Supersymmetric Quantum Mechanics and Morse Theory

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Solution to Homework 4: Supersymmetric quantum mechanics

1 Fermionic coherent states

Let us consider a quantum mechanics of complex fermion. Let b and $b^\dagger$ be fermionic creation annihilation operators i.e. they obey anticommutation relation

$$\{b, b^\dagger\} = bb^\dagger + b^\dagger b = 1 \tag{1.1}$$

while the Hilbert space $\mathcal{H}$ is two dimensional with basis

$$|0\rangle, \;\; |1\rangle \equiv b^\dagger|0\rangle, \;\; (-)^F|n\rangle = (-1)^n|n\rangle \tag{1.2}$$

The vacuum state obeys

$$b|0\rangle = 0, \;\; \langle 0|0\rangle = 1. \tag{1.3}$$

We can introduce complex Grassmnann-odd variable ψ and define fermionic coherent states and adjoints

$$|\psi\rangle \equiv e^{b^\dagger\psi}|0\rangle, \quad \langle\bar{\psi}| \equiv \langle 0|e^{\bar{\psi}b} \tag{1.4}$$

1. (*10 Points*) Show that $|\psi\rangle$ are eigenstates of b i.e

$$b|\psi\rangle = \psi|\psi\rangle, \quad \langle\bar{\psi}|b^\dagger = \langle\bar{\psi}|\bar{\psi} \tag{1.5}$$

Solution: Let us rewrite the coherent state in particle number basis $|n\rangle$

$$|\psi\rangle \equiv e^{b^\dagger\psi}|0\rangle = (1 + b^\dagger\psi)|0\rangle = (1 - \psi b^\dagger)|0\rangle = |0\rangle - \psi|1\rangle, \tag{1.6}$$

while the adjoint

$$\langle\bar{\psi}| \equiv \langle 0|e^{b\bar{\psi}} = \langle 0|(1 + \bar{\psi}b) = \langle 0|(1 - b\bar{\psi}) = \langle 0| - \langle 1|\bar{\psi}. \tag{1.7}$$

The eigenstates relation in number basis

$$\begin{aligned} b|\psi\rangle &= b(|0\rangle - \psi|1\rangle) = b|0\rangle - b\psi b^\dagger|0\rangle = \psi bb^\dagger|0\rangle \\ &= \psi(1 - b^\dagger b)|0\rangle = \psi|0\rangle = \psi(|0\rangle - \psi|1\rangle) = \psi|\psi\rangle \end{aligned} \tag{1.8}$$

and

$$\begin{aligned} \langle\bar{\psi}|b^\dagger &= \langle 0|(1 - b\bar{\psi})b^\dagger = \langle 0|b^\dagger - \langle 0|b\bar{\psi}b^\dagger = \langle 0|bb^\dagger\bar{\psi} \\ &= \langle 0|(1 + b^\dagger b)\bar{\psi} = \langle 0|\bar{\psi} = \langle 0|(1 - b\bar{\psi})\bar{\psi} = \langle\bar{\psi}|\bar{\psi}. \end{aligned} \tag{1.9}$$

2. (*10 Points*) Evaluate the paring

$$\langle \bar{\chi}|\psi\rangle \tag{1.10}$$

Solution: Explicit evaluation in number basis

$$\begin{aligned}
\langle \bar{\chi}|\psi\rangle &= \langle 0|(1-b\bar{\chi})(1-\psi b^\dagger)|0\rangle \\
&= \langle 0|0\rangle - \langle 0|\psi b^\dagger|0\rangle - \langle 0|b\bar{\chi}|0\rangle + \langle 0|b\bar{\chi}\psi b^\dagger|0\rangle \\
&= 1 - \psi\langle 0|b^\dagger|0\rangle + \bar{\chi}\langle 0|b|0\rangle + \bar{\chi}\psi\langle 0|bb^\dagger|0\rangle \\
&= 1 + \bar{\chi}\psi\langle 0|(1+b^\dagger b)|0\rangle = 1 + \bar{\chi}\psi = e^{\bar{\chi}\psi}.
\end{aligned} \tag{1.11}$$

3. (*10 Points*) Show completeness of the $|\psi\rangle$ basis i.e.

