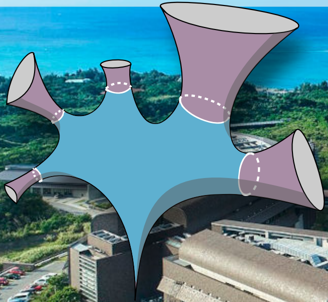
03-08-2023 New trends of conformal theory from probability to gravity @ OIST

Tree bijections and the geometry of random hyperbolic surfaces

Timothy Budd



w.i.p. with T. Meeusen & B. Zonneveld
(and earlier work with N. Curien)

Radboud University

T.Budd@science.ru.nl
http://hef.ru.nl/~tbudd/

Photo credit: By OIST from Onna Village, Japan - Aerial View of Campus Buildings

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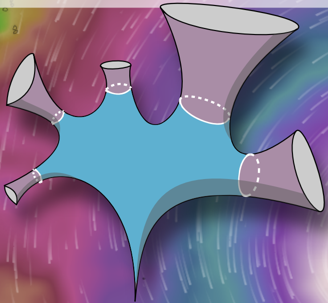

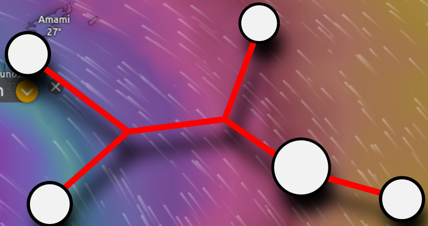

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Image credit: windy.com

Hyperbolic surfaces: a motivation from JT gravity

2D quantum gravity

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[Teitelboim, '83] [Jackiw, '85]

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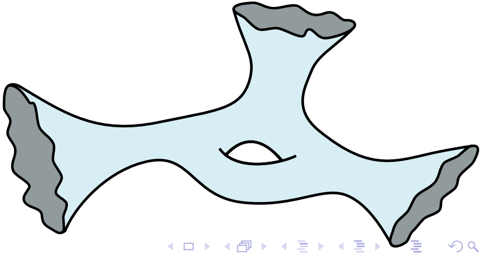

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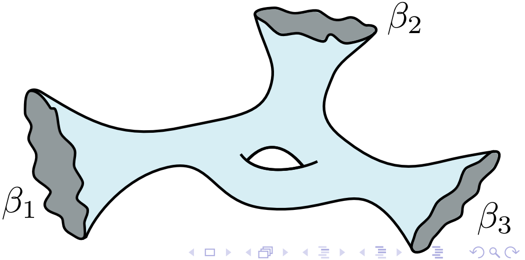

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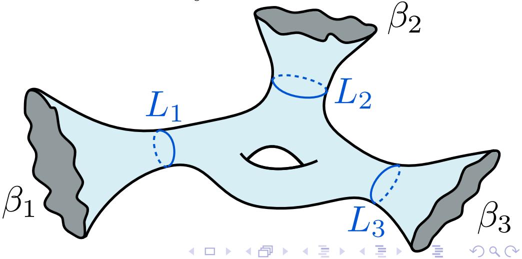

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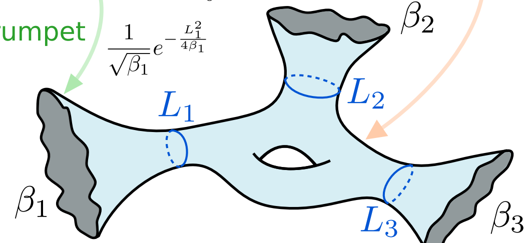
trumpet $\quad \dfrac{1}{\sqrt{\beta_1}} e^{-\frac{L_1^2}{4\beta_1}}$

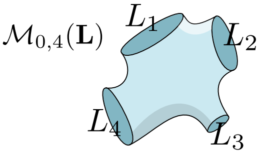


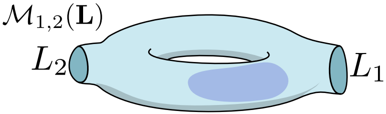
The partition function of hyperbolic surfaces: WP volumes

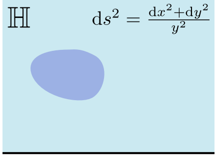
[Wolpert, Penner, Zograf, Witten, Kontsevich, Mirzakhani, ...]

- Consider the Moduli space

$$\mathcal{M}_{g,n}(\mathbf{L}) = \left\{ \begin{array}{l} \text{hyperbolic metrics on genus-}g\text{ surface with } n \\ \text{geodesic boundaries of lengths } \mathbf{L} = (L_1, \ldots, L_n) \end{array} \right\} / \mathrm{Diff}^+$$







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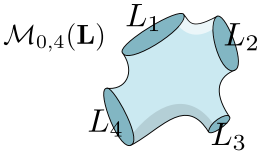

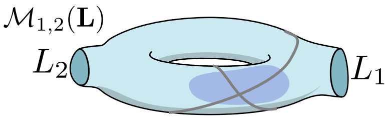

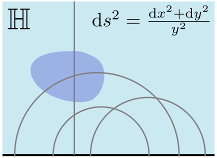

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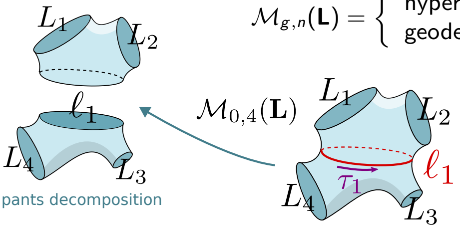

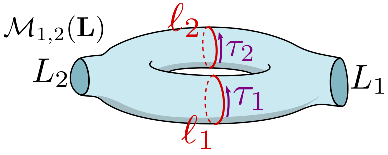

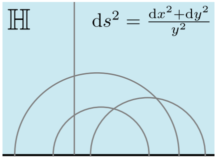

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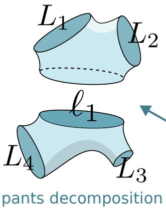

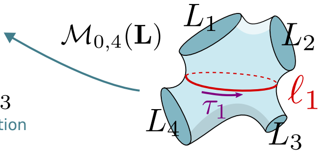

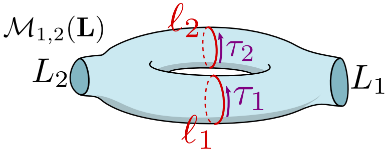

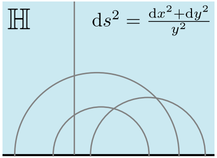

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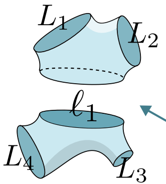
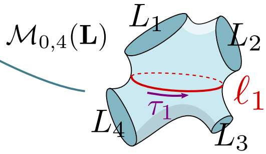
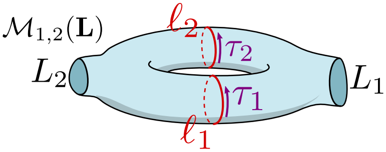
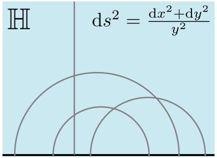


pants decomposition







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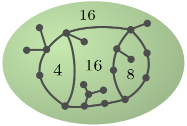
- Examples: $V_{0,3}(\mathbf{L}) = 1, \qquad V_{0,4}(\mathbf{L}) = \frac{1}{2}(L_1^2 + L_2^2 + L_3^2 + L_4^2) + 2\pi^2,$
  $V_{1,2}(\mathbf{L}) = \frac{1}{192}(L_1^2 + L_2^2 + 4\pi^2)(L_1^2 + L_2^2 + 12\pi^2).$

Bipartite maps on surfaces



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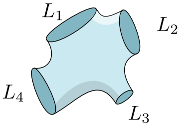

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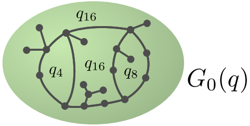
$$G_g(q) = \sum_{n\geq 1} \frac{1}{n!} \sum_{d_1=0}^{\infty} q_{2d_i} \cdots \sum_{d_n=0}^{\infty} q_{2d_n} \# \left\{ \begin{array}{l} \text{genus-}g \text{ maps with} \\ \text{face degrees } 2d_1, \ldots, 2d_n \end{array} \right\}$$



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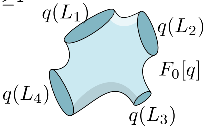

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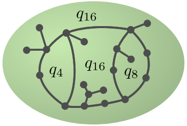

$G_0(q)$

$e^{\sum_g \lambda^g G_g}$ is a τ-function of KP hierarchy

[Kadomtsev, Petriashvili, Panharipande, Okounkov, Kazarian, ...]

