Supersymmetric Quantum Mechanics and Morse Theory

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Solution to Homework 2: Differential forms

1 Differential forms

Let us consider Euclidean space $\mathbb{R}^3$ with coordinates x^1, x^2, x^3 endowed with Riemann metric metric $g(\partial_i, \partial_j) = g_{ij} = \delta_{ij}$. For a vector field

$$\vec{v} = v^i(x)\partial_i \in Vect(\mathbb{R}^3) \tag{1.1}$$

we can associate two differential forms

$$\omega^2_{\vec{v}} = \frac{1}{2}\sum_{ijk} \epsilon_{ijk} v^i dx^j \wedge dx^k \in \Omega^2(\mathbb{R}^3), \quad \omega^1_{\vec{v}} = \sum_{ij} g_{ij} v^j dx^i \in \Omega^1(\mathbb{R}^3). \tag{1.2}$$

Show that the following relations are identities:

1. (*5 points*) Hodge dual

$$\omega^2_{\vec{v}} = *\omega^1_{\vec{v}}, \quad \omega^1_{\vec{v}} = *\omega^2_{\vec{v}}. \tag{1.3}$$

Solution: Using the definition of Hodge star

$$\begin{aligned} *\omega^1_{\vec{v}} &= \frac{1}{2}\sum_{ij} (*\omega^1_{\vec{v}})_{ij} dx^i \wedge dx^j = \frac{1}{2}\sum_{ijklm} \epsilon_{ijk} g^{kl} g_{lm} v^m dx^i \wedge dx^j \\ &= \frac{1}{2}\sum_{ijk} \epsilon_{ijk} v^k dx^i \wedge dx^j = \omega^2_{\vec{v}}. \end{aligned} \tag{1.4}$$

Similarly

$$\begin{aligned} *\omega^2_{\vec{v}} &= \sum_{i} (*\omega^2_{\vec{v}})_i dx^i = \sum_{ii'jj'kk'} \frac{1}{2} \epsilon_{ijk} g^{jj'} g^{kk'} \epsilon_{i'j'k'} v^{i'} dx^i \\ &= \sum_{ii'} \delta_{ii'} v^{i'} dx^i = \omega^1_{\vec{v}}. \end{aligned} \tag{1.5}$$

Both formulas are consistent with

$$*^2 = (-1)^{k(n-k)} = 1 \;:\; \Omega^1(\mathbb{R}^3) \to \Omega^1(\mathbb{R}^3). \tag{1.6}$$

2. (*5 points*) Gradient formula

$$\omega^1_{\vec{\nabla} f} = df. \tag{1.7}$$

Solution: The gradient vector field in x^1, x^2, x^3 coordinates is

$$\vec{\nabla} f = \sum_{ij} (g^{ij} \partial_j f) \partial_i, \tag{1.8}$$

while the corresponding 1-form

$$\omega^1_{\vec{\nabla} f} = \sum_{ijk} g_{ij} (g^{jk} \partial_k f) dx^i = \sum_i \partial_i f dx^i = df. \tag{1.9}$$

3. (*5 points*) Curl formula

$$d\omega^1_{\vec{v}} = \omega^2_{\vec{\nabla} \times \vec{v}}. \tag{1.10}$$

Solution: The curl of the vector field in x^1, x^2, x^3 coordinates is

$$\vec{\nabla} \times \vec{v} = \sum_{ijk} (\epsilon^{ijk} \partial_i v_j) \partial_k, \tag{1.11}$$

where we introduced notation

$$v_i \equiv \sum_j g_{ij} v^j = \sum_j \delta_{ij} v^j = v^i. \tag{1.12}$$

The sides of the relation

$$d\omega^1_{\vec{v}} = d \sum_{ij} g_{ij} v^j dx^i = d \sum_j v_j dx^j = \sum_{ij} \partial_i v_j dx^i \wedge dx^j \tag{1.13}$$

and

$$\begin{aligned} \omega^2_{\vec{\nabla} \times \vec{v}} &= \frac{1}{2} \sum_{ijj'kk'} \epsilon_{ijk} (\epsilon^{ij'k'} \partial_{j'} v_{k'}) dx^j \wedge dx^k = \frac{1}{2} \sum_{ijj'kk'} (\delta^{j'}_j \delta^{k'}_k - \delta^{k'}_j \delta^{j'}_k) \partial_{j'} v_{k'} dx^j \wedge dx^k \\ &= \frac{1}{2} \sum_{jk} (\partial_j v_k - \partial_k v_j) dx^j \wedge dx^k = \sum_{ij} \partial_i v_j dx^i \wedge dx^j = d\omega^1_{\vec{v}}. \end{aligned} \tag{1.14}$$

4. (*5 points*) Divergence

$$* \, d\omega^2_{\vec{v}} = \vec{\nabla} \cdot \vec{v}. \tag{1.15}$$

Solution: The divergence of vector field in x^1, x^2, x^3 coordinates is of the form

$$\vec{\nabla} \cdot \vec{v} = \sum_i \partial_i v^i, \tag{1.16}$$

while the relation in question becomes

$$\begin{aligned} *d\omega^2_{\vec{v}} &= \frac{1}{2} * d \sum_{ijk} \epsilon_{ijk} v^i dx^j \wedge dx^k = \frac{1}{2} * \sum_{ijkl} \epsilon_{ijk} \partial_l v^i \, dx^l \wedge dx^j \wedge dx^k \\ &= \frac{1}{3!} * \sum_{ijkl} 3\epsilon_{ijk} \partial_l v^i \, dx^l \wedge dx^j \wedge dx^k = \sum_{ijkll'j'k'} \frac{1}{3!} 3\epsilon_{ijk} \partial_l v^i g^{ll'} g^{jj'} g^{kk'} \epsilon_{l'j'k'} \\ &= \sum_{ill'} \delta_i^{l'} \partial_l v^i g^{ll'} = \sum_i \partial_i v^i = \vec{\nabla} \cdot \vec{v}. \end{aligned} \tag{1.17}$$

