- 3:00 pm	Jana Björn	Olli Saari	Naotaka Kajino	
3:10-4:10 pm	3:10-3:30 Antoni Kijowski 3:45-4:05 Cintia Pacchiano	Xining Li (online)	3:10-3:30 Meng Yang 3:45-4:05 Riku Anttila	
4:10-4:30 pm	Coffee break	Coffee break	Coffee break	
4:30-5:30 pm	Takanobu Hara (4:30-4:50 pm)	Ye Zhang	Dimitrios Ntalampekos (online)	
5:40-6:00 pm	Yannick Sire (5:00-6:00 pm, online)	Josh Kline	Behnam Esmayli	

Organizers: Sylvester Eriksson-Bique (University of Jyväskylä), Panu Lahti (Chinese Academy of Sciences), Xiaodan Zhou (OIST)

Tuesday, May 30th

Nageswari Shanmugalingam, University of Cincinnati

Title: Newton-Sobolev classes of functions on domains in metric measure spaces of controlled geometry, and their traces to the boundary of the domain

Abstract: In considering Dirichlet boundary value problem for a domain, the Dirichlet boundary condition corresponds to functions that arise in some sense as traces, at the boundary, of Sobolev functions in a domain. In the Euclidean and more general metric measure spaces where the measure is doubling and supports a p-Poincar´e inequality, it is now well-established that traces of Sobolev functions on a bounded uniform domain belong to certain Besov classes of functions on the boundary of the domain. In this talk we will give a brief overview of this link between Sobolev spaces and Besov spaces, and discuss recent work on extending the setting to unbounded domains. This talk is based on collaborative works with Lukas Maly, Riikka Korte, Ryan Gibara, Josh Kline, Anders Bj¨orn, and Jana Bj¨orn.

Mathav Murugan, University of British Columbia

Title: Stability results for symmetric random walks and diffusions I

Abstract: Which properties of space are stable under perturbations? We survey known results, relevant open problems, and some methods developed to answer this question. The lectures will focus on properties of random walks and diffusions such as recurrence/transience, Harnack inequalities, heat kernel bounds, and the Liouville property.

Jana Björn, Linköping University

Title: Analysis on metric measure spaces - outside the realm of open and locally compact sets

Abstract:Analysis on metric measure spaces makes it possible to define and study Sobolev spaces and minimization problems on very general sets, seen as metric spaces in their own right. At the same time, notions such as upper gradients and capacity, as well as some properties of Sobolev functions, are highly dependent on the choice of the metric space. They are preserved under suitable conditions on the involved sets, such as quasiopen, finely open and p-path (almost) open sets. In the talk we discuss these properties and provide some equivalent characterizations for them, as well as concrete examples.

Antoni Kijowski, OIST

Title: AMV harmonic functions

Abstract: The asymptotic mean value property holds true for any harmonic function $u : \mathbb{R}^n \supset \Omega \to \mathbb{R}$ and reads

$$\Delta_r u(x) := \frac{u_{B(x,r)} - u(x)}{r^2} \xrightarrow{r\to 0^+} 0, \tag{1}$$

where $u_{B(x,r)}$ denotes the average of u over ball $B(x,r)$. The convergence in (1) is an easy consequence of the Taylor's expansion. The Blaschke–Privaloff–Zaremba theorem asserts the opposite: every continuous function u attaining the asymptotic mean value property property (1) is harmonic. Notice, that $\Delta_r u$ in (1) makes sense formally in a metric measure space. The operator $\Delta_r u$ was first studied by Burago–Kurylev–Ivanov in this setting. In a natural way one defines *amv-harmonic functions* as those for which $\Delta_r u \to 0$ in an appropriate sense as $r \to 0^+$. In a joint project with Tomasz Adamowicz and Elefterios Soultanis [1, 2] we discuss properties of the class of amv-harmonic functions in doubling spaces and compare them to other notions of harmonicity in weighted Euclidean setting, Carnot groups and RCD spaces.