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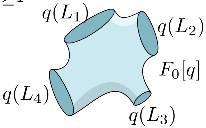

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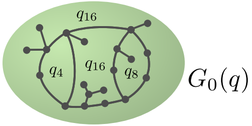

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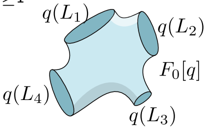

$e^{\sum_g \lambda^g F_g}$ is a τ-function of KdV hierarchy

[Witten, '91][Kontsevich, '92][Kaufmann, Manin, Zagier, '96][Mirzakhani, '07]

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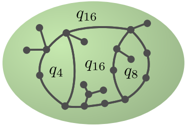
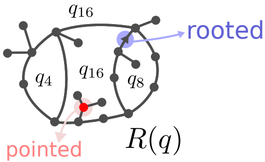


$G_0(q)$

- G_0 determined by string eq. for $R(q) := \frac{\partial G_0}{\partial q_0 \partial q_1}$

$$R = 1 + \sum_{k=1}^{\infty} \binom{2k-1}{k} q_{2k} R^k$$



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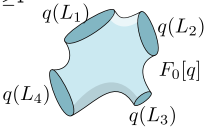

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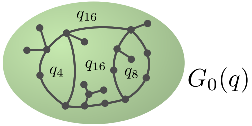

- G_0 determined by string eq. for $R(q) := \frac{\partial G_0}{\partial q_0 \partial q_1}$

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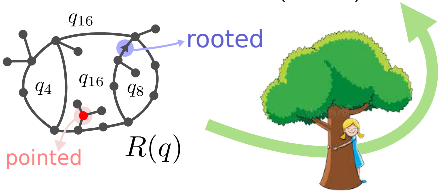

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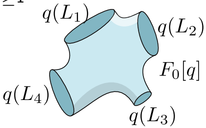

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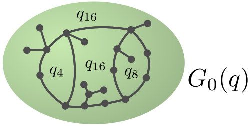

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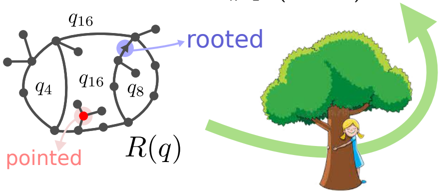

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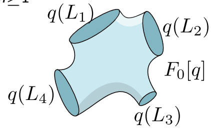

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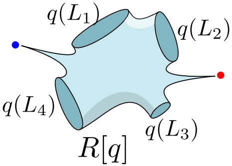

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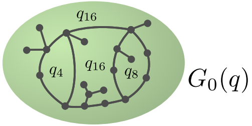

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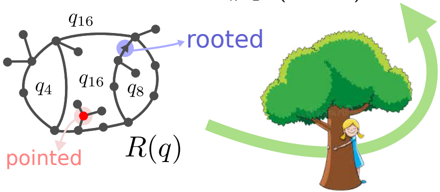

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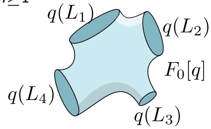

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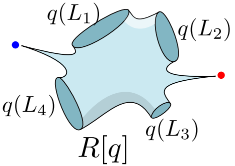

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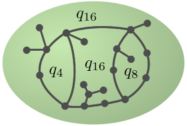

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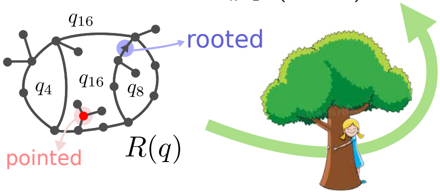

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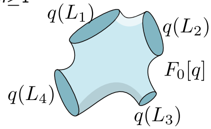

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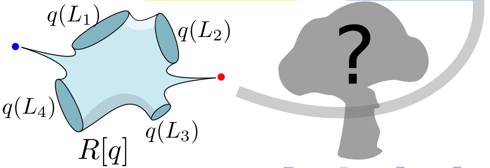

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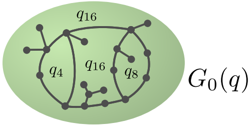

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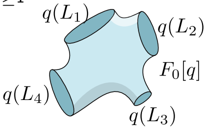

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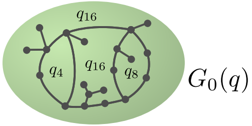

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- Scaling limit (if q sufficiently regular):

$$(\mathfrak{m}, n^{-\frac{1}{4}} d_{\mathrm{gr}}) \xrightarrow[n\to\infty]{\text{(d) GH}} \text{Brownian sphere}$$

- Random metric space
- Hausdorff dimension 4
- Topology of 2-sphere [Le Gall, Miermont, Marckert, Marzouk, ...]
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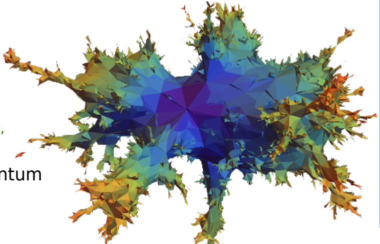

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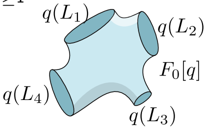

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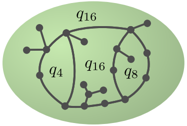

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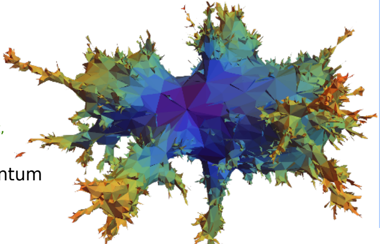

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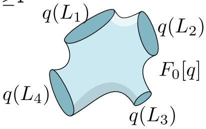

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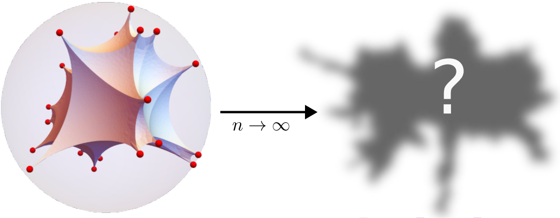
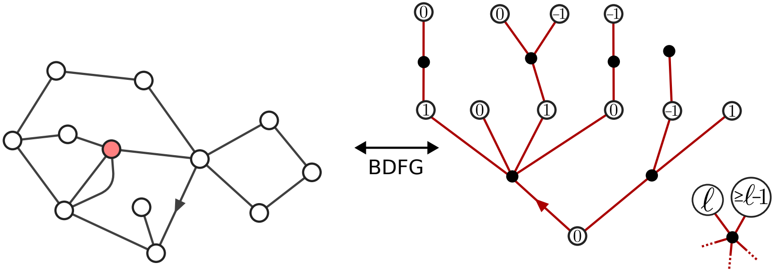


Bouttier–Di Francesco–Guitter bijection [BDFG, '04]

$$\left\{\begin{array}{c}\text{rooted bipartite planar maps}\\\text{with marked vertex ("origin")}\end{array}\right\} \overset{\text{2-to-1}}{\longleftrightarrow} \left\{\begin{array}{c}\text{mobiles (bicolored plane trees}\\\text{with labeled white vertices)}\end{array}\right\}$$



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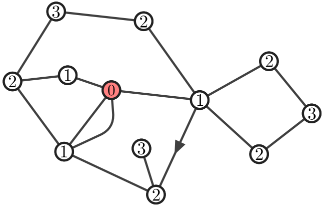

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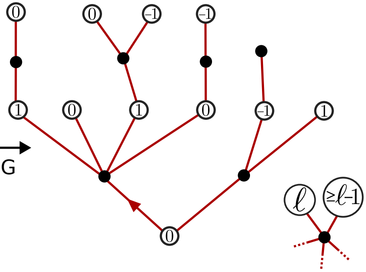

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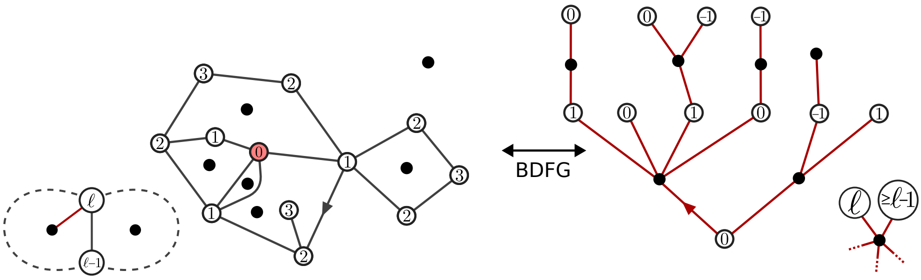

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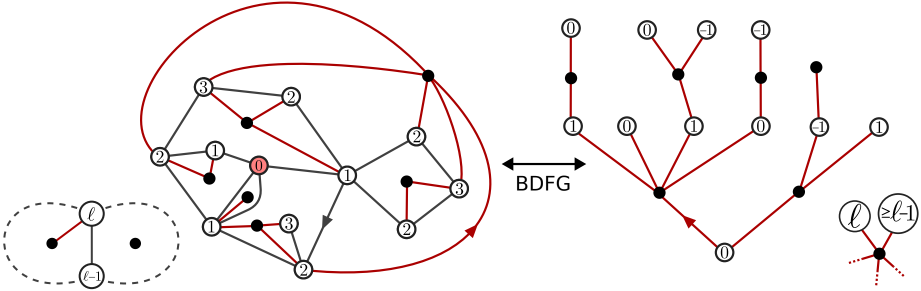

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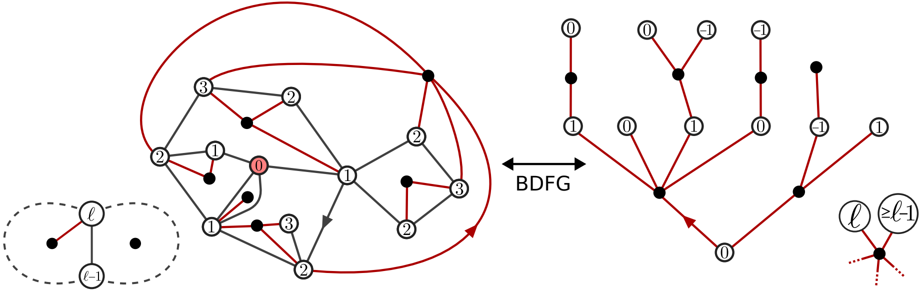

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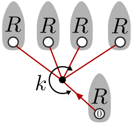

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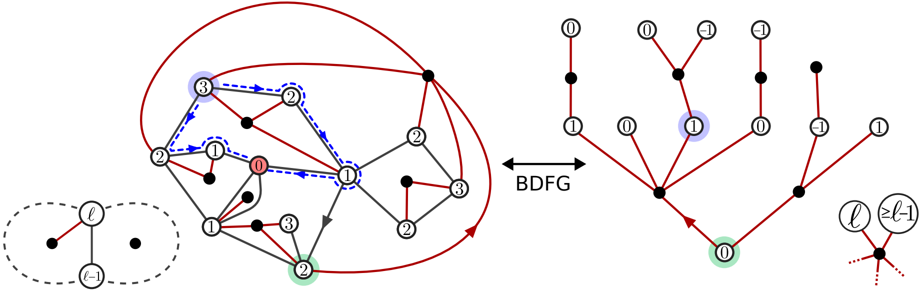