5. (*10 points*) Maxwell equations: Let us consider 4d Minkowski space $\mathbb{R}^{3,1} = \mathbb{R}^3 \times \mathbb{R}$ with coordinates $t = x^0, x^1, x^2, x^3$ and metric

$$g(\partial_i, \partial_j) = g_{ij} = \delta_{ij}, \quad g(\partial_i, \partial_0) = 0, \quad g(\partial_0, \partial_0) = -1. \tag{1.18}$$

For electromagnetic field in terms of electric field 3-vector $E^i(x)$ and magnetic field 3-vector $B^i(x)$ we can associate a 2-form

$$F = g_{ij} E^i dx^0 \wedge dx^j + \epsilon_{ijk} B^i dx^j \wedge dx^k = dx^0 \wedge \omega^1_{\vec{E}} + \omega^2_{\vec{B}}. \tag{1.19}$$

Show that Maxwell equations in vacuum can be written in the form

$$dF = d * F = 0. \tag{1.20}$$

with $*$ being Hodge dual in four-dimensional space. We assumed the choice of units where the speed of light $c = 1$.

Solution: Maxwell equations in 4d Minkowski space with coordinates x^0, x^1, x^2, x^3

$$\begin{aligned} \vec{\nabla} \cdot \vec{E} &= \partial_i E^i = 0, \\ \vec{\nabla} \cdot \vec{B} &= \partial_i B^i = 0, \\ \vec{\nabla} \times \vec{E} &= \epsilon^{ijk} \partial_j E_i \partial_k = -\partial_t \vec{B} = -\partial_0 B^k \partial_k, \\ \vec{\nabla} \times \vec{B} &= \epsilon^{ijk} \partial_j B_i \partial_k = \partial_t \vec{E} = \partial_0 E^k \partial_k. \end{aligned} \tag{1.21}$$

Let us rewrite the equations in components

$$\begin{aligned} 0 = dF = d\left(-\omega^1_{\vec{E}} \wedge dx^0 + \omega^2_{\vec{B}}\right) &= -d\omega^1_{\vec{E}} \wedge dx^0 + d\omega^2_{\vec{B}} \\ &= -\omega^2_{\vec{\nabla}\times\vec{E}} \wedge dx^0 + dx^0 \wedge \partial_0 \omega^2_{\vec{B}} + (\vec{\nabla}\cdot\vec{B}) \cdot \mathrm{vol}_{\mathbb{R}^3} \\ &= (-\omega^2_{\vec{\nabla}\times\vec{E}} - \omega^2_{\partial_t\vec{B}}) \wedge dx^0 + (\vec{\nabla}\cdot\vec{B}) \cdot \mathrm{vol}_{\mathbb{R}^3} \\ &= -\omega^2_{\vec{\nabla}\times\vec{E}+\partial_t\vec{B}} \wedge dx^0 + (\vec{\nabla}\cdot\vec{B}) \cdot \mathrm{vol}_{\mathbb{R}^3}. \end{aligned} \tag{1.22}$$

Among 3-forms the volume form $\mathrm{vol}_{\mathbb{R}^3}$ and the $\omega^2_{\vec{v}} \wedge dx^0$ are linearly independent and

$$\omega^2_{\vec{v}} = 0 \Leftrightarrow \vec{v} = 0, \tag{1.23}$$

so we arrive into

$$dF = 0 \Longleftrightarrow \vec{\nabla} \times \vec{E} + \partial_t \vec{B} = 0, \quad \vec{\nabla} \cdot \vec{B} = 0. \tag{1.24}$$

Let u rewrite $*F = 0$ in components

$$\begin{aligned} *F = *\left(-\omega^1_{\vec{E}} \wedge dx^0 + \omega^2_{\vec{B}}\right) &= \frac{1}{2}\sum_{ab} (*F)_{ab}\, dx^a dx^b \\ &= \frac{1}{2}\sum_{abcdc'd'} \frac{1}{2}\epsilon_{abcd} g^{cc'} g^{dd'} F_{c'd'} dx^a dx^b \\ &= \frac{1}{4}\sum_{abcd} \epsilon_{abcd} dx^a dx^b \left(\sum_i g^{c0} g^{di} F_{0i} + \sum_i g^{ci} g^{d0} F_{i0} + \sum_{ij} g^{ci} g^{dj} F_{ij}\right) \\ &= \frac{1}{4}\sum_{abcd} \epsilon_{abcd} dx^a dx^b \left(2\sum_i g^{c0} g^{di} g_{ij} E^j + \sum_{ij} g^{ci} g^{dj} \epsilon_{ijk} B^k\right) \\ &= \frac{1}{2}\sum_{abiji'} \epsilon_{ab0i'} dx^a dx^b g^{00} \delta^{i'i} \delta_{ij} E^j + \frac{1}{4}\sum_{abiji'j'} \epsilon_{abi'j'} dx^a dx^b \delta^{i'i} \delta^{j'j} \epsilon_{ijk} B^k \\ &= -\frac{1}{2}\sum_{lmj} \epsilon_{0lmj} dx^l dx^m E^j + \frac{1}{2}\sum_{liji'j'} \epsilon_{0li'j'} dx^0 dx^l \delta^{i'i} \delta^{j'j} \epsilon_{ijk} B^k \\ &= -\frac{1}{2}\sum_{lmj} \epsilon_{lmj} dx^l dx^m E^j + \sum_{lk} dx^0 dx^l \delta_{lk} B^k = -\omega^2_{\vec{E}} + dx^0 \wedge \omega^1_{\vec{B}}. \end{aligned} \tag{1.25}$$

