Invitation to topological recursion and its ramifications: Exercises

Weil–Petersson volumes

Let $\mathcal{M}_{g,n}(L1,\ldots,L_n)$ be the moduli space of oriented hyperbolic surfaces of genus g with n boundary components of geodesic lengths $L_1,\ldots,L_n$. Every Riemann surface can be decomposed into $2g-2+n$ pairs of pants with $3g-3+n$ internal lengths. These lengths and the gluing angles give coordinates in this moduli space (called *Fenchel–Nielsen* coordinates). Using them we locally define a 2-form ω_{WP} on the moduli space. Taking care of some issues, we can use it to define Weil–Petersson volumes:

$$V_{g,n} := \frac{1}{(3g-3+n)!}\int_{\mathcal{M}_{g,n}(L_1,\ldots,L_n)} \omega_{\mathrm{WP}}^{3g-3+n}.$$

It is only possible to compute $V_{g,n}$ directly for very small (g,n), but Mirzakhani (2004) found a recursion for them:

$$V_{g,n} = \text{“expression in terms of } V_{g',n'}\text{”} \quad \text{(complicated looking integral formula)},$$

with $2g'-2+n' < 2g-2+n$.
Consider the Laplace transforms of the WP volumes:

$$W_{g,n}(z_1,\ldots,z_n) = \int_0^\infty \cdots \int_0^\infty e^{-z_1L_1-\ldots-z_nL_n} V_{g,n}(L_1,\ldots,L_n) \prod_{i=1}^n L_i \,\mathrm{d}L_i.$$

In 2006, Eynard and Orantin proved that Mirzakhani's recursion is actually an instance of TR, by Laplace transform.

Theorem. *The TR applied to the following spectral curve*

$$\mathcal{S}_{\mathrm{WP}} = \begin{cases} x(z) = z^2, \\ y(z) = \frac{-1}{4\pi}\sin(2\pi z), \\ \omega_{0,2}(z_1,z_2) = B(z_1,z_2) = \frac{\mathrm{d}z_1\mathrm{d}z_2}{(z_1-z_2)^2} \end{cases}$$

governs Weil–Petersson volumes, in the sense that the TR multi-differentials are $\omega_{g,n}(z_1,\ldots,z_n) = W_{g,n}(z_1,\ldots,z_n)\,\mathrm{d}z_1\cdots\mathrm{d}z_n$.

Exercise 1. Using topological recursion on the spectral curve of the previous theorem, compute $W_{0,3}(z_1,z_2,z_3)$ and $W_{1,1}(z_1)$. (*Bonus:* Compute $W_{0,4}(z_1,z_2,z_3,z_4)$ and more.)
Start by showing that the recursion kernel is:

$$K(z_1,z) := \frac{\int_{-z}^{z}\omega_{0,2}(z_1,\cdot)}{2(\omega_{0,1}(z)-\omega_{0,1}(\sigma(z)))} = \frac{-\pi\,\mathrm{d}z_1}{(z_1^2-z^2)\sin(2\pi z)\,\mathrm{d}z},$$

with $\sigma(z) = -z$.
Deduce that $V_{0,3}(L_1,L_2,L_3) = 1$, $V_{1,1}(L_1) = \frac{1}{24}(2\pi^2 + \frac{1}{2}L_1^2)$ and $V_{0,4}(L_1,L_2,L_3,L_4) = 2\pi^2 + \frac{1}{2}\sum_{i=1}^4 L_i^2$. (You may use a CAS for this computation).

GUE = Maps without internal faces

Exercise 2. Consider maps without internal faces, that is $t_k = 0$ in the generating series. This corresponds to the Gaussian Unitary Ensemble (GUE).

1. Deduce from combinatorics that $\mathrm{Map}_\ell^{[g]} = 0$ for all ℓ odd. (*Bonus*) Can you also justify this vanishing making use of the correspondence to a certain expectation value?

2. (*Bonus*) Determine the constant Z_{GUE} such that $\int_{\mathcal{H}_N} \mathrm{d}\nu_{GUE} = 1$, with

$$\mathrm{d}\nu_{GUE}(A) = \frac{1}{Z_{GUE}} e^{-C\,\mathrm{Tr}(A^2)}\,\mathrm{d}A.$$

3. Write the Tutte's recursion for the generating series $W_{g,n}(x_1,\ldots,x_n)$ of maps without internal faces.

4. From the Tutte's recursion for the disk, i.e. for $(0,1)$, compute $W(x) = W_{0,1}(x)$. Can you recognise the coefficients of $x^{-(1+2m)}$?

5. Show that $W(x)$ is an analytic function of $x \in \mathbb{C} \setminus [a_-, a_+]$. Compute $a_\pm$.

6. Compute $\omega_{0,1}(z)$ and $\omega_{1,1}(z)$. Would you know how to deduce the number of maps of topology $(1,1)$ with boundary length up to 10? (*Indication:* To compute $\omega_{1,1}(z)$, you can use the topological recursion formula and a CAS. To compute the number of maps of topology $(1,1)$, you can use a CAS.)

Reminder: $W_{g,1}(x) = \sum_{\ell\geq 0} \frac{\mathrm{Map}_\ell^{[g]}}{x^{\ell+1}}$, $\omega_{g,1}(z) = W_{g,1}(x(z))\,\mathrm{d}x(z)$, with $x(z) = z + \frac{1}{z}$.