$$1_{\mathcal{H}} = \int d\psi d\bar{\psi}\; e^{\psi\bar{\psi}}|\psi\rangle\langle\bar{\psi}| \tag{1.12}$$

Solution: The number basis $|0\rangle$ and $|1\rangle$ is a complete, orthonormal basis in $\mathcal{H} = \mathbb{C}^2$

$$\langle n|m\rangle = \delta_{nm}, \quad 1_{\mathcal{H}} = |0\rangle\langle 0| + |1\rangle\langle 1| = \sum_{n=0,1} |n\rangle\langle n| \tag{1.13}$$

Let us evaluate the completeness formula in number basis

$$\begin{aligned}
1_{\mathcal{H}} &= \int d\psi d\bar{\psi}\; e^{\psi\bar{\psi}}|\psi\rangle\langle\bar{\psi}| = \int d\psi d\bar{\psi}\; e^{\psi\bar{\psi}}(|0\rangle - \psi|1\rangle)(\langle 0| - \langle 1|\bar{\psi}) \\
&= \int d\psi d\bar{\psi}\; e^{\psi\bar{\psi}}\left(|0\rangle\langle 0| - |0\rangle\langle 1|\bar{\psi} - \psi|1\rangle\langle 0| + \psi|1\rangle\langle 1|\bar{\psi}\right) \\
&= |0\rangle\langle 0|\int d\psi d\bar{\psi}\; e^{\psi\bar{\psi}} - |0\rangle\langle 1|\int d\psi d\bar{\psi}\; e^{\psi\bar{\psi}}\bar{\psi} \\
&- |1\rangle\langle 0|\int d\psi d\bar{\psi}\; e^{\psi\bar{\psi}}\psi + |1\rangle\langle 1|\int d\psi d\bar{\psi}\; e^{\psi\bar{\psi}}\psi\bar{\psi} \\
&= |0\rangle\langle 0|\int d\psi d\bar{\psi}\; \psi\bar{\psi} - |0\rangle\langle 1|\int d\psi d\bar{\psi}\; \bar{\psi} - |1\rangle\langle 0|\int d\psi d\bar{\psi}\; \psi + |1\rangle\langle 1|\int d\psi d\bar{\psi}\; \psi\bar{\psi} \\
&= |0\rangle\langle 0| + |1\rangle\langle 1| = 1_{\mathcal{H}}.
\end{aligned}$$

4. (*10 Points*) Prove the trace formula

$$\mathrm{Tr}(\mathcal{O}) = \int d\psi d\bar{\psi}\; e^{\psi\bar{\psi}}\langle -\bar{\psi}|\mathcal{O}|\psi\rangle \tag{1.14}$$

Solution: The trace in orthonormal number basis is

$$\mathrm{Tr}(\mathcal{O}) = \sum_{n=0,1} \langle n|\mathcal{O}|n\rangle = \langle 0|\mathcal{O}|0\rangle + \langle 1|\mathcal{O}|1\rangle. \tag{1.15}$$

The coherent state integral formula for trace can be evaluated in number basis

$$\begin{aligned}
\mathrm{Tr}(\mathcal{O}) &= \int d\psi d\bar{\psi}\; e^{\psi\bar{\psi}} \langle -\bar{\psi}|\mathcal{O}|\psi\rangle \\
&= \int d\psi d\bar{\psi}\; e^{\psi\bar{\psi}} [\langle 0| - \langle 1|(-\bar{\psi})]\mathcal{O}(|0\rangle - \psi|1\rangle) \\
&= \int d\psi d\bar{\psi}\; e^{\psi\bar{\psi}} \big(\langle 0|\mathcal{O}|0\rangle - \psi\langle 0|\mathcal{O}|1\rangle + \bar{\psi}\langle 1|\mathcal{O}|0\rangle - \bar{\psi}\psi\langle 1|\mathcal{O}|1\rangle\big) \\
&= \langle 0|\mathcal{O}|0\rangle \int d\psi d\bar{\psi}\; e^{\psi\bar{\psi}} - \langle 0|\mathcal{O}|1\rangle \int d\psi d\bar{\psi}\; e^{\psi\bar{\psi}}\psi \\
&+ \langle 1|\mathcal{O}|0\rangle \int d\psi d\bar{\psi}\; e^{\psi\bar{\psi}}\bar{\psi} - \langle 1|\mathcal{O}|1\rangle \int d\psi d\bar{\psi}\; e^{\psi\bar{\psi}}\bar{\psi}\psi \\
&= \langle 0|\mathcal{O}|0\rangle \int d\psi d\bar{\psi}\; \psi\bar{\psi} - \langle 1|\mathcal{O}|1\rangle \int d\psi d\bar{\psi}\; \bar{\psi}\psi \\
&= \langle 0|\mathcal{O}|0\rangle + \langle 1|\mathcal{O}|1\rangle.
\end{aligned} \tag{1.16}$$