We used the $a, b, c, d...$ indices which run over $0, 1, 2, 3$, while keeping i, j, k, k, l to run over $1, 2, 3$. The inverse metric

$$g^{ij} = \delta^{ij}, \quad g^{i0} = g^{0i} = 0, \quad g^{00} = -1 \tag{1.26}$$

and antisymmetric tensor

$$\epsilon_{0ijk} = \epsilon_{ijk}. \tag{1.27}$$

The second equation in components

$$\begin{aligned} 0 = d * F = d\left(-\omega^2_{\vec{E}} + dx^0 \wedge \omega^1_{\vec{B}}\right) &= -d\omega^1_{\vec{B}} \wedge dx^0 - d\omega^2_{\vec{E}} \\ &= -\omega^2_{\vec{\nabla}\times\vec{B}} \wedge dx^0 - dx^0 \wedge \partial_0 \omega^2_{\vec{E}} - (\vec{\nabla}\cdot\vec{E}) \cdot \mathrm{vol}_{\mathbb{R}^3} \\ &= (-\omega^2_{\vec{\nabla}\times\vec{B}} + \omega^2_{\partial_t \vec{E}}) \wedge dx^0 - (\vec{\nabla}\cdot\vec{E}) \cdot \mathrm{vol}_{\mathbb{R}^3} \\ &= -\omega^2_{\vec{\nabla}\times\vec{B} - \partial_t \vec{E}} \wedge dx^0 + (\vec{\nabla}\cdot\vec{E}) \cdot \mathrm{vol}_{\mathbb{R}^3}. \end{aligned} \tag{1.28}$$

Among 3-forms the volume form $\mathrm{vol}_{\mathbb{R}^3}$ and the $\omega^2_{\vec{v}} \wedge dx^0$ are linearly independent so

$$d * F = 0 \Longleftrightarrow \vec{\nabla} \times \vec{B} - \partial_t \vec{E} = 0, \quad \vec{\nabla} \cdot \vec{E} = 0. \tag{1.29}$$

2 Differential forms as functions on supermanifold

In class we established a relation between differential forms and functions on supergaminold

$$F : \Omega^*(\mathbb{R}^n) \to C^\infty(\Pi T\mathbb{R}^n) : \omega \to F_\omega. \tag{2.1}$$

Let us choose $x^1, ..., x^n$ and $\psi^1, .., \psi^n$ as even and odd coordinates on $\Pi T\mathbb{R}^n$. The differential p-form

$$\omega = \sum_{i_1 \dots i_p} \frac{1}{p!} \omega_{i_1 \dots i_p} dx^{i_1} ... dx^{i_p} \in \Omega^p(\mathbb{R}^n) \tag{2.2}$$

maps to a function

$$F_\omega(x, \psi) = \sum_{i_1 \dots i_p} \frac{1}{p!} \omega_{i_1 \dots i_p} \psi^{i_1} ... \psi^{i_p} \in C^\infty(\Pi T\mathbb{R}^n). \tag{2.3}$$

1. (*5 points*) Show that the differential d and vector field substitution ι_v are first order differential operators on $C^\infty(\Pi T\mathbb{R}^n)$

$$F_{d\omega} = D_d F_\omega, \quad F_{\iota_v \omega} = D_{\iota_v} F_\omega \tag{2.4}$$

and evaluate D_d and D_{ι_v} in (x, ψ) coordinates.

Solution: For a differential form

$$\omega = \sum_{i_1...i_p} \frac{1}{p!} \omega_{i_1...i_p} dx^{i_1}...dx^{i_p} \in \Omega^p(\mathbb{R}^n) \tag{2.5}$$

we can evaluate

$$\begin{aligned} F_{d\omega} &= \sum_{ji_1...i_p} \frac{1}{p!} \partial_j \omega_{i_1...i_p} \psi^j \psi^{i_1}...\psi^{i_p} = \sum_{j=1}^n \left(\psi^j \frac{\partial}{\partial x^j}\right) \sum_{i_1..i_p} \frac{1}{p!} \omega_{i_1...i_p} \psi^{i_1}...\psi^{i_p} \\ &= \sum_{j=1}^n \left(\psi^j \frac{\partial}{\partial x^j}\right) F_\omega = D_d F_\omega. \end{aligned} \tag{2.6}$$

and

$$\begin{aligned} F_{\iota_v\omega} = &\sum_{kii_1...i_{p-1}} \frac{1}{p!} (-1)^k v^i \omega_{i_1...i_{k-1} i i_{k+1}...i_{p-1}} \psi^{i_1}...\psi^{i_{p-1}} = \sum_{j=1}^n \left(v^j \frac{\partial}{\partial \psi^j}\right) \sum_{i_1..i_p} \frac{1}{p!} \omega_{i_1...i_p} \psi^{i_1}...\psi^{i_p} \\ &= \sum_{j=1}^n \left(v^j \frac{\partial}{\partial \psi^j}\right) F_\omega = D_{\iota_v} F_\omega. \end{aligned} \tag{2.7}$$

Summarizing the answers

$$D_d = \sum_{j=1}^n \psi^j \frac{\partial}{\partial x^j}, \quad D_{\iota_v} = \sum_{j=1}^n v^j \frac{\partial}{\partial \psi^j}. \tag{2.8}$$

2. (*5 points*) Using the differential operator representation verify the properties of external derivative d and ι_v

- Differential

$$d^2 = 0. \tag{2.9}$$

Solution: By definition

$$F_{d^2\omega} = D_d F_{d\omega} = D_d^2 F_\omega \tag{2.10}$$

while the differential operator

$$D_d^2 = \left(\sum_{j=1}^n \psi^j \frac{\partial}{\partial x^j}\right)^2 = \sum_{ij} \psi^i \psi^j \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j} = -\sum_{ij} \psi^j \psi^i \frac{\partial}{\partial x^j} \frac{\partial}{\partial x^i} = 0, \tag{2.11}$$

where we used the

$$\psi^i\psi^j = -\psi^j\psi^i, \quad \frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j} = \frac{\partial}{\partial x^j}\frac{\partial}{\partial x^i}. \tag{2.12}$$