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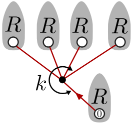

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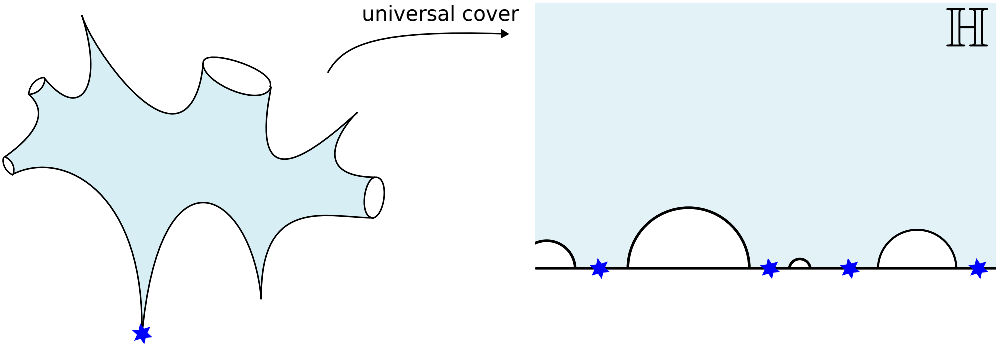
Tree in a hyperbolic surface?



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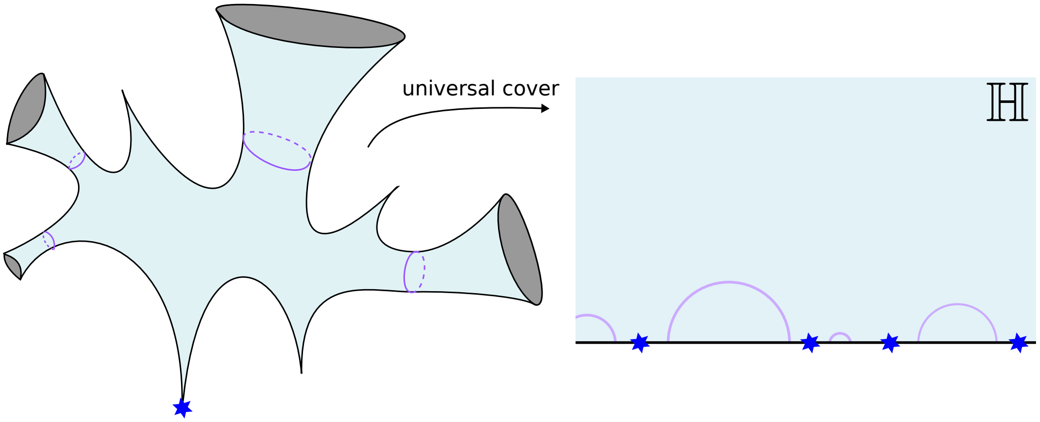

- Extend boundaries with hyperbolic funnels/trumpets.

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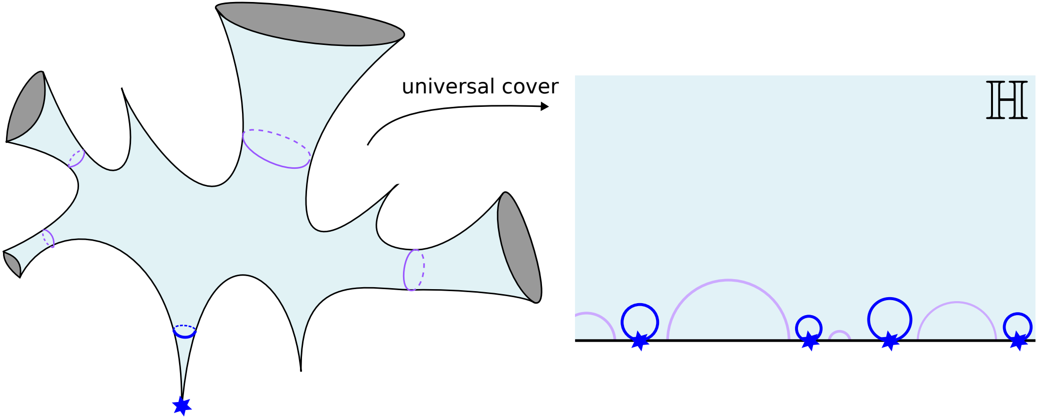

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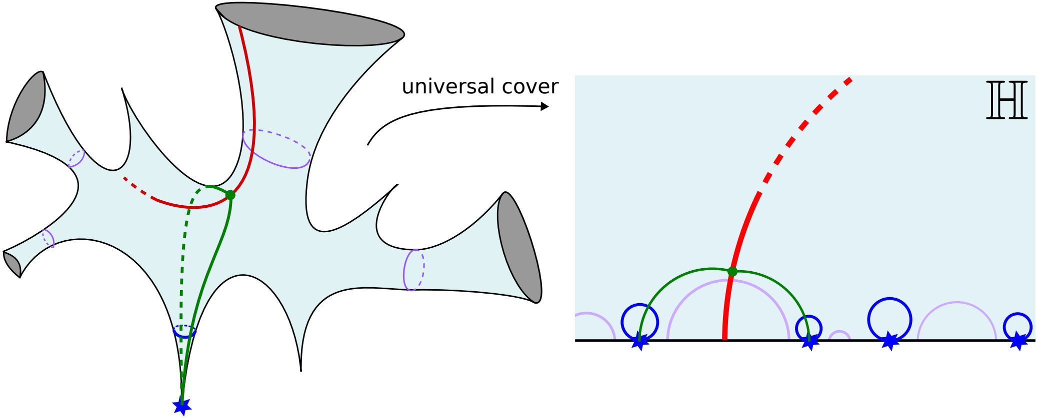

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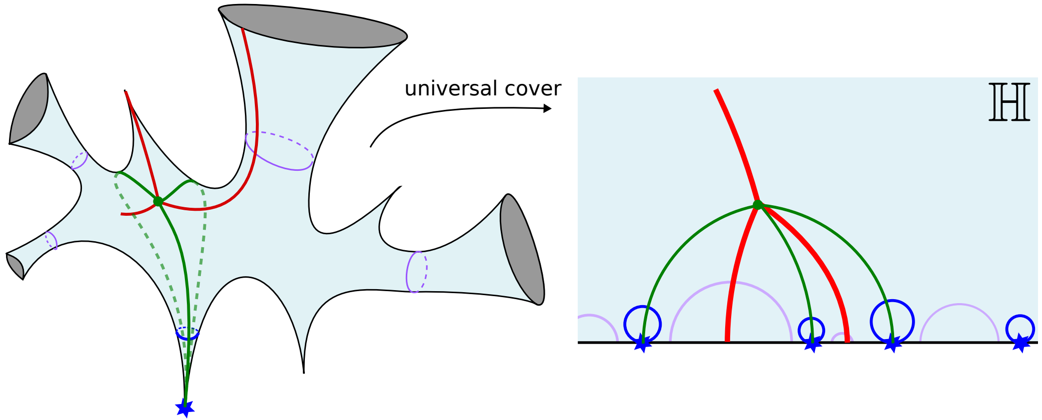

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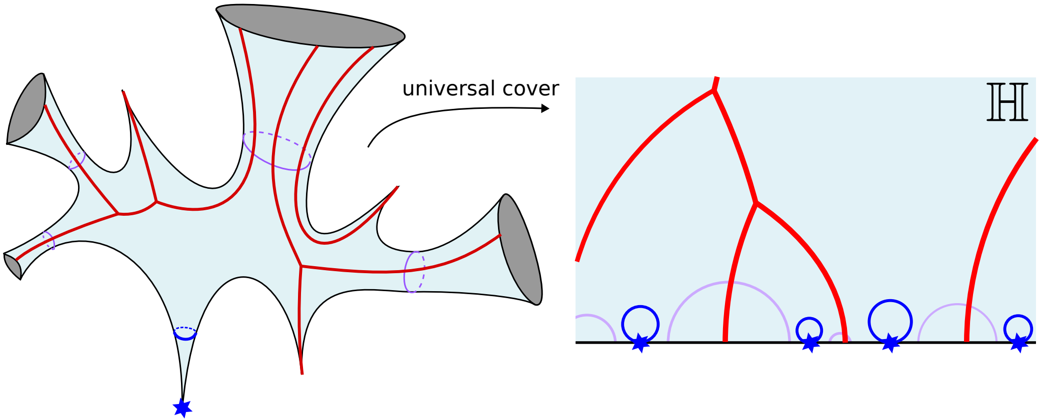

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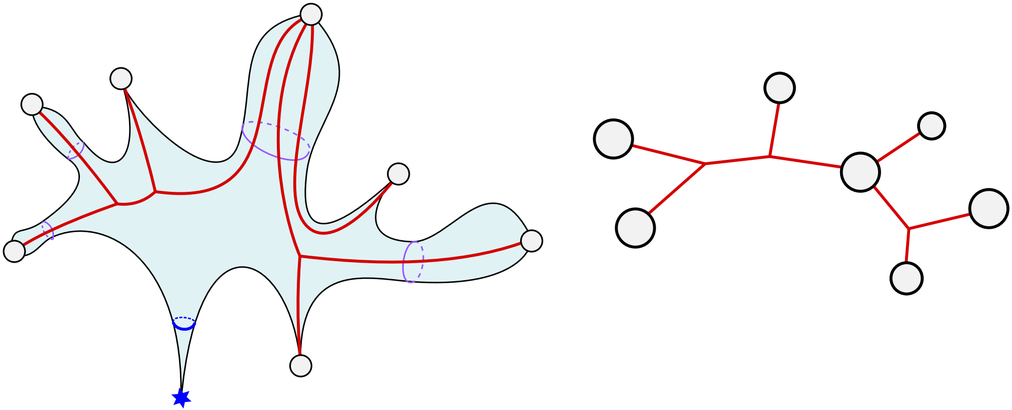