5. (*10 Points*) Prove the supertrace formula

$$\mathrm{Str}(\mathcal{O}) \equiv \mathrm{Tr}((-)^F\mathcal{O}) = \int d\psi d\bar{\psi}\; e^{\psi\bar{\psi}} \langle \bar{\psi}|\mathcal{O}|\psi\rangle \tag{1.17}$$

Solution: The supertrace in number basis

$$\mathrm{Str}(\mathcal{O}) = \mathrm{Tr}((-)^F\mathcal{O}) = \sum_{n=0,1} \langle n|(-)^F\mathcal{O}|n\rangle = \langle 0|\mathcal{O}|0\rangle - \langle 1|\mathcal{O}|1\rangle, \tag{1.18}$$

while the coherent states integral

$$
\begin{aligned}
\mathrm{Str}(\mathcal{O}) &= \int d\psi d\bar{\psi}\ e^{\psi\bar{\psi}} \langle\bar{\psi}|\mathcal{O}|\psi\rangle \\
&= \int d\psi d\bar{\psi}\ e^{\psi\bar{\psi}} (\langle 0| - \langle 1|\bar{\psi})\mathcal{O}(|0\rangle - \psi|1\rangle) \\
&= \int d\psi d\bar{\psi}\ e^{\psi\bar{\psi}} \big(\langle 0|\mathcal{O}|0\rangle - \psi\langle 0|\mathcal{O}|1\rangle - \bar{\psi}\langle 1|\mathcal{O}|0\rangle + \bar{\psi}\psi\langle 1|\mathcal{O}|1\rangle\big) \\
&= \langle 0|\mathcal{O}|0\rangle \int d\psi d\bar{\psi}\ e^{\psi\bar{\psi}} - \langle 0|\mathcal{O}|1\rangle \int d\psi d\bar{\psi}\ e^{\psi\bar{\psi}}\psi \\
&- \langle 1|\mathcal{O}|0\rangle \int d\psi d\bar{\psi}\ e^{\psi\bar{\psi}}\bar{\psi} + \langle 1|\mathcal{O}|1\rangle \int d\psi d\bar{\psi}\ e^{\psi\bar{\psi}}\bar{\psi}\psi \\
&= \langle 0|\mathcal{O}|0\rangle \int d\psi d\bar{\psi}\ \psi\bar{\psi} + \langle 1|\mathcal{O}|1\rangle \int d\psi d\bar{\psi}\ \bar{\psi}\psi \\
&= \langle 0|\mathcal{O}|0\rangle - \langle 1|\mathcal{O}|1\rangle.
\end{aligned}
\tag{1.19}
$$

2 1d Sigma Model

Christoffel symbols Γ for Levi-Cevita connection of the Riemann metric g_{ij} are

$$
\Gamma^k_{ij} = \frac{1}{2} g^{kl}(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}) \tag{2.1}
$$

The Riemann tensor for the metric g_{ij} can be expressed via

$$
R_{abcd} = \frac{1}{2}(\partial^2_{bc} g_{ad} + \partial^2_{ad} g_{bc} - \partial^2_{bd} g_{ac} - \partial^2_{ac} g_{bd}) + g_{ij}(\Gamma^i_{bc}\Gamma^j_{ad} - \Gamma^i_{bd}\Gamma^j_{ac}) \tag{2.2}
$$

The $N = 2$ superfield $\hat{x}^i$ in components

$$
\hat{x}^j(t, \theta, \bar{\theta}) = x^j(t) + \theta\bar{\psi}^j(t) - \bar{\theta}\psi^j(t) + \theta\bar{\theta}F^j(t) \tag{2.3}
$$

with supercovariant derivatives

$$
\begin{aligned}
\mathfrak{D} &= \frac{\partial}{\partial\theta} - i\bar{\theta}\frac{\partial}{\partial t}, \\
\bar{\mathfrak{D}} &= \frac{\partial}{\partial\bar{\theta}} - i\theta\frac{\partial}{\partial t}
\end{aligned}
\tag{2.4}
$$