- Antysymmetry

$$\iota_v\iota_w\omega = -\iota_w\iota_v\omega. \tag{2.13}$$

Solution: By definition

$$F_{\iota_v\iota_w\omega} = D_{\iota_v}F_{\iota_w\omega} = D_{\iota_v}D_{\iota_w}F_\omega \tag{2.14}$$

while the differential operators

$$\begin{aligned} D_{\iota_v}D_{\iota_w} &= \left(\sum_{j=1}^n v^j\frac{\partial}{\partial\psi^j}\right)\left(\sum_{i=1}^n w^i\frac{\partial}{\partial\psi^i}\right) \\ &= \sum_{ij} v^j w^i \frac{\partial}{\partial\psi^j}\frac{\partial}{\partial\psi^i} = -\sum_{ij} v^j w^i \frac{\partial}{\partial\psi^i}\frac{\partial}{\partial\psi^j} \\ &= \left(\sum_{i=1}^n w^i\frac{\partial}{\partial\psi^i}\right)\left(\sum_{j=1}^n v^j\frac{\partial}{\partial\psi^j}\right) = -D_{\iota_w}D_{\iota_v}, \end{aligned} \tag{2.15}$$

where we used

$$\frac{\partial}{\partial\psi^j}\frac{\partial}{\partial\psi^i} = -\frac{\partial}{\partial\psi^i}\frac{\partial}{\partial\psi^j}. \tag{2.16}$$

- Derivation: Let $\omega \in \Omega^k(\mathbb{R}^n)$ and $\mu \in \Omega^m(\mathbb{R}^n)$

$$d(\omega\wedge\mu) = d\omega\wedge\mu + (-1)^k\omega\wedge d\mu, \tag{2.17}$$

Solution: By definition

$$\begin{aligned} F_{d(\omega\wedge\mu)} &= D_dF_{\omega\wedge\mu} = D_d(F_\omega\cdot F_\mu) \\ &= D_dF_\omega\cdot F_\mu + (-1)^{|F_\omega|}F_\omega\cdot D_dF_\mu \\ &= F_{d\omega}\cdot F_\mu + (-1)^kF_\omega\cdot F_{d\mu} = F_{d\omega\wedge\mu} + (-1)^kF_{\omega\wedge d\mu} \\ &= F_{d\omega\wedge\mu+(-1)^k\omega\wedge d\mu} \end{aligned} \tag{2.18}$$

where we used Leibnitz

$$
\begin{aligned}
D_d(F_\omega \cdot F_\mu) &= \left(\sum \psi^i \partial_i\right)(F_\omega \cdot F_\mu) = \sum \psi^i(\partial_i F_\omega \cdot F_\mu + F_\omega \cdot \partial_i F_\mu) \\
&= \left(\sum \psi^i \partial_i F_\omega\right) \cdot F_\mu + (-1)^{|F_\omega|\cdot 1} F_\omega \cdot \left(\sum \psi^i \partial_i F_\omega\right) F_\mu \\
&= D_d F_\omega \cdot F_\mu + (-1)^{|F_\omega|} F_\omega \cdot D_d F_\mu,
\end{aligned}
\tag{2.19}
$$

graded commutators

$$\psi^i F_\omega = (-1)^{|F_\omega|\cdot|\psi|} F_\omega \psi^i = (-1)^{|F_\omega|\cdot 1} F_\omega \psi^i \tag{2.20}$$

and grading relations

$$|F_\omega| = |\omega| = k. \tag{2.21}$$

- $$\iota_v(\omega \wedge \mu) = \iota_v \omega \wedge \mu + (-1)^k \omega \wedge \iota_v \mu. \tag{2.22}$$

$$
\begin{aligned}
F_{d(\omega\wedge\mu)} &= D_d F_{\omega\wedge\mu} = D_d(F_\omega \cdot F_\mu) \\
&= D_d F_\omega \cdot F_\mu + (-1)^{|F_\omega|} F_\omega \cdot D_d F_\mu \\
&= F_{d\omega} \cdot F_\mu + (-1)^k F_\omega \cdot F_{d\mu} = F_{d\omega\wedge\mu} + (-1)^k F_{\omega\wedge d\mu} \\
&= F_{d\omega\wedge\mu + (-1)^k \omega\wedge d\mu}
\end{aligned}
\tag{2.23}
$$

where we used graded Leibnitz for Grassmann-odd variables

$$\frac{\partial}{\partial \psi^i}(F_\omega \cdot F_\mu) = \frac{\partial}{\partial \psi^i} F_\omega \cdot F_\mu + (-1)^{|F_\omega|} F_\omega \cdot \frac{\partial}{\partial \psi^i} F_\mu \tag{2.24}$$

and grading relations

$$|F_\omega| = |\omega| = k. \tag{2.25}$$

3. (*5 points*) We can use Cartan formula to describe Lie derivative via

$$\mathcal{L}_v = \{d, \iota_v\} \tag{2.26}$$

In terms of the functions on $C^\infty(\Pi T\mathbb{R}^n)$ the D_{L_v} is the graded commutator of two first order differential operators, so it is also a first order differential operator

$$D_{L_v} = \{D_d, D_{\iota_v}\} \tag{2.27}$$

Evaluate the D_{L_v} in coordinates x, ψ.