References

[1] T. Adamowicz, A. Kijowski, E. Soultanis, *Asymptotically Mean Value Harmonic Functions in Doubling Metric Measure Spaces Measure Spaces.* Anal. Geom. Metr. Spaces 10 (2022), no. 1, 344–372.

[2] T. Adamowicz, A. Kijowski, E. Soultanis, *Asymptotically Mean Value Harmonic Functions in Subriemannian and RCD Settings.* J Geom Anal 33, 80 (2023).

Cintia Pacchiano, University of Calgary

Title: Regularity Results for Double Phase Problems on Metric Measure Spaces

Abstract: In this talk, we present local and global higher integrability properties for quasiminimizers of a class of double phase integrals characterized by non-standard growth conditions. We work purely on a variational level in the setting of a doubling metric measure space supporting a Poincar´e inequality. The main novelty is the use of an intrinsic approach, based on a double phase Sobolev-Poincar´e inequality.

During the past two decades, a theory of Sobolev functions and first degree calculus has been developed in this abstract setting. A central motivation for developing such a theory has been the desire to unify the assumptions and methods employed in various specific spaces, such as weighted Euclidean spaces, Riemannian manifolds, Heisenberg groups, graphs, etc.

Analysis on metric spaces is nowadays an active and independent field, bringing together researchers from different parts of the mathematical spectrum. It has applications to disciplines as diverse as geometric group theory, nonlinear PDEs, and even theoretical computer science. This can offer us a better understanding of the phenomena and also lead to new results, even in the classical Euclidean case.

Takanobu Hara, Hokkaido University

Title: Barrier construction methods and weighted Hardy inequalities

Abstract: We discuss weighted p-Poisson equations on domains with capacity density conditions. Extending results of Ancona ('86), we construct barriers using Wiener regularity estimates. As a consequence, we prove weighted Hardy inequalities. Furthermore, we prove an existence result of global continuous solutions for boundary singular data.

Wednesday, May 31th

Kuwae Kazuhiro, Fukuoka University

Title: The Littlewood-Paley-Stein inequality for diffusion processes on metric measure spaces with variable curvature lower bounds

Abstract: The notion of tamed Dirichlet space was proposed by Erbar-Rigoni-SturmTamanini, Tamed spaces — Dirichlet spaces with distribution-valued Ricci bounds, J. Math. Pures Appl. (9) 161 (2022), 1–69 as a Dirichlet space having weak form of BakryEmery curvature lower bounds in distribution sense. After their work, Mathias Braun,´ Vector calculus for tamed Dirichlet spaces, preprint 2021, to appear in Mem. Amer. Math. Soc. established a vector calculus for it. In this framework, we establish the LittlewoodPaley-Stein inequality for L^p-functions and its applications, which partially generalizes the result given by Kawabi-Miyokawa, The Littlewood-Paley-Stein inequality for diffusion processes on general metric spaces, J. Math. Sci. Univ. Tokyo 14 (2007), 1–30.

Mathav Murugan, University of British Columbia

Title: Stability results for symmetric random walks and diffusions II

Abstract: Which properties of space are stable under perturbations? We survey known results, relevant open problems, and some methods developed to answer this question. The lectures will focus on properties of random walks and diffusions such as recurrence/transience, Harnack inequalities, heat kernel bounds, and the Liouville property.

Thursday, June 1st

Takashi Kumagai, Waseda University

Title: Gradient estimates of the heat kernel for random walks in time-dependent random environments

Abstract: We consider a random walk among time-dependent random conductances. In recent years the long-time behavior of this model under diffusive rescaling has been intensively studied, and it is well understood. In this talk, we will discuss how to obtain first and second space derivatives of the annealed transition density. We use entropy estimates that has been developed in the time-independent setting by Benjamini, DuminilCopin, Kozma and Yadin (2016). This is a joint work with J-D. Deuschel (Berlin) and M. Slowik (Mannheim).