- (*30 Points*)The superspace realization of sigma-model Lagrangian is given by

$$
L[x, \psi, \bar{\psi}, F] = \int d^2\theta\ \left(\frac{1}{2} g_{jk}(\hat{x})\mathfrak{D}\hat{x}^j\bar{\mathfrak{D}}\hat{x}^k\right) \tag{2.5}
$$

Perform the superspace integration and show that

$$\begin{aligned} L[x,\psi,\bar\psi,F] = g_{jk}\left(i\bar\psi^j\nabla_t\psi^k - i\nabla_t\bar\psi^j\psi^k + \dot x^j\dot x^k\right) \\ + g_{lm}(F^l - \psi^k\bar\psi^j\Gamma^l_{kj})(F^m - \psi^k\bar\psi^j\Gamma^m_{kj}) + \psi^a\bar\psi^b\psi^c\bar\psi^d R_{abcd} \end{aligned} \tag{2.6}$$

where

$$\nabla_t\psi^k = \dot\psi^k + \Gamma^k_{lm}\dot x^l\psi^m \tag{2.7}$$

Solution: The supercovariant derivatives are

$$\begin{aligned} \mathfrak{D}\hat x^j &= \bar\psi^j + \bar\theta F^j - i\bar\theta\dot x^j - i\bar\theta\theta\dot{\bar\psi}^j \\ \bar{\mathfrak{D}}\hat x^k &= -\psi^k - \theta F^k - i\theta\dot x^k + i\theta\bar\theta\dot\psi^k. \end{aligned} \tag{2.8}$$

The metric expansion is

$$\begin{aligned} g_{jk}(\hat x) &= g_{jk}(x) + (\theta\bar\psi^l - \bar\theta\psi^l + \theta\bar\theta F^l)\partial_l g_{jk}(x) + \frac{1}{2}(\theta\bar\psi^l - \bar\theta\psi^l)(\theta\bar\psi^m - \bar\theta\psi^m)\partial^2_{lm}g_{jk} \\ &= g_{jk}(x) + (\theta\bar\psi^l - \bar\theta\psi^l + \theta\bar\theta F^l)\partial_l g_{jk}(x) - \frac{1}{2}\bar\theta\theta(\bar\psi^l\psi^m - \psi^l\bar\psi^m)\partial^2_{lm}g_{jk} \\ &= g_{jk}(x) + (\theta\bar\psi^l - \bar\theta\psi^l + \theta\bar\theta F^l)\partial_l g_{jk}(x) + \theta\bar\theta\bar\psi^l\psi^m\partial^2_{lm}g_{jk}. \end{aligned} \tag{2.9}$$

The superspace integration picks up only $\theta\bar\theta$ terms, so we can drop the other terms

$$\begin{aligned} g_{jk}(\hat x)\mathfrak{D}\hat x^j\bar{\mathfrak{D}}\hat x^k &= g_{jk}(x)\left(\bar\psi^j i\theta\bar\theta\dot\psi^k - \bar\theta(F^j - i\dot x^j)\theta(F^k + i\dot x^k) + i\bar\theta\theta\dot{\bar\psi}^j\psi^k\right) \\ &+ \partial_l g_{jk}(x)\left(\theta\bar\psi^l\bar\theta(F^j - i\dot x^j)(-\psi^k) + \bar\theta\psi^l\bar\psi^j\theta(F^k + i\dot x^k) - \theta\bar\theta F^l\bar\psi^j\psi^k\right) \\ &+ \theta\bar\theta\bar\psi^l\psi^m\partial^2_{lm}g_{jk}\bar\psi^j(-\psi^k) + \mathcal{O}(\theta,\bar\theta) \\ &= \theta\bar\theta g_{jk}(x)\left(i\bar\psi^j\dot\psi^k - i\dot{\bar\psi}^j\psi^k + \dot x^j\dot x^k + F^jF^k\right) \\ &+ \theta\bar\theta\partial_l g_{jk}(x)\left(\bar\psi^l(F^j - i\dot x^j)\psi^k - \psi^l\bar\psi^j(F^k + i\dot x^k) - F^l\bar\psi^j\psi^k\right) \\ &- \theta\bar\theta\bar\psi^l\psi^m\partial^2_{lm}g_{jk}\bar\psi^j\psi^k + \mathcal{O}(\theta,\bar\theta) \end{aligned} \tag{2.10}$$