Solution: Using results from previous parts

$$
\begin{aligned}
D_{L_v} &= \{D_d, D_{\iota_v}\} = \Big\{\sum \psi^i \partial_i, \sum v^j \partial_{\psi^j}\Big\} = \sum_{ij} \psi^i \partial_i \cdot v^j \partial_{\psi^j} + \sum_{ij} v^j \partial_{\psi^j} \cdot \psi^i \partial_i \\
&= \sum_{ij} \left(\partial_i v^j \psi^i \partial_{\psi^j} + v^j \psi^i \partial_i \partial_{\psi^j} + v^j \delta^i_j \partial_i - v^j \psi^i \partial_i \partial_{\psi^j}\right) \\
&= \sum_i v^i \partial_i + \sum_{ij} \partial_i v^j \psi^i \partial_{\psi^j},
\end{aligned}
\tag{2.28}
$$

where we used graded commutators between derivatives and multiplication by a function

$$
\partial_i \cdot v^j = \partial_i v^j + v^j \partial_i, \quad \{\partial_{\psi^j}, \psi^i\} = \delta^i_j. \tag{2.29}
$$

4. (*5 points*) Using differential operator representation verify the properties of Lie derivative

- Derivation property: Let $\omega \in \Omega^k(\mathbb{R}^n)$ and $\mu \in \Omega^m(\mathbb{R}^n)$

$$
\mathcal{L}_v(\omega \wedge \mu) = \mathcal{L}_v \omega \wedge \mu + \omega \wedge \mathcal{L}_v \mu.
$$

Solution: By definition

$$
\begin{aligned}
F_{\mathcal{L}_v(\omega\wedge\mu)} &= D_{L_v} F_{\omega\wedge\mu} = D_{L_v}(F_\omega \cdot F_\mu) = (D_{\iota_v} D_d + D_d D_{\iota_v})(F_\omega \cdot F_\mu) \\
&= D_{\iota_v}(D_d F_\omega \cdot F_\mu + (-1)^{|F_\omega|} F_\omega \cdot D_d F_\mu) + D_d(D_{\iota_v} F_\omega \cdot F_\mu + (-1)^{|F_\omega|} F_\omega \cdot D_{\iota_v} F_\mu) \\
&= D_{\iota_v} D_d F_\omega \cdot F_\mu + (-1)^{|D_d F_\omega|} D_d F_\omega \cdot D_{\iota_v} F_\mu + (-1)^{|F_\omega|} D_{\iota_v} F_\omega \cdot D_d F_\mu \\
&+ (-1)^{2|F_\omega|} F_\omega \cdot D_{\iota_v} D_d F_\mu) + D_d D_{\iota_v} F_\omega \cdot F_\mu + (-1)^{|D_{\iota_v} F_\omega|} D_{\iota_v} F_\omega \cdot D_d F_\mu \\
&+ (-1)^{|F_\omega|} D_d F_\omega \cdot D_{\iota_v} F_\mu + (-1)^{2|F_\omega|} F_\omega \cdot D_d D_{\iota_v} F_\mu) \\
&= (D_{\iota_v} D_d + D_d D_{\iota_v}) F_\omega \cdot F_\mu + F_\omega \cdot (D_{\iota_v} D_d + D_d D_{\iota_v}) F_\mu \\
&= D_{L_v} F_\omega \cdot F_\mu + F_\omega \cdot D_{L_v} F_\mu = F_{\mathcal{L}_v \omega \wedge \mu + \omega \wedge \mathcal{L}_v \mu},
\end{aligned}
\tag{2.30}
$$

where we used graded Leibnitz for D_d and D_{ι_v}

$$
D_d(F_\omega \cdot F_\mu) = D_d F_\omega \cdot F_\mu + (-1)^{|F_\omega|} F_\omega \cdot D_d F_\mu \tag{2.31}
$$

and grading relations

$$|D_d F_\omega| = |F_\omega| + 1, \quad |D_{\iota_v} F_\omega| = |F_\omega| + 1. \tag{2.32}$$

- Lie algebra :

$$(\mathcal{L}_v \mathcal{L}_w - \mathcal{L}_w \mathcal{L}_v)\omega = \mathcal{L}_{[v,w]}\omega.$$

Solution: By definition

$$\begin{aligned} F_{(\mathcal{L}_v\mathcal{L}_w - \mathcal{L}_w\mathcal{L}_v)\omega} &= D_{L_v} F_{\mathcal{L}_w\omega} - D_{L_w} F_{\mathcal{L}_v\omega} = D_{L_v} D_{L_w} F_\omega - D_{L_w} D_{L_v} F_\omega \\ &= (D_{L_v} D_{L_w} - D_{L_w} D_{L_v}) F_\omega = [D_{L_v}, D_{L_w}] F_\omega \end{aligned} \tag{2.33}$$