Properties

For all $2g-2+n>0$, we have *dilaton equation*

$$\sum_{\alpha\in\mathrm{Ram}} \operatorname*{Res}_{z=\alpha} \Phi(z)\omega_{g,n+1}(z_1,\ldots,z_n) = (2g-2+n)\omega_{g,n}(z_1,\ldots,z_n),$$

where $\mathrm{d}\Phi = y\mathrm{d}x$ (defined up to a constant, but $\operatorname*{Res}_{z=\alpha} \omega_{g,n}(z_1,\ldots,z_n) = 0$.

Exercise 3. Deduce a reasonable definition for $\omega_{g,0}$ from dilaton equation.

Exercise 4. Prove homogeneity: if we rescale $y(z) \mapsto \lambda y(z)$, the output of TR rescales as $\omega_{g,n} \mapsto \lambda^{2-2g-n}\omega_{g,n}$.

Exercise 5. (*Symplectic invariance*) Prove that the following transformations of spectral curves (x,y) are symplectic, i.e. they preserve the symplectic form $\mathrm{d}x \wedge \mathrm{y}$:

1. $y \mapsto y + R(x)$, where $R(x)$ is a rational function of x;
2. $y \mapsto \lambda y$, $x \mapsto \frac{x}{\lambda}, \lambda \in \mathbb{C}^*$;
3. $x \mapsto \frac{ax+b}{ax+d}$, $y \mapsto \frac{(cx+d)^2}{ad-bc}y$;
4. $(x,y) \mapsto (y,-x)$ (*exchange transformation*).

Prove that the first three leave the output of topological recursion invariant. The transformation of the output when applying the last transformation is highly non-trivial, being actively studied and still to be properly understood (more on talks next week).

Bonus exercises on maps

There are a special type of maps, very related to the exchange transformation (symplectic invariance), called *fully simple maps*. In some sense, fully simple maps are the dual of maps; you will learn more on this duality in the 3 of the TR talks of next week. Simple maps are defined as maps in which when going around the edges of a boundary, one only enters every vertex once. Fully simple maps are simple maps in which the boundaries don't share any vertex. Do you understand the definitions? Consider the analogue of the generating series of maps in which we only sum over fully simple maps: $\mathrm{FSMap}^{[g]}_{\ell_1,\ldots,\ell_n}$.

Exercise 6. Consider the generating series of disks $W(x)$ and fully simple disks $X(w)$:

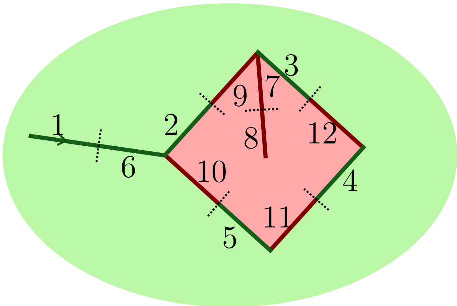
$$W(x) = \textstyle\sum_{\ell\geq 0} \frac{\mathrm{Map}_\ell^{[0]}}{x^{\ell+1}} \quad \text{and} \quad X(w) = \sum_{\ell\geq 0} \mathrm{FSMap}_\ell^{[0]} w^{\ell-1}\,.$$

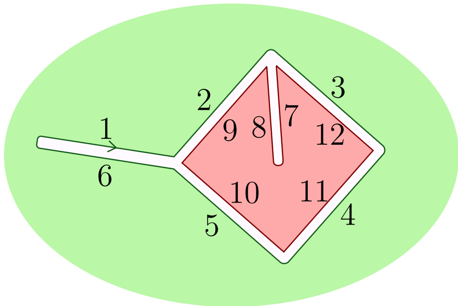
Prove that $X(W(x)) = x$.

Exercise 7. Consider quadrangulations, i.e. maps with $t_k = 0$ for $k \neq 4$. Draw all the disks with one boundary of length 4 and none or 1 internal quadrangle. How many are there? How many of those are simple? And fully simple? Compute the number of quadrangulations with a boundary of length 2 and none or 1 internal quadrangle. How many cylinders with boundaries of lengths 2 and 4 are there? How many of those are simple/fully simple? How are the series $F^{[0]}_{\ell_1}$ and $F^{[0]}_{\ell_1,4}$ related? Do you observe some general phenomenon?

Maps with labeled half-edges are in one-to-one correspondence with triples of permutations σ, α, φ on the set of half-edges. The permutation σ is defined as the permutation whole cycles go around the half-edges of vertices (counter-clockwise), α is an involution that pairs half-edges and φ has cycles that go around the half-edges of faces (counter-clockwise).

Exercise 8. Consider the following map with labeled half-edges. Find the associated permutations σ, α, φ that encode it. Check that you get $\sigma \circ \alpha \circ \varphi = 1$ and that the number of vertices, edges and faces of the map coincide with the number of cycles of σ, α, φ, respectively.





Exercise 9. Formalise the definitions of isomorphisms of maps and autormorphisms of maps using the permutational model and play with them.

Bonus TR exercise

Exercise 10. Compute the spectral curve of quadrangulations (in terms of the parameter z). Would you know how to extract the number of quadrangulations, first for disks and then for the next topologies?