After the integration

$$\begin{aligned} 2L = g_{jk}(x)\left(i\bar\psi^j\dot\psi^k - i\dot{\bar\psi}^j\psi^k + \dot x^j\dot x^k + F^jF^k\right) \\ + \partial_l g_{jk}(x)\left(\bar\psi^l(F^j - i\dot x^j)\psi^k - \psi^l\bar\psi^j(F^k + i\dot x^k) - F^l\bar\psi^j\psi^k\right) \\ - \bar\psi^l\psi^m\partial^2_{lm}g_{jk}\bar\psi^j\psi^k \end{aligned} \tag{2.11}$$

The last line in the expression above contains second derivatives of metric, so we can use the Riemann tensor formula

$$R_{abcd} = \frac{1}{2}(\partial^2_{bc}g_{ad} + \partial^2_{ad}g_{bc} - \partial^2_{bd}g_{ac} - \partial^2_{ac}g_{bd}) + g_{ij}(\Gamma^i_{bc}\Gamma^j_{ad} - \Gamma^i_{bd}\Gamma^j_{ac}) \tag{2.12}$$

to simplify

$$\begin{aligned} &-\bar{\psi}^l\psi^m\partial^2_{lm}g_{jk}\bar{\psi}^j\psi^k = -\psi^m\bar{\psi}^l\psi^k\bar{\psi}^j\partial^2_{lm}g_{jk} = \psi^m\bar{\psi}^j\psi^k\bar{\psi}^l\partial^2_{lm}g_{jk} \\ &= \frac{1}{2}\psi^a\bar{\psi}^b\psi^c\bar{\psi}^d\left(\partial^2_{bc}g_{da} - \partial^2_{ab}g_{dc}\right) = \psi^a\bar{\psi}^b\psi^c\bar{\psi}^d(R_{abcd} - g_{ij}\Gamma^i_{bc}\Gamma^j_{ad}) \end{aligned} \tag{2.13}$$

The second line contain derivatives of metric, which should become the Levi-Cevita connection coefficients

$$\begin{aligned} &\partial_l g_{jk}\left(\bar{\psi}^l(F^j - i\dot{x}^j)\psi^k - \psi^l\bar{\psi}^j(F^k + i\dot{x}^k) - F^l\bar{\psi}^j\psi^k\right) \\ &= F^l\psi^k\bar{\psi}^j\left(-\partial_j g_{lk} - \partial_k g_{jl} + \partial_l g_{jk}\right) + i\dot{x}^l\bar{\psi}^j\psi^k(-\partial_j g_{lk} + \partial_k g_{jl}) \\ &= -F^l\psi^k\bar{\psi}^j g_{ml}2\Gamma^m_{jk} + i\dot{x}^l\bar{\psi}^j\psi^k(\partial_l g_{jk} + \partial_k g_{jl} - \partial_j g_{lk} + \partial_k g_{jl} - \partial_l g_{jk} - \partial_k g_{jl}) \\ &= -2F^l\psi^k\bar{\psi}^j g_{ml}\Gamma^m_{jk} + ig_{jk}\bar{\psi}^j\Gamma^k_{lm}\dot{x}^l\psi^m - ig_{jk}\Gamma^j_{lm}\dot{x}^l\bar{\psi}^m\psi^k \end{aligned} \tag{2.14}$$