The commutator of vector fields

$$\begin{aligned} [D_{L_v}, D_{L_w}] &= \left[\sum_i v^i \partial_i + \sum_{ij} \partial_i v^j \psi^i \partial_{\psi^j}, \sum_k w^k \partial_k + \sum_{kl} \partial_k w^l \psi^k \partial_{\psi^l} \right] \\ &= \left[\sum_i v^i \partial_i, \sum_k w^k \partial_k \right] + \left[\sum_i v^i \partial_i, \sum_{kl} \partial_k w^l \psi^k \partial_{\psi^l} \right] \\ &+ \left[\sum_{ij} \partial_i v^j \psi^i \partial_{\psi^j}, \sum_k w^k \partial_k \right] + \left[\sum_{ij} \partial_i v^j \psi^i \partial_{\psi^j}, \sum_{kl} \partial_k w^l \psi^k \partial_{\psi^l} \right] \\ &= \sum_i [v,w]^i \partial_i + \sum_{ikl} v^i \partial_i \partial_k w^l \psi^k \partial_{\psi^l} - \sum_{ijk} w^k \partial_k \partial_i v^j \psi^i \partial_{\psi^j} \\ &+ \sum_{ijkl} \partial_i v^j \partial_k w^l (\delta^k_j \ \psi^i \partial_{\psi^l} - \delta^i_l \ \psi^k \partial_{\psi^j}) \\ &= \sum_i [v,w]^i \partial_i + \sum_{ijk} (v^k \partial_k \partial_i w^j - w^k \partial_k \partial_i v^j) \psi^i \partial_{\psi^j} \\ &+ \sum_{ijk} (\partial_i v^k \partial_k w^j \ \psi^i \partial_{\psi^j} - \partial_k v^j \partial_i w^k \ \psi^i \partial_{\psi^j}) \\ &= \sum_i [v,w]^i \partial_i + \sum_{ijk} (v^k \partial_k \partial_i w^j - w^k \partial_k \partial_i v^j + \partial_i v^k \partial_k w^j - \partial_k v^j \partial_i w^k) \psi^i \partial_{\psi^j} \\ &= \sum_i [v,w]^i \partial_i + \sum_{ijk} \partial_i (v^k \partial_k w^j - w^k \partial_k v^j) \psi^i \partial_{\psi^j} \\ &= \sum_i [v,w]^i \partial_i + \sum_{ij} \partial_i [v,w]^j \psi^i \partial_{\psi^j} = D_{L_{[v,w]}} \end{aligned} \tag{2.34}$$

where we used

$$[\partial_i, f] = \partial_i f, \ \ [\psi^i \partial_{\psi^j}, \psi^k \partial_{\psi^l}] = \delta^k_j \ \psi^i \partial_{\psi^l} - \delta^i_l \ \psi^k \partial_{\psi^j} \tag{2.35}$$

- Lie algebra representation:

$$(\mathcal{L}_v \iota_w - \iota_w \mathcal{L}_v)\omega = \iota_{[v,w]}\omega.$$

Solution: By definition

$$\begin{aligned} F_{(\mathcal{L}_v \iota_w - \iota_w \mathcal{L}_v)\omega} &= D_{L_v} F_{\iota_w \omega} - D_{\iota_w} F_{L_v \omega} = D_{L_v} D_{\iota_w} F_\omega - D_{\iota_w} D_{L_v} F_\omega \\ &= (D_{L_v} D_{\iota_w} - D_{\iota_w} D_{L_v}) F_\omega = [D_{L_v}, D_{\iota_w}] F_\omega \end{aligned} \tag{2.36}$$

The commutator of vector fields

$$\begin{aligned} [D_{L_v}, D_{\iota_w}] &= \left[\sum_i v^i \partial_i + \sum_{ij} \partial_i v^j \psi^i \partial_{\psi^j}, \sum_k w^k \partial_{\psi^k} \right] \\ &= \left[\sum_i v^i \partial_i, \sum_k w^k \partial_{\psi^k} \right] + \left[\sum_{ij} \partial_i v^j \psi^i \partial_{\psi^j}, \sum_k w^k \partial_{\psi^k} \right] \\ &= \sum_{ik} v^i \partial_i w^k \partial_{\psi^k} - \sum_{ijk} w^k \partial_i v^j \delta^k_i \partial_{\psi^j} = \sum_{ik} (v^i \partial_i w^k \partial_{\psi^k} - w^i \partial_i v^k \partial_{\psi^k}) \\ &= \sum_k [v,w]^k \partial_{\psi^k} = D_{\iota_{[v,w]}} \end{aligned} \tag{2.37}$$

where we used

$$[\partial_i, f] = \partial_i f, \ \ [\psi^i \partial_{\psi^j}, \partial_{\psi^k}] = -\delta^i_k \partial_{\psi^j} \tag{2.38}$$

5. (*10 points*) Let us restrict our attention to the differential forms μ, ω that go to zero at infinity sufficiently fast to the integrals

$$\langle \omega, \mu \rangle = \int_{\mathbb{R}^n} \omega \wedge *\mu \tag{2.39}$$

are finite and we can define the Hodge-dual operators. Verify the Hodge dual of a differential p-form ω as a function on supermanifold $\Pi T \mathbb{R}^n$

$$(-1)^{\frac{1}{2}(n-p)(n+p-1)} F_{*\omega}(\psi, x) = \int_{\mathbb{R}^{0|n}} d^n \eta \ \ e^{\sum \delta_{ij} \psi^i \eta^j} F_\omega(x, \eta) \tag{2.40}$$

with integration over $d^n\eta$ being Grassmann integration.

Hint: Let α be a monomial expression of Grassmann variables, not depending on η^i then

$$\int_{\mathbb{R}^{0|n}} d^n\eta \;\; \alpha\eta^1...\eta^n = \alpha \tag{2.41}$$

Solution: Let us consider

$$\omega = \frac{1}{p!}\sum_{i_1..i_p} \omega_{i_1..i_p}(x) dx^{i_1}\wedge ... \wedge dx^{i_p} \in \Omega^p(\mathbb{R}^n) \tag{2.42}$$

with corresponding function on $\Pi T\mathbb{R}^n$

$$F_\omega(x,\psi) = \frac{1}{p!}\sum_{i_1..i_p} \omega_{i_1...i_p}(x)\psi^{i_1}...\psi^{i_p} \in C^\infty(\Pi T\mathbb{R}^n). \tag{2.43}$$