Mathav Murugan, University of British Columbia

Title: Stability results for symmetric random walks and diffusions III

Abstract: Which properties of space are stable under perturbations? We survey known results, relevant open problems, and some methods developed to answer this question. The lectures will focus on properties of random walks and diffusions such as recurrence/transience, Harnack inequalities, heat kernel bounds, and the Liouville property.

Naotaka Kajino, Kyoto University

Title: Impossibility of quasisymmetric Gaussian uniformization, via decay rates of harmonic functions, for Brownian motion on some planar Sierpin´ski carpets

Abstract: It is an established result in the field of analysis of diffusion processes on fractals, that the transition density of the diffusion typically satisfies analogs of Gaussian bounds

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which involve a space-time scaling exponent β greater than two and thereby are called SUB-Gaussian bounds. The exponent β, called the walk dimension of the diffusion, could be considered as representing "how close the geometry of the fractal is to being smooth". It has been observed by Kigami in [Math. Ann. **340** (2008), 781–804] that, in the case of the standard two-dimensional Sierpin´ski gasket, one can decrease this exponent to two (so that Gaussian bounds hold) by suitable changes of the metric and the measure while keeping the associated Dirichlet form (the quadratic energy functional) the same. Then it is natural to ask how general this phenomenon is for diffusions on fractals.
In fact, it turns out that the above phenomenon, that one can decrease the exponent β to two so that Gaussian bounds hold, seems to happen only for a very limited class of self-similar fractals. This talk is aimed at presenting the result that this phenomenon indeed does NOT happen for the Brownian motion on a class of two-dimensional Sierpinski carpets, as well as for the Brownian motion on the standard three- and higher-dimensional Sierpinski gaskets. The key to the proof is some knowledge about decay rates of harmonic functions, which for Sierpin´ski carpets seems new and is of independent interest. This talk is based on joint works with Mathav Murugan (University of British Columbia). The results for planar Sierpinski carpets is in progress, and that for the standard higherdimensional Sierpin´ski gaskets is given in [Invent. math. **231** (2023), 263–405].

Meng Yang, University of Bonn, Hausdorff Center for Mathematics

Title: Gradient Estimate for the Heat Kernel on Some Fractal-Like Cable Systems

Abstract: We give pointwise upper estimate for the gradient of the heat kernel on some fractal-like cable systems including the Vicsek and the Sierpinski cable systems. Based on arXiv: 2103.10181 joint with Baptiste Devyver (Institut Fourier) and Emmanuel Russ (Institut Fourier).

Riku Anttila, Tampere University

Title: Symmetric fractal spaces and the combinatorial Loewner property

Abstract: In quasisymmetric geometry we study properties which are preserved under quasisymmetries. Roughly speaking quasisymmetry is a homeomorphism of metric spaces that preserves ratios of distances. A lot of recent research in quasisymmetric geometry has focused on understanding the Ahlfors regular conformal dimension, which is a quasisymmetric invariant that measures how much a metric space can be simplified while preserving its quasisymmetric geometry.

In this talk we will focus on the quasisymmetric properties of fractal spaces that arise from self-similar replacement rules for graphs. We will present regularity- and symmetricity conditions for these spaces to have the combinatorial Loewner property. Furthermore, we will present a computational method that can be used to approximate the Ahlfors regular conformal dimension of these spaces.

Dimitrios Ntalampekos, Stony Brook University

Title: Uniformization of metric surfaces of finite area

Abstract: The classical uniformization theorem for Riemann surfaces implies that every smooth two-dimensional sphere can be conformally parametrized by the Euclidean sphere. The recent developments in the field of Analysis on Metric Spaces have allowed the extension of this result beyond the smooth setting. Sufficient geometric conditions have been established so that a fractal sphere can be transformed to the Euclidean sphere with a bi-Lipschitz, quasisymmetric, or quasiconformal map. The milestones in this direction are the works of Bonk-Kleiner on the quasisymmetric uniformization of spheres and of Rajala on the characterization of quasiconformal spheres. It was conjectured by Rajala and Wenger that, under no assumption, every metric two-dimensional sphere of finite area can be parametrized by the Euclidean sphere with a weakly quasiconformal map. In this talk we present an affirmative answer to this conjecture. The talk is based on joint work with Matthew Romney.