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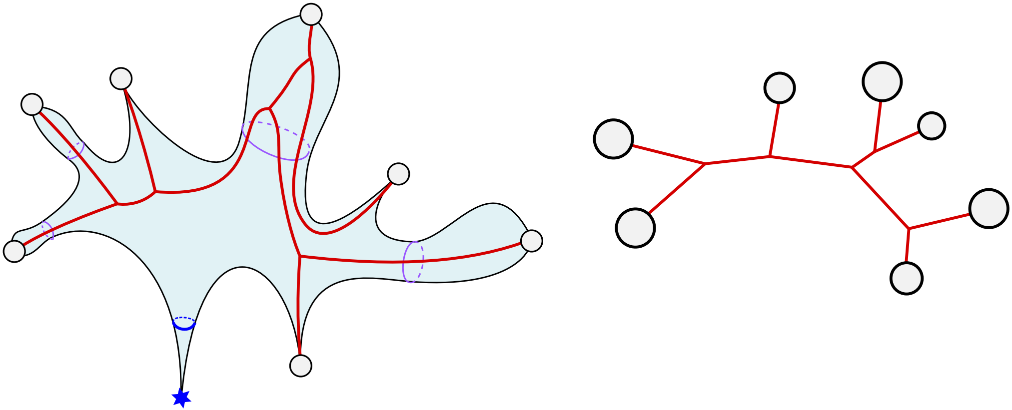

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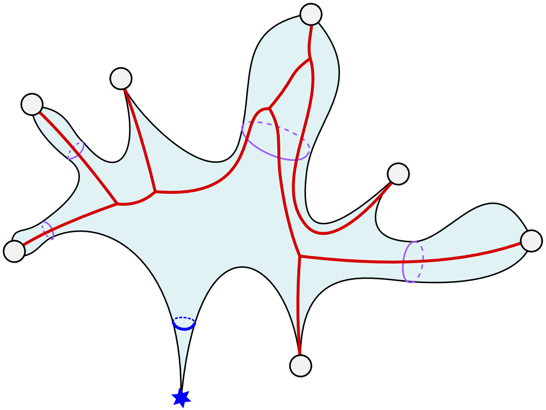
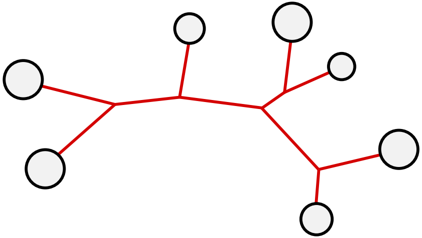
- Note: spine edges can meet in cylinders!

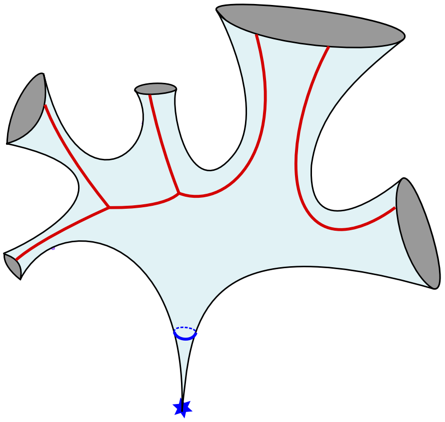
Tree in a hyperbolic surface?

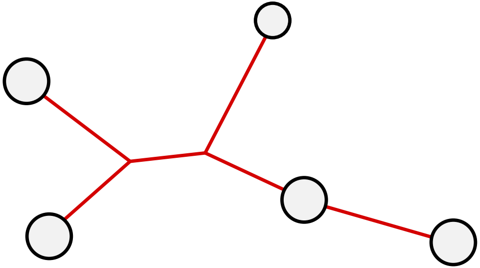




- Extend boundaries with hyperbolic funnels/trumpets.
- Determine spine of origin ★: points with more than one shortest geodesic to ★. [Bowditch, Epstein, '88]
- Upon compactification it is a plane tree!
- Note: spine edges can meet in cylinders!
- Can we label the tree to make a bijection?

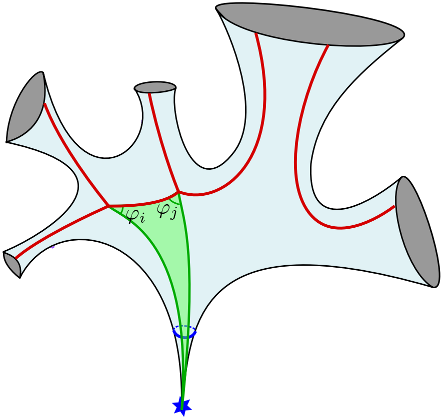
Labels: angles on half edges

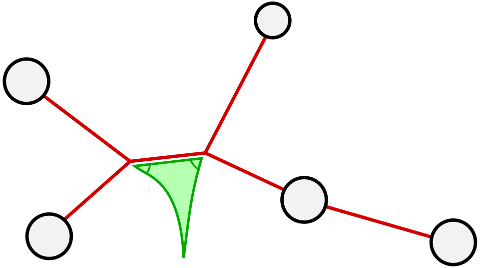




- The surface is canonically triangulated by

Labels: angles on half edges





- The surface is canonically triangulated by
  - for each spine edge: triangle with angles $\varphi_i, \varphi_j, 0$ (so $\varphi_i + \varphi_j < \pi$)

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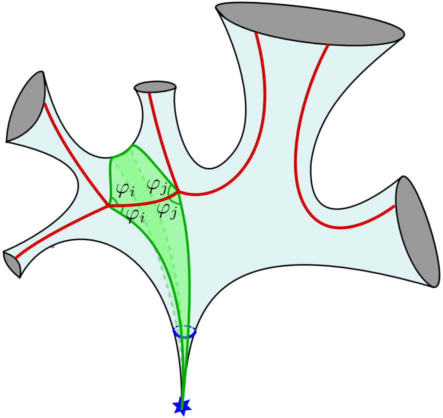

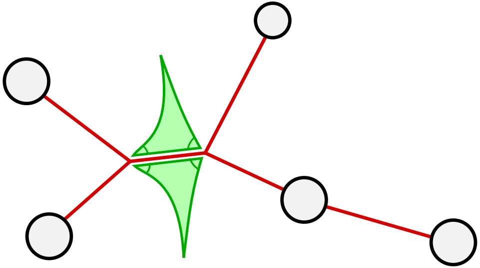

- The surface is canonically triangulated by
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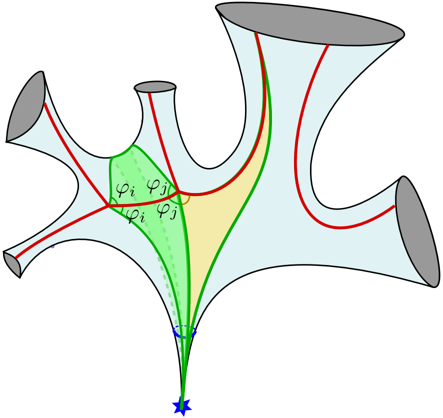
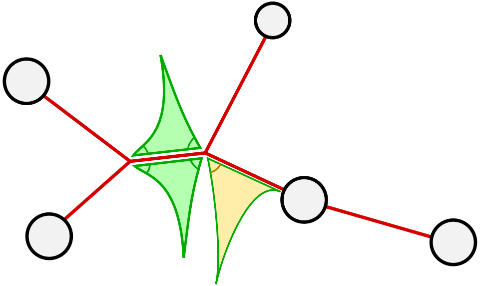
Labels: angles on half edges





- The surface is canonically triangulated by
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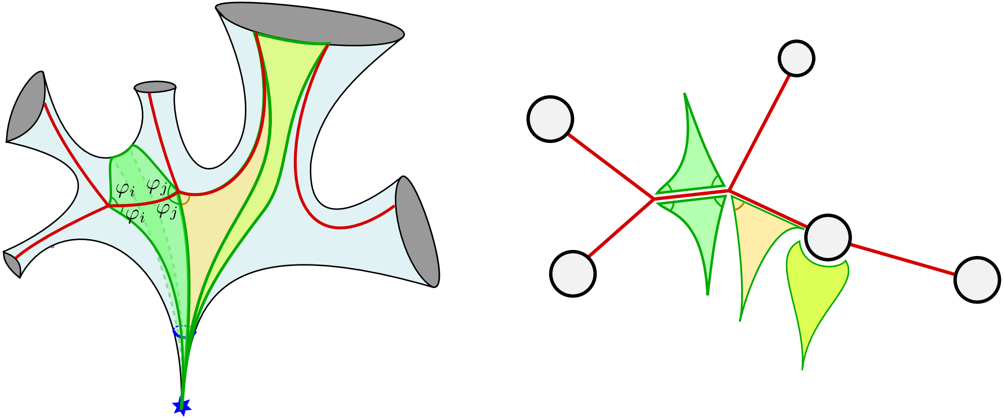