The integral

$$\begin{aligned}
I_\omega(x,\psi) &= \int d^n\eta \;\; e^{\sum \delta_{ij}\psi^i\eta^j} F_\omega(x,\eta) = \int d^n\eta \;\; e^{\sum \delta_{ij}\psi^i\eta^j} \frac{1}{p!}\sum_{i_1..i_p}\omega_{i_1...i_p}\eta^{i_1}...\eta^{i_p} \\
&= \frac{1}{p!}\sum_{i_1..i_p}\omega_{i_1...i_p}(x)\int d^n\eta \;\; e^{\sum \delta_{ij}\psi^i\eta^j}\eta^{i_1}...\eta^{i_p} \\
&= \frac{1}{p!}\sum_{i_1..i_p}\omega_{i_1...i_p}(x)\int d^n\eta \;\; \frac{1}{(n-p)!}\left(\sum \delta_{ij}\psi^i\eta^j\right)^{n-p}\eta^{i_1}...\eta^{i_p} \\
&= \frac{1}{p!(n-p)!}\sum_{i_1..i_p}\omega_{i_1...i_p}(x)\sum_{i_{p+1}...i_n j_{p+1}...j_n}\delta_{i_{p+1}j_{p+1}}...\delta_{i_n j_n}\int d^n\eta \;\; \eta^{i_1}...\eta^{i_p}\psi^{j_{p+1}}\eta^{i_{p+1}}...\psi^{j_n}\eta^{i_n} \\
&= (-1)^{\frac{1}{2}(n-p)(n+p-1)}\frac{1}{p!(n-p)!}\sum_{i_1..i_n j_{p+1}..j_n}\omega_{i_1...i_p}(x)\delta_{i_{p+1}j_{p+1}}...\delta_{i_n j_n}\epsilon^{i_1....i_n}\psi^{j_{p+1}}...\psi^{j_n} \\
&= (-1)^{\frac{1}{2}(n-p)(n+p-1)}\frac{1}{(n-p)!}\sum_{j_{p+1}...j_n}\left(\sum_{i_1 j_1...i_p j_p}\frac{1}{p!}\omega_{i_1...i_p}(x)\delta^{i_1 j_1}...\delta^{i_p j_p}\epsilon_{j_1....j_n}\right)\psi^{j_{p+1}}...\psi^{j_n} \\
&= (-1)^{\frac{1}{2}(n-p)(n+p-1)}\frac{1}{(n-p)!}\sum_{j_{p+1}...j_n}(*\omega)_{j_{p+1}...j_n}(x)\psi^{j_{p+1}}...\psi^{j_n} \\
&= (-1)^{\frac{1}{2}(n-p)(n+p-1)}F_{*\omega}(x,\psi).
\end{aligned} \tag{2.44}$$

The sign comes from reordering

$$\begin{aligned}
&\eta^{i_1}...\eta^{i_p}\psi^{j_{p+1}}\eta^{i_{p+1}}...\psi^{j_n}\eta^{i_n} = (-1)^p\psi^{j_{p+1}}\eta^{i_1}...\eta^{i_p}\eta^{i_{p+1}}\psi^{j_{p+2}}\eta^{i_{p+2}}...\psi^{j_n}\eta^{i_n} \\
&= (-1)^{p+(p+1)+...+(n-1)}\psi^{j_{p+1}}\psi^{j_{p+2}}....\psi^{j_n}\cdot\eta^{i_1}...\eta^{i_p}\eta^{i_{p+1}}....\eta^{i_n}
\end{aligned} \tag{2.45}$$

6. (*10 points*) The hodge dual d^* of external derivative d is the second order differential operator D_{d^*} on $C^\infty(\Pi T\mathbb{R}^n)$. Use the Hodge star formula (2.40) to evaluate this operator in x^i, ψ^i coordinates

$$F_{d^*\omega} = D_{d^*}F_\omega. \tag{2.46}$$

Solution: Using the definitions for $\omega \in \Omega^p(\mathbb{R}^n)$

$$\begin{aligned}
D_{d^*}F_\omega(\psi, x) &= F_{d^*\omega}(\psi, x) = (-1)^{\cdots}F_{*d*\omega}(\psi, x) \\
&= (-1)^{\cdots}\int d^n\xi \;\; e^{\sum\delta_{ij}\psi^i\xi^j}\left(\sum\xi^i\partial_i\right)\int d^n\eta \;\; e^{\sum\delta_{ij}\xi^i\eta^j}F_\omega(x,\eta) \\
&= (-1)^{\cdots}\int d^n\xi \;\; \left(\sum\delta^{ij}\partial_{\psi^j}\partial_i\right)(e^{\sum\delta_{ij}\psi^i\xi^j})\int d^n\eta \;\; e^{\sum\delta_{ij}\xi^i\eta^j}F_\omega(x,\eta) \\
&= (-1)^{\cdots}\left(\sum\delta^{ij}\partial_{\psi^j}\partial_i\right)\int d^n\xi \;\; e^{\sum\delta_{ij}\psi^i\xi^j}\int d^n\eta \;\; e^{\sum\delta_{ij}\xi^i\eta^j}F_\omega(x,\eta) \\
&= (-1)^{\cdots}\left(\sum\delta^{ij}\partial_{\psi^j}\partial_i\right)\int d^n\xi \;\; e^{\sum\delta_{ij}\psi^i\xi^j}F_{*\omega}(x,\xi) \\
&= (-1)^{\cdots}\left(\sum\delta^{ij}\partial_{\psi^j}\partial_i\right)F_{**\omega}(x,\psi) = (-1)^{\cdots}\left(\sum\delta^{ij}\partial_{\psi^j}\partial_i\right)F_\omega(x,\psi)
\end{aligned}$$

From the relation above we conclude that

$$D_{d^*} = (-1)^{\cdots}\sum_{i=1}^{n}\delta^{ij}\frac{\partial}{\partial\psi^i}\frac{\partial}{\partial x^j} \tag{2.47}$$

3 Laplacian with superpotential

Given a smooth function h on closed smooth manifold M with Riemann metric g we can construct a differential operator

$$d_h = e^{-h}de^h = d + dh\wedge : \Omega^p(M) \to \Omega^{p+1}(M). \tag{3.1}$$

1. (*5 points*) Show that d_h is the differential.