Let us put all three expression together and further simplify the Lagrangian

$$\begin{aligned} 2L =& g_{jk}\left(i\bar{\psi}^j\dot{\psi}^k - i\dot{\bar{\psi}}^j\psi^k + \dot{x}^j\dot{x}^k + F^jF^k\right) + ig_{jk}\bar{\psi}^j\Gamma^k_{lm}\dot{x}^l\psi^m - ig_{jk}\Gamma^j_{lm}\dot{x}^l\bar{\psi}^m\psi^k \\ &- 2F^l\psi^k\bar{\psi}^j g_{ml}\Gamma^m_{kj} + \psi^a\bar{\psi}^b\psi^c\bar{\psi}^d(R_{abcd} - g_{ij}\Gamma^i_{bc}\Gamma^j_{ad}) \\ &= g_{jk}\left(i\bar{\psi}^j\dot{\psi}^k - i\dot{\bar{\psi}}^j\psi^k + \dot{x}^j\dot{x}^k + i\bar{\psi}^j\Gamma^k_{lm}\dot{x}^l\psi^m - i\Gamma^j_{lm}\dot{x}^l\bar{\psi}^m\psi^k\right) \\ &+ g_{lm}(F^l - \psi^k\bar{\psi}^j\Gamma^l_{kj})(F^m - \psi^k\bar{\psi}^j\Gamma^m_{kj}) + \psi^a\bar{\psi}^b\psi^c\bar{\psi}^d R_{abcd} \\ &= g_{jk}\left(i\bar{\psi}^j\nabla_t\psi^k - i\nabla_t\bar{\psi}^j\psi^k + \dot{x}^j\dot{x}^k\right) \\ &+ g_{lm}(F^l - \psi^k\bar{\psi}^j\Gamma^l_{kj})(F^m - \psi^k\bar{\psi}^j\Gamma^m_{kj}) + \psi^a\bar{\psi}^b\psi^c\bar{\psi}^d R_{abcd}. \end{aligned} \tag{2.15}$$

- (*5 Points*) Describe the SUSY transformations for $x, F, \psi, \bar{\psi}$.

Solution: The SUSY transformations for each individual component are same as the ones we discussed on lectures, so that

$$\begin{aligned}
\delta_\epsilon x^j &= \epsilon\bar{\psi}^j - \bar{\epsilon}\psi^j \\
\delta_\epsilon \psi^j &= \epsilon(i\dot{x}^j + F^j) \\
\delta_\epsilon \bar{\psi}^j &= \bar{\epsilon}(-i\dot{x}^j + F^j) \\
\delta_\epsilon F^j &= -i\epsilon\dot{\bar{\psi}}^j - i\bar{\epsilon}\dot{\psi}^j
\end{aligned} \tag{2.16}$$

- (*5 Points*) "Integrate out" the auxiliary fields F^i, i.e. solve the equations of motion for F^i as functional of the remaining fields and evaluate the action on this solution.

$$L[x,\psi,\bar{\psi}] = \frac{1}{2}g_{jk}\left(i\bar{\psi}^j\nabla_t\psi^k - i\nabla_t\bar{\psi}^j\psi^k + \dot{x}^j\dot{x}^k\right) + \frac{1}{2}\psi^a\bar{\psi}^b\psi^c\bar{\psi}^d R_{abcd} \tag{2.17}$$

Solution: The equations of motion for field F^l are of the form

$$\frac{\delta L}{\delta F^l} = g_{lm}\left(F^m - \psi^k\bar{\psi}^j\Gamma^m_{kj}\right) = 0, \tag{2.18}$$

with solution

$$F^m = \psi^k\bar{\psi}^j\Gamma^m_{kj}. \tag{2.19}$$

The Lagrangian evaluated on classical solution

$$L[x,\psi,\bar{\psi}] = \frac{1}{2}g_{jk}\left(i\bar{\psi}^j\nabla_t\psi^k - i\nabla_t\bar{\psi}^j\psi^k + \dot{x}^j\dot{x}^k\right) + \frac{1}{2}\psi^a\bar{\psi}^b\psi^c\bar{\psi}^d R_{abcd}. \tag{2.20}$$

- (*10 Points*) Describe the SUSY transformations in $x, \psi, \bar{\psi}$ variables.

Solution: We can take the SUSY transformations from previous section and substitute the classical solution for F to arrive into

$$\begin{aligned}
\delta_\epsilon x^j &= \epsilon\bar{\psi}^j - \bar{\epsilon}\psi^j \\
\delta_\epsilon \psi^j &= \epsilon\left(i\dot{x}^j + \psi^k\bar{\psi}^l\Gamma^j_{kl}\right) \\
\delta_\epsilon \bar{\psi}^j &= \bar{\epsilon}\left(-i\dot{x}^j + \psi^k\bar{\psi}^l\Gamma^j_{kl}\right).
\end{aligned} \tag{2.21}$$