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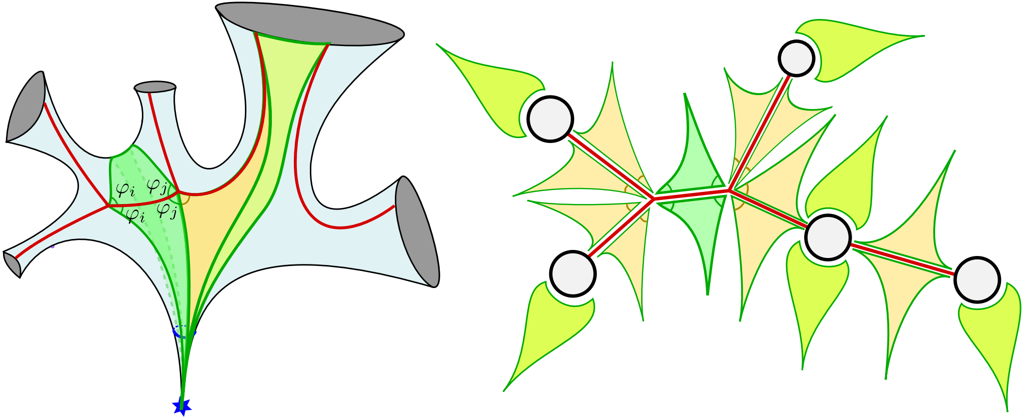

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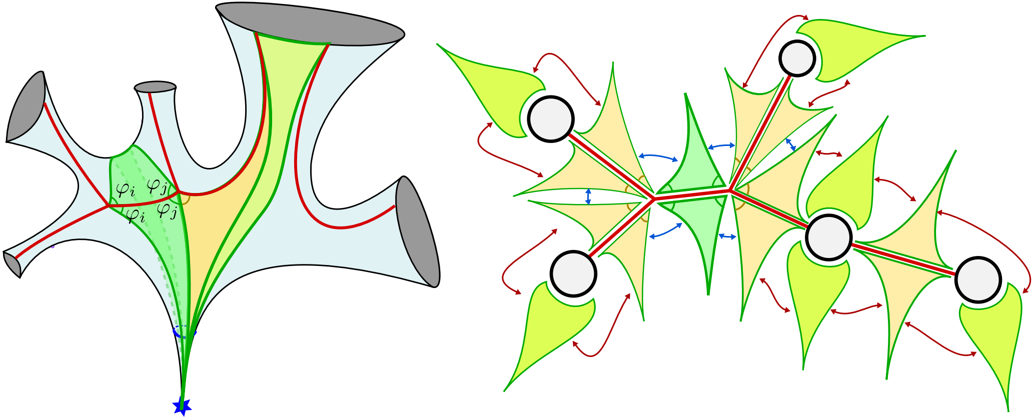

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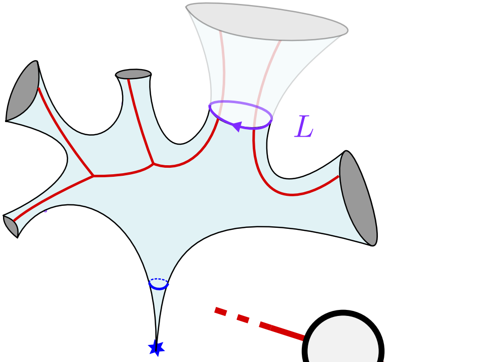
  - for each corner of white vertex: an ideal wedge.
- Gluing of triangles is unique, except for bi-infinite sides: need extra parameters for injectivity.

Labels: geometry around boundary



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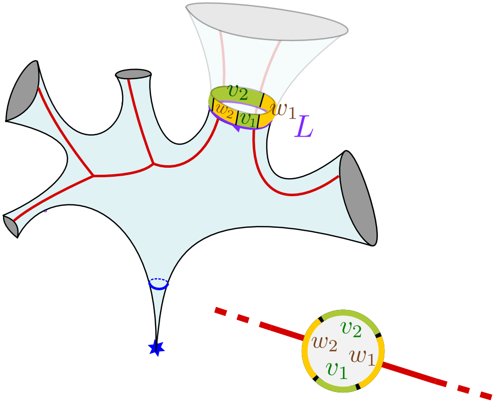
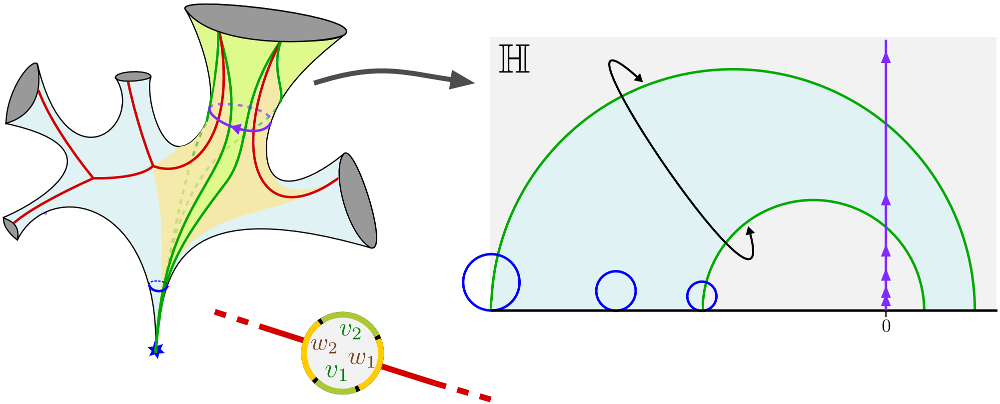


- Boundary of degree k partitions into $2k$ segments of lengths $v_1, \ldots, v_k$, $w_1, \ldots, w_k$.

Labels: geometry around boundary



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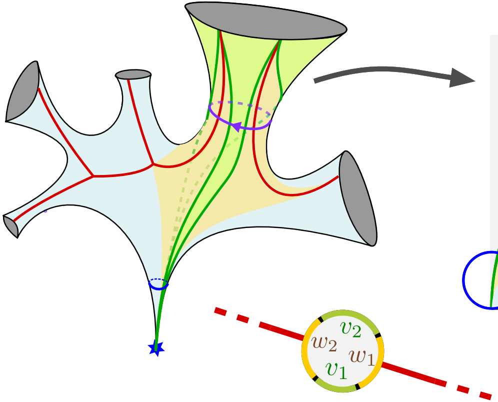
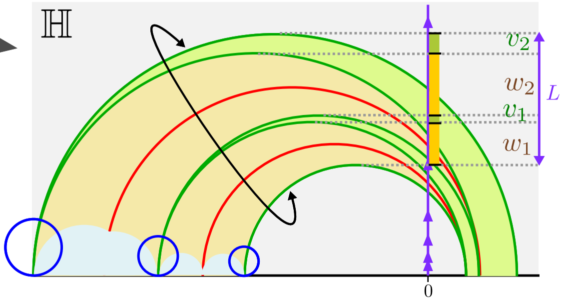
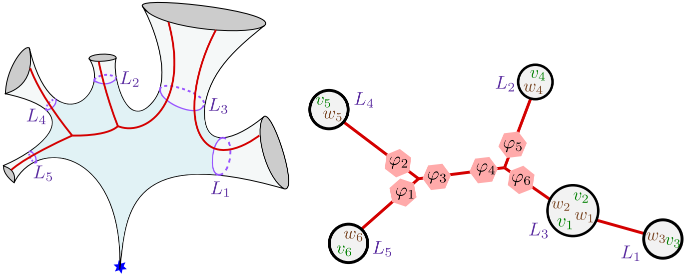




- Boundary of degree k partitions into $2k$ segments of lengths $v_1, \ldots, v_k$, $w_1, \ldots, w_k$.
- Uniquely determines gluing, so should label vertex by

$$\left\{ (v_i, w_i)_{i=1}^{k} : \sum_{i=1}^{k} v_i = \sum_{i=1}^{k} w_i = \frac{L}{2} \right\}.$$

Bijective result



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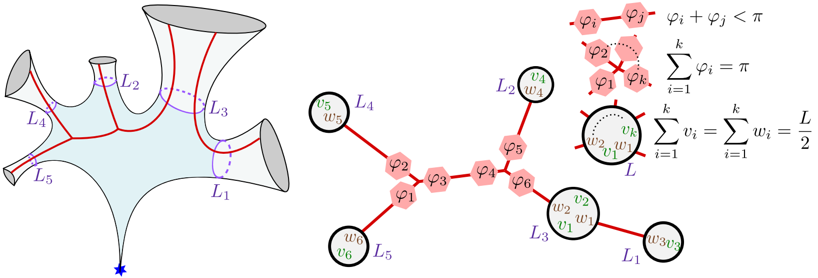

- For tree $\mathfrak{t}$ with n white vertices (deg ≥ 1) and red vertices (deg ≥ 3),

$$\mathcal{A}_{\mathfrak{t}}(L_1, \ldots, L_n) = \{(\phi_i, v_i, w_i) : \phi_i > 0, v_i \geq 0, w_i > 0, \text{constraints above}\}.$$

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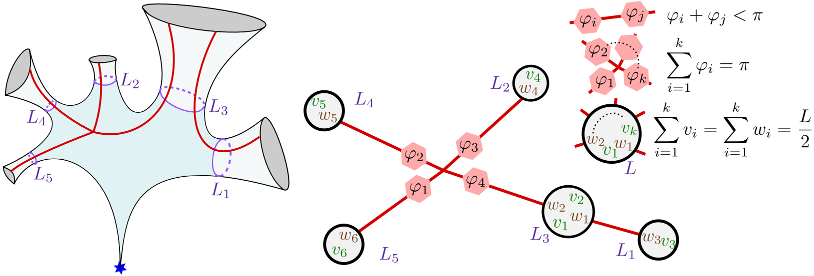

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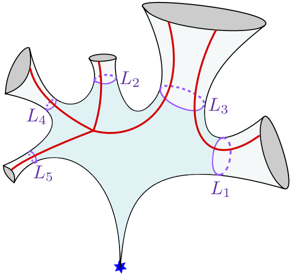