Solution In representation

$$d_h = e^{-h} d e^h \tag{3.2}$$

the differential property is manifest

$$d_h^2 = e^{-h} d e^h e^{-h} d e^h = e^{-h} d^2 e^h = 0 \tag{3.3}$$

2. (*5 points*) Show that

$$H^k(\Omega^*(M), d_h) = H^k_{DR}(M). \tag{3.4}$$

Solution: Let us introduce notations :

$$D^\bullet = \ldots \longrightarrow \Omega^k(M) \xrightarrow{d} \Omega^{k+1}(M) \xrightarrow{d} \Omega^{k+2}(M) \longrightarrow \ldots \tag{3.5}$$

for de Rham complex and

$$C^\bullet = \ldots \longrightarrow \Omega^k(M) \xrightarrow{d_h} \Omega^{k+1}(M) \xrightarrow{d_h} \Omega^{k+2}(M) \longrightarrow \ldots \tag{3.6}$$

for complex with differential d_h The collections of maps

$$f^k : D^k \to C^k : \Omega^k(M) \to \Omega^k(M) : \omega \mapsto e^h \omega \tag{3.7}$$

and

$$g^k : C^k \to D^k : \Omega^k(M) \to \Omega^k(M) : \omega \mapsto e^{-h} \omega \tag{3.8}$$

obey

$$d f^k \omega = d e^h \omega = f^k d_h \omega = e^h d_h \omega = e^h e^{-h} d e^h \omega \tag{3.9}$$

and

$$d_h g^k \omega = d_h e^{-h} \omega = e^{-h} d e^h e^{-h} \omega = e^{-h} d\omega = g^k d\omega = e^{-h} d\omega \tag{3.10}$$

The two relations above tell us that f and g are chain maps between two complexes $C^\bullet$ and $D^\bullet$.

$$\begin{array}{ccccccccc}
\ldots & \longrightarrow & C_{k+1} & \xrightarrow{\partial^C_{k+1}} & C_k & \xrightarrow{\partial^C_k} & C_{k-1} & \longrightarrow & \ldots \\
 & & \downarrow{\scriptstyle f_{k+1}} & & \downarrow{\scriptstyle f_k} & & \downarrow{\scriptstyle f_{k-1}} & & \\
\ldots & \longrightarrow & D_{k+1} & \xrightarrow[\partial^D_{k+1}]{} & D_k & \xrightarrow[\partial^D_k]{} & D_{k-1} & \longrightarrow & \ldots
\end{array}$$

3. (*10 points*) Let us for simplicity assume that $M = \mathbb{R}^n$ with Euclidean metric $g_{ij} = \delta_{ij}$ and we restrict our attention to the forms, that vanish sufficiently fast at infinity, so the integrals over $\mathbb{R}^n$ are finite. Describe the dual operator d_h^* defined via

$$\langle d_h\omega, \mu\rangle = \langle \omega, d_h^*\mu\rangle, \quad \mu \in \Omega^k(M) \tag{3.11}$$

in terms of d^* and ι_v.

Hint: It might be helpful to represent the differential forms as a functions on supermanifold $\Pi T\mathbb{R}^n$.

Solution By definition

$$\begin{aligned}\langle d_h\omega, \mu\rangle &= \int_M e^{-h} d(e^h\omega) \wedge *\mu = \int_M d(e^h\omega) \wedge e^{-h} * \mu \\ &= (-1)^k \int_M (e^h\omega) \wedge d(e^{-h} * \mu) = (-1)^k \int_M \omega \wedge e^h d(e^{-h} * \mu)\end{aligned} \tag{3.12}$$

So we arrive into

$$d_h^* = e^h d^* e^{-h} \tag{3.13}$$

Using the supermanifold description of the differential forms

$$D_{d^*} = \sum \delta^{ij} \frac{\partial}{\partial x^i} \frac{\partial}{\partial \psi^j} \tag{3.14}$$

we arrive into

$$\begin{aligned}D_{d_h^*} &= e^{h(x)} D_{d^*} e^{-h(x)} = e^{h(x)} \sum \delta^{ij} \frac{\partial}{\partial x^i} \frac{\partial}{\partial \psi^j} e^{-h(x)} \\ &= D_{d^*} - \sum \delta^{ij} \partial_i h \frac{\partial}{\partial \psi^j} = D_{d^*} - D_{\iota_v}\end{aligned} \tag{3.15}$$

with vector field v being the gradient vector field for h

$$\sum_i v^i \partial_i = \sum_{ij} \delta^{ij} \partial_j h \tag{3.16}$$

Summarizing the argument above we arrive into nice geometric formula

$$d_h^* = d^* - \iota_v, \quad g(v, \cdot) = dh \tag{3.17}$$

4. (*10 points*) Let us for simplicity assume that $M = \mathbb{R}^n$ with Euclidean metric $g_{ij} = \delta_{ij}$ and we restrict our attention to the forms, that vanish sufficiently fast at infinity, so the integrals over $\mathbb{R}^n$ are finite. Evaluate the Laplacian

$$\Delta_h = d_h d_h^* + d_h^* d_h \tag{3.18}$$

in the form of differential operator acting on $C^\infty(\Pi T\mathbb{R}^n)$.

Solution

$$\begin{aligned} D_{\Delta_h} &= D_{d_h} D_{d_h^*} + D_{d_h^*} D_{d_h} \\ &= \left(e^{-h} \sum \psi^k \partial_k e^h\right) e^h \sum \delta^{ij} \frac{\partial}{\partial x^i} \frac{\partial}{\partial \psi^j} e^{-h} + e^h \sum \delta^{ij} \frac{\partial}{\partial x^i} \frac{\partial}{\partial \psi^j} e^{-h} \left(e^{-h} \sum \psi^k \partial_k e^h\right) \end{aligned}$$