Behnam Esmayli, University of Jyvaskyla

Title: Poincare Inequality on a Square Pinched on a Product Cantor Set

Abstract: The fewer paths there exist on a metric-measure space, the less likely it is for the space to support Poincare inequalities. We consider a space that is constructed by removing from a solid square in the plane very many ”diamond” shaped pieces. Paths can connect separate manifold parts of the space only through a product of Cantor sets. Under right conditions on the Cantor sets, we prove that the space still satisfies some Poincare inequality with exponent less than 2.

Friday, June 2nd

Qing Liu, OIST

Title: Discontinuous eikonal equations in metric measure spaces

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Abstract: In this talk, we study the eikonal equation in metric measure spaces, where the inhomogeneous term is allowed to be discontinuous, unbounded and merely p-integrable in the domain. For continuous eikonal equations, it is known that the notion of Monge solutions is equivalent to the standard definition of viscosity solutions. Generalizing the notion of Monge solutions in our setting, we establish uniqueness and existence results for the associated Dirichlet boundary problem. The key in our approach is to adopt a new metric, based on an optimal control interpretation, which incorporates the discontinuous term and converts the eikonal equation to a standard form. We also discuss the Holder continuity of the unique solution with respect to the original metric under regularity assumptions on the space and the inhomogeneous term. This talk is based on joint work with Nageswari Shanmugalingam (U. Cincinnati) and Xiaodan Zhou (OIST).

Olli Saari, Universitat Politecnica de Catalunya

Title: Quantitative characterization of reverse H¨older weights through Carleson conditions

Abstract: Weights (non-negative locally integrable functions) satisfying a reverse H¨older condition are important in the study of harmonic measure and boundary value problems for elliptic partial differential equations. In this talk, I will explain a quantitative version of their characterization through a Carleson condition, originally observed by Fefferman, Kenig and Pipher without focus on estimation of constants. The quantitative estimate of this talk applies to almost flat weights and has direct consequences for elliptic measures on the boundary of the Euclidean upper half space for a certain class of differential operators with rough coefficients. This is based on joint work with Simon Bortz and Moritz Egert.

Xining Li, Sun Yat-sen University

Title: Characterization of H^p spaces in Quasiconformal Mappings

Abstract: We study integral characterizations of weighted Hardy spaces of quasiconformal mappings on the n-dimensional unit ball using the weight $(1 - r)^{n-2+\alpha}$. And we extend known results for univalent functions on the unit disk. Some of our results are new even in the unweighted setting for quasiconformal mappings.

Ye Zhang, OIST

Title: On the Centered Hardy-Littlewood Maximal Functions on H-Type Groups

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Abstract: We establish the weak type (1,1) bound estimate, which is given by a universal constant C multiplying the dimension of the underlying group, for the centered HardyLittlewood maximal function defined by the Kor´anyi norm. Corresponding results for maximal functions defined by sub-Riemannian distance and Riemannian distance are also discussed in this talk. This is a joint work with Cheng Bi and Hong-Quan Li.

Josh Kline, University of Cincinnati

Title: On the nonlocal least gradient problem in metric measure spaces

Abstract: In the metric setting, we consider minimizers of the nonlocal perimeter functional, as studied in $\mathbb{R}^n$ by Cafarelli, Roquejoffre, and Savin. Consisting of (part) of the Besov seminorm, this functional measures the interaction, roughly speaking, between a set and its complement with respect to a given domain. In this talk, we discuss boundary regularity for minimizers, as well as a generalized version of this problem, for which we study existence and related trace results.