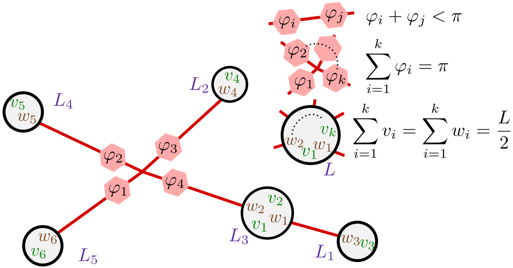

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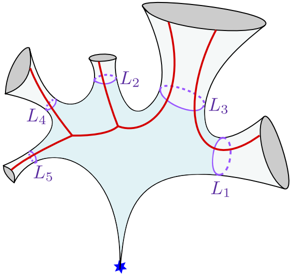
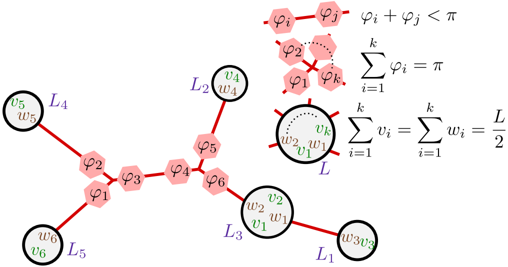
Theorem (TB, Meeusen, Zonneveld, '23+)

This determines a bijection

$$\Phi : \mathcal{M}_{0,n+1}(0, L_1, \ldots, L_n) \longleftrightarrow \bigsqcup_{\mathfrak{t}} \mathcal{A}_{\mathfrak{t}}(L_1, \ldots, L_n).$$

Bijective result





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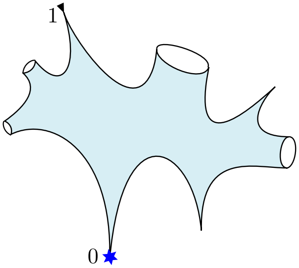
The push-forward of the WP volume is simply the Euclidean volume on the polytope $\mathcal{A}_{\mathfrak{t}} \subset \mathbb{R}^{2n-4}$,

$$\Phi_* \mu_{\mathrm{WP}} = \prod_{\circ} 2^{k-1} \mathrm{d}w_1 \mathrm{d}v_1 \cdots \mathrm{d}w_{k-1} \mathrm{d}v_{k-1} \prod_{\bullet} 2\mathrm{d}\phi_1 \mathrm{d}\phi_2.$$

WP volume generating function

- Why does $R = \sum_{n\geq 1} \frac{1}{n!} \int_0^\infty \mathrm{d}q(L_1)\cdots\mathrm{d}q(L_n) V^{\mathrm{WP}}_{0,n+2}(0,0,\mathbf{L})$ satisfy

$$R = \sum_{k=0}^{\infty} \frac{2^{k-1}}{k!} t_k R^k + \sum_{k=2}^{\infty} \frac{2^{k-1}}{k!} \gamma_k R^k, \qquad t_k = \frac{2}{k!} \int_0^\infty \left(\tfrac{L}{2}\right)^{2k} \mathrm{d}q(L), \quad \gamma_k = \frac{(-1)^k \pi^{2k-2}}{(k-1)!}$$



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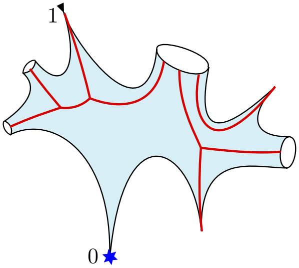

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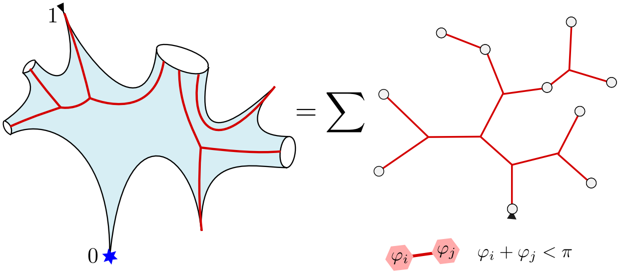

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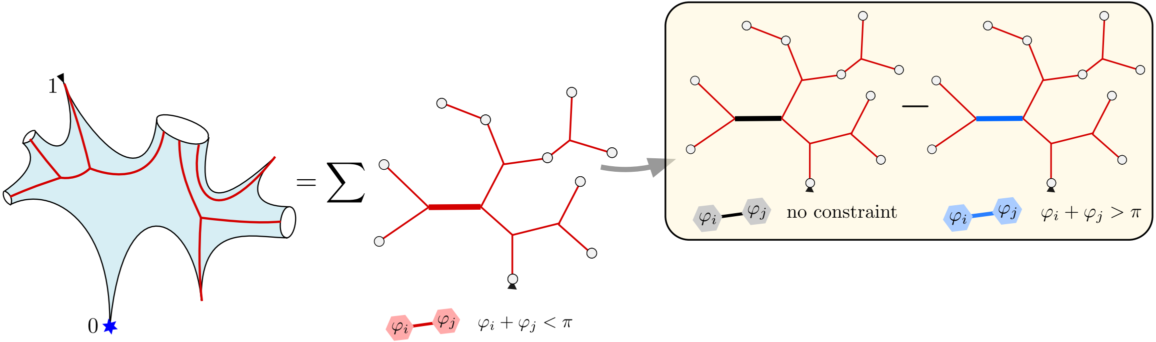

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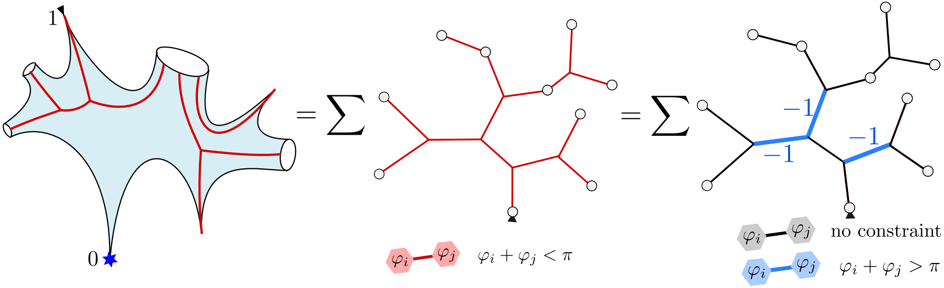

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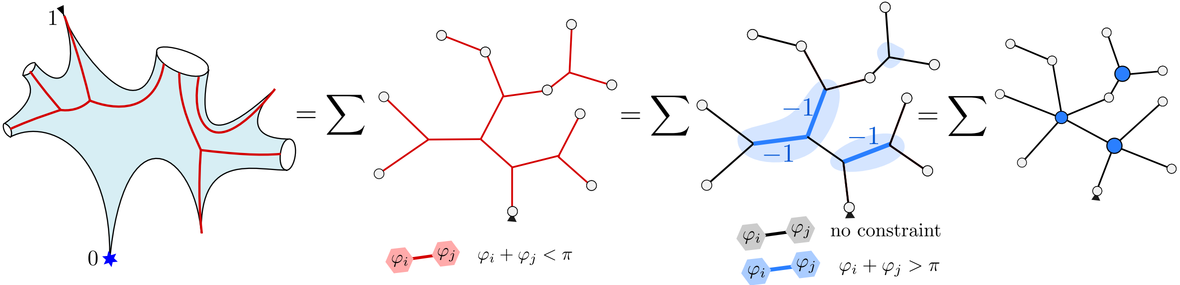

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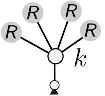
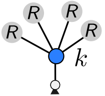
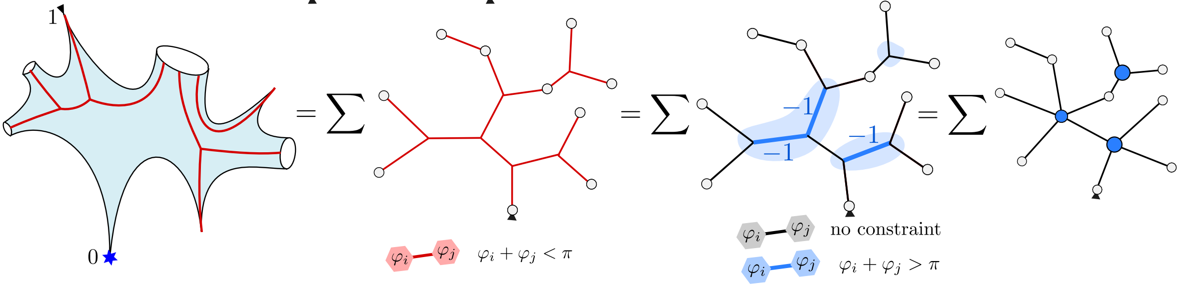
$= \sum_{k=0}^{\infty}$ 

$+ \sum_{k=2}^{\infty}$ 



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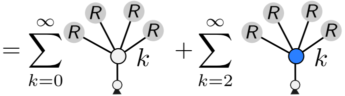
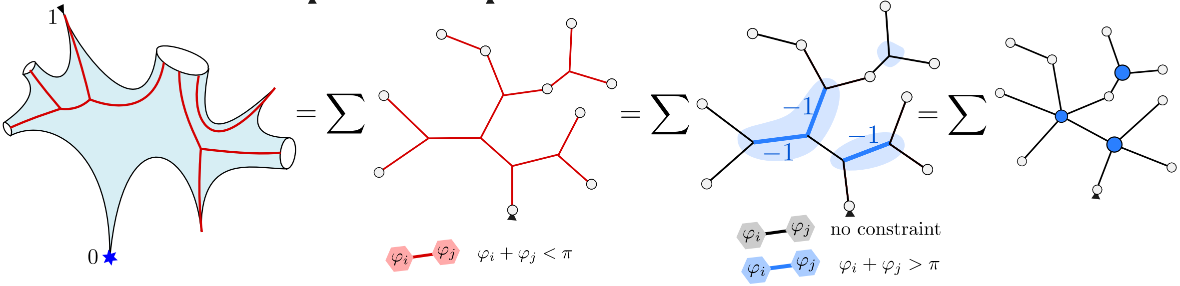
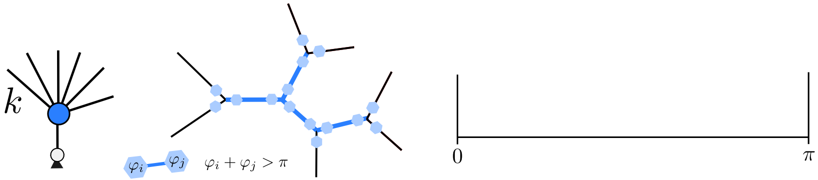




WP volume of blue vertices

- The reversed condition $\varphi_i + \varphi_j > \pi$ is simpler, because WP volume is independent of tree structure:



$$\frac{\gamma_k}{k!} = (-1)^k \sum_{\text{binary trees}} \int d\varphi_1 \cdots d\varphi_{2k-2} =$$

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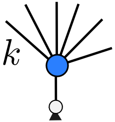

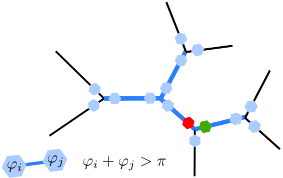

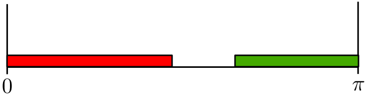

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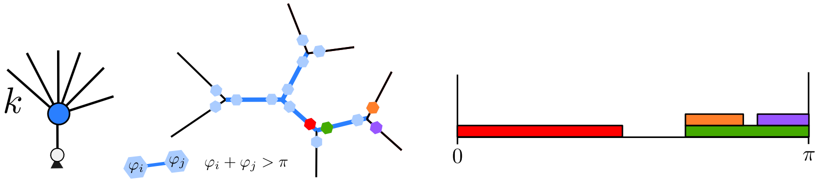

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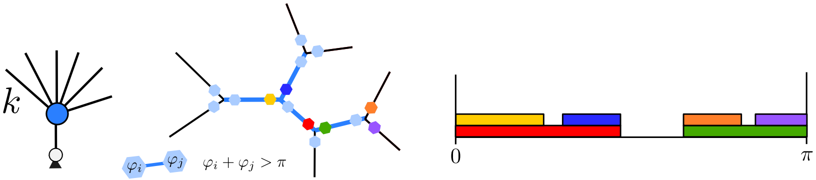

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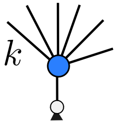

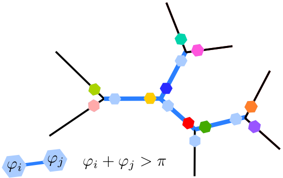

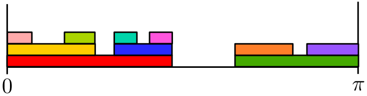

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WP volume of blue vertices

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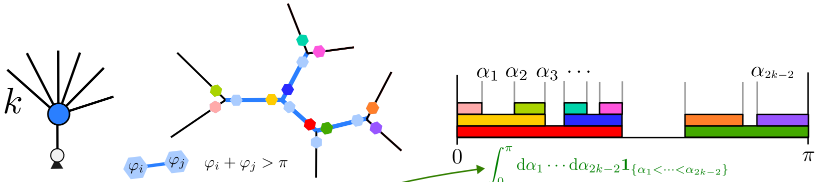

$$\frac{\gamma_k}{k!} = (-1)^k \sum_{\text{binary trees}} \int d\varphi_1 \cdots d\varphi_{2k-2} = \int_0^\pi d\alpha_1 \cdots d\alpha_{2k-2} \mathbf{1}_{\{\alpha_1 < \cdots < \alpha_{2k-2}\}}$$

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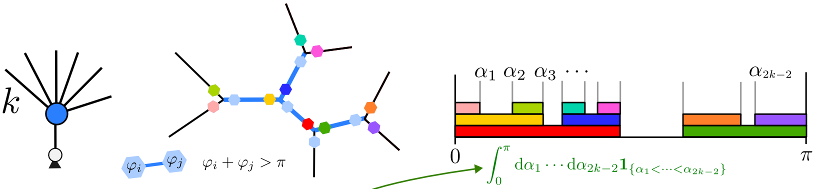
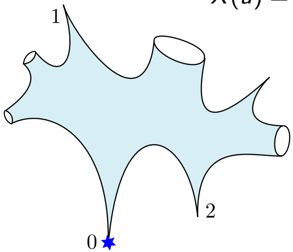


$$\frac{\gamma_k}{k!} = (-1)^k \sum_{\text{binary trees}} \int \mathrm{d}\varphi_1 \cdots \mathrm{d}\varphi_{2k-2} = (-1)^k \,\mathrm{Cat}(k-1)\, \frac{\pi^{2k-2}}{(2k-2)!} = (-1)^k \frac{\pi^{2k-2}}{k!(k-1)!}$$

Not just volumes: geodesic distance control!

- Consider the distance-dependent generating function of triply-cusped surfaces

$$X(u) = \sum_{n=0}^{\infty} \frac{1}{n!} \int_0^{\infty} \Big[\prod_{i=1}^{n} \mathrm{d}q(L_i) \Big] \int_{\mathcal{M}_{n+3}(0,0,0,\mathbf{L})} e^{2u(d_1 - d_2)} \mathrm{d}\mu_{\mathrm{WP}}.$$



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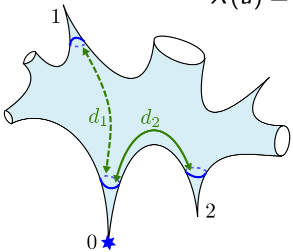

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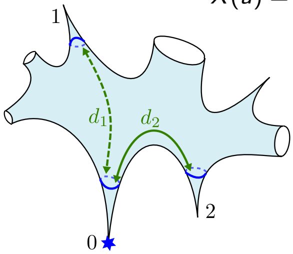

Theorem (TB, Meeusen, Zonneveld, '23+)

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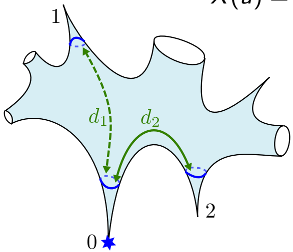

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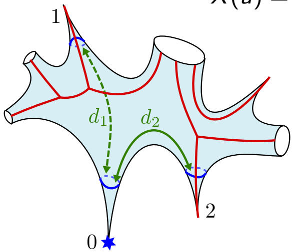

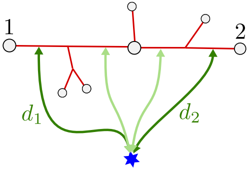

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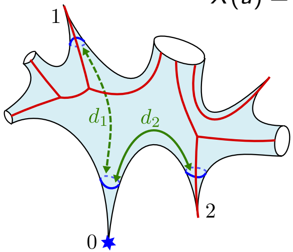

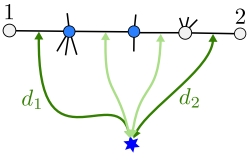

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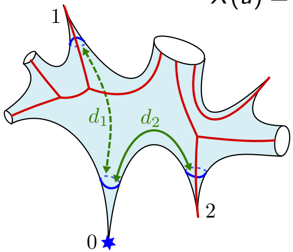

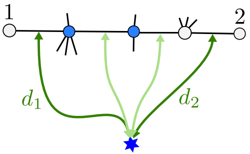

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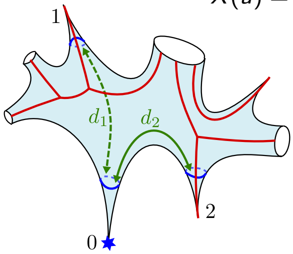

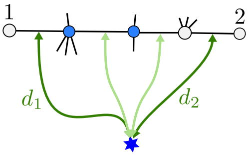

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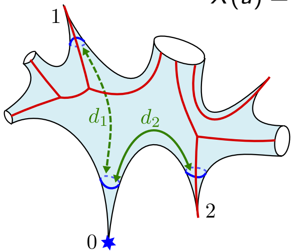

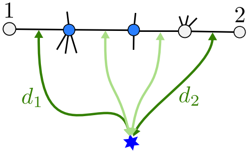

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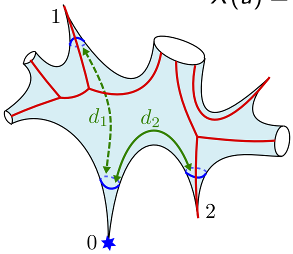

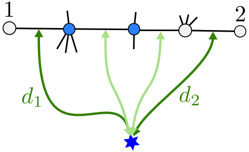

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- Singularity analysis: $d_1 - d_2 \approx n^{1/4}$ in Boltzmann hyperbolic sphere for n large. Same universality class as Boltzmann planar map?

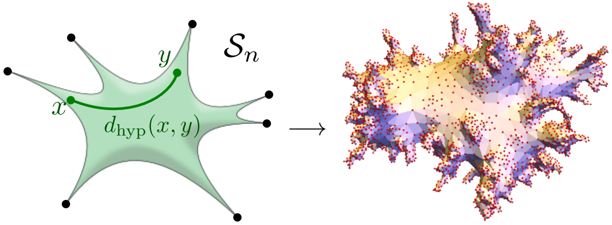
Control on hyperbolic distances

- In the case of only cusps, $q(L) = x\delta(L)$, this is indeed true:

Theorem (TB, Curien, '23+)

If $S_n \in \mathcal{M}_{0,n}(0)$ is sampled with probability density $\mu_{WP}/V_{0,n}(0)$, then we have the convergence in distribution of the random metric space in the Gromov–Prokhorov topology

$$\left(S_n, \frac{d_{\mathrm{hyp}}}{c\, n^{1/4}}\right) \xrightarrow[n\to\infty]{(\mathrm{d})} \textit{Brownian sphere}, \quad c = 2.339\ldots.$$



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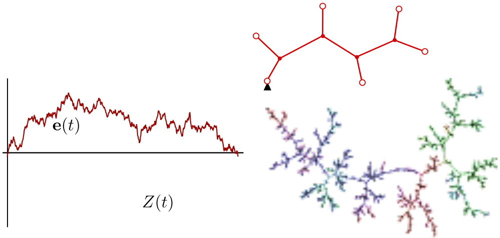
Proof ingredients: Le Gall's strategy [Le Gall, '11]

convergence of labeled tree to Brownian snake



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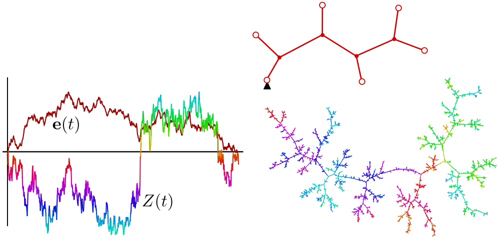

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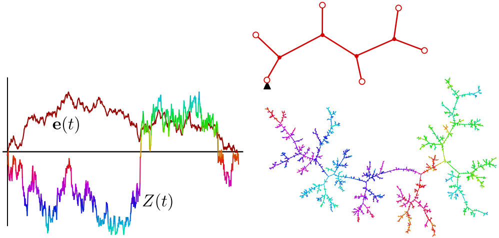
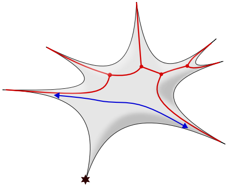


+

bound on distances between arbitrary cusps



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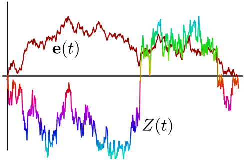
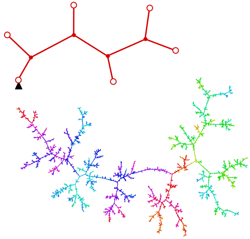
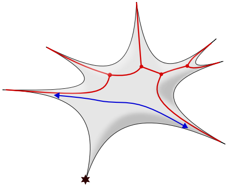
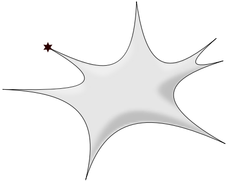
convergence of labeled tree to Brownian snake + bound on distances between arbitrary cusps + invariance under change of origin

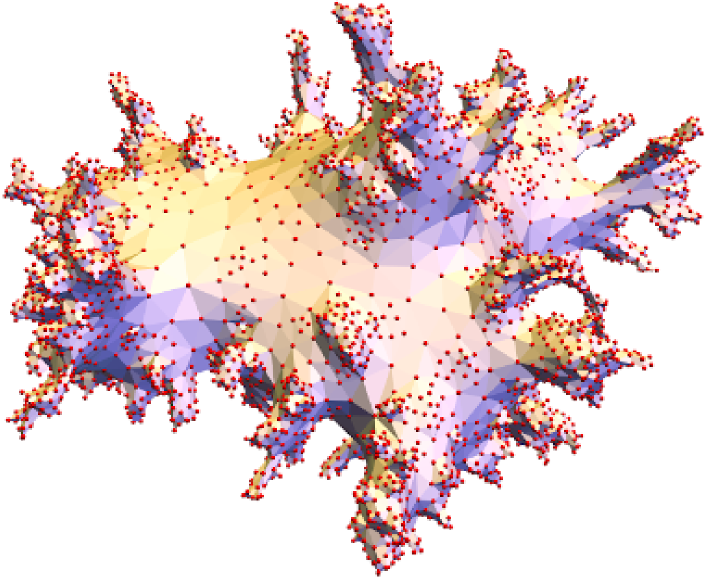




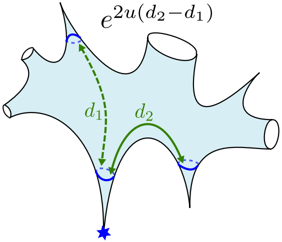




Thanks for your attention!



Topological recursion



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Theorem (TB, Zonneveld, '23+)

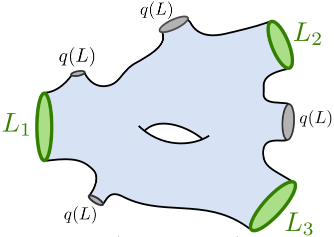
The invariants $\omega_{g,n}(\mathbf{z})$ *of the curve* $(x(u) = u^2, y(u))$ *with initial condition* $\omega_{0,2}(z) = (z_1 - z_2)^2$ *and topological recursion*

$$\omega_{g,n}(\mathbf{z}) = \operatorname*{Res}_{u \to 0} \frac{1}{(z_1^2 - u^2)y(u)} \left[\omega_{g-1,n+1}(u, -u, \mathbf{z}_{\widehat{\{1\}}}) + \sum_{\substack{g_1+g_2=g \\ I \amalg J = \{2,\ldots,n\}}} \omega_{g_1,n_1}(u, \mathbf{z}_I)\omega_{g_2,n_2}(-u, \mathbf{z}_J) \right]$$

give the Laplace transforms of "Tight Weil–Petersson volumes" $T_{g,n}(L_1, \ldots, L_n)$,

$$\omega_{g,n}(\mathbf{z}) = \int_0^\infty \mathrm{d}L_1 L_1 e^{-z_1 L_1} \cdots \int_0^\infty \mathrm{d}L_n L_n e^{-z_n L_n} T_{g,n}(L_1, \ldots, L_n).$$

Tight Weil-Petersson volumes



$T_{1,3}(L_1, L_2, L_3)$

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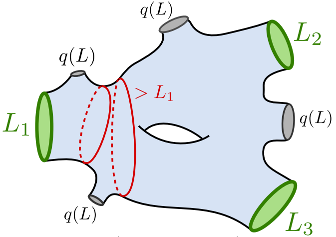

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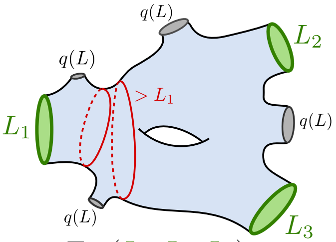
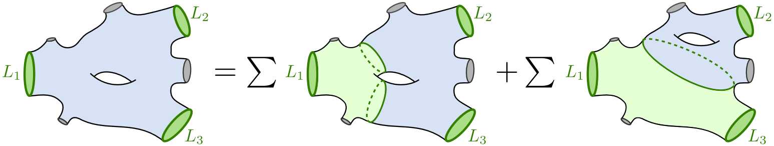
Tight Weil-Petersson volumes



$T_{1,3}(L_1, L_2, L_3)$

$$T_{g,n}(\mathbf{L}) = \sum_{p=0}^{\infty} \frac{1}{p!} \int \mathrm{d}q(L_{n+1}) \int \mathrm{d}q(L_{n+p}) \int_{\mathcal{M}_{g,n+p}(\mathbf{L},\mathbf{L})} \mathrm{d}\mu_{\mathrm{WP}} \mathbf{1}_{\{\mathrm{tight}\}}$$

Topological recursion pictorially:



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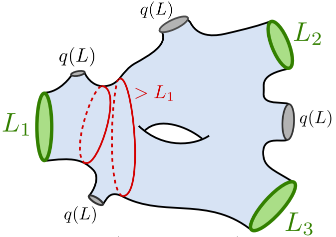

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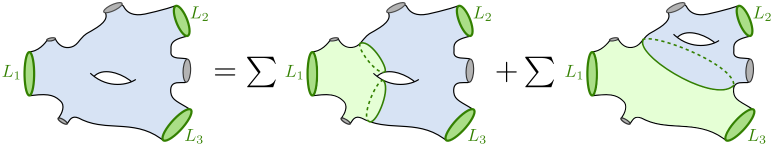

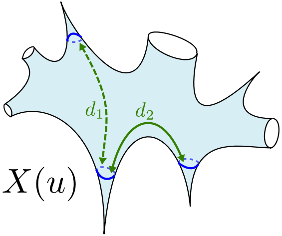

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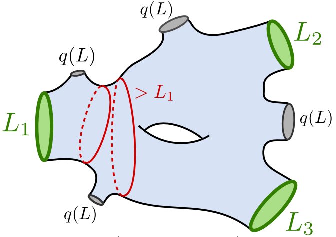

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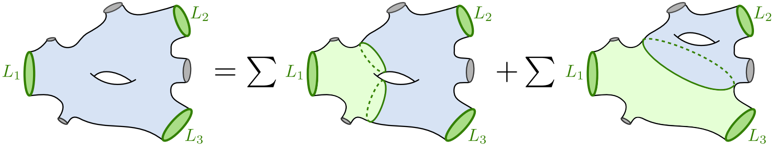

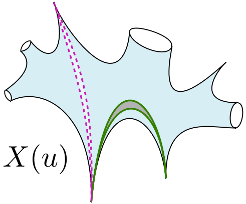

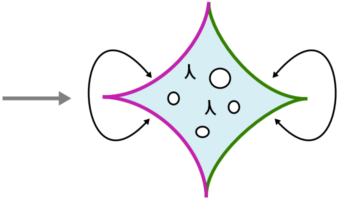

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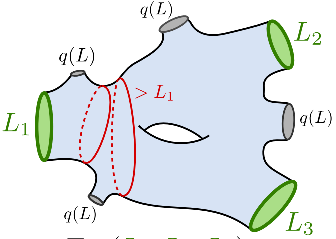

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