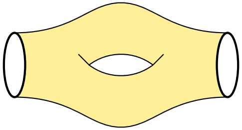
Ciliated maps, minimal models coupled to gravity and topological gravity

Okinawa Institute of Science and Technology
CFT, Probability, Gravity
Séverin Charbonnier – Université de Genève

August 2nd 2023

Discretisation



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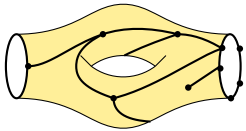

$\rightarrow$ Count maps. Generating functions F_g.
Partition function

$$Z = \exp\left(\sum_{g\geq 0} \hbar^{g-1} F_g\right)$$

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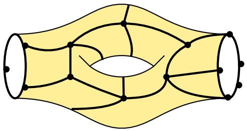

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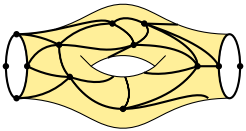

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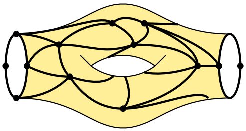

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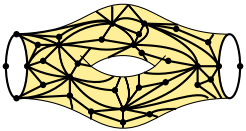

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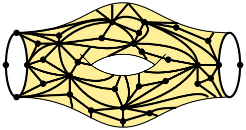

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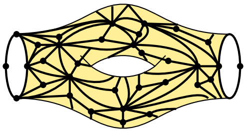

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$\overline{\mathcal{M}}_{g,n}$: {stable Riemann surfaces, genus g, n marked points}/ $\sim$.

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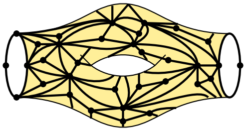

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$\overline{\mathcal{M}}_{g,n}$: {stable Riemann surfaces, genus g, n marked points}/ $\sim$.
Observables $\psi_i \in H^2(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$: Chern classes.
Intersection numbers:

$$\langle \tau_{d_1} \dots \tau_{d_n} \rangle_g := \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \dots \psi_n^{d_n}$$

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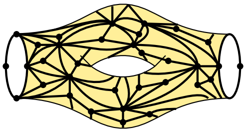

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$$Z^{\psi} = \exp\left(\sum_{g\geq 0} \hbar^{g-1} \left\langle e^{\sum_{d\geq 0} t_d \tau_d} \right\rangle_g\right)$$

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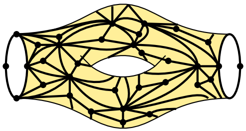

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[Witten '90] conjecture; [Konsevich '91] theorem. Both approaches are consistent: $Z = Z^\psi \Rightarrow$ Z^ψ solution of KdV integrable hierarchy.

Plan

1. Ciliated maps: definitions and enumeration results
   - Ciliated maps
   - Topological Recursion
   - Enumeration results
2. Large maps from ciliated maps and minimal models
   - Asymptotics of large maps
   - Singular spectral curve and minimal models
   - KPZ exponents
3. Topological gravity associated to ciliated maps
   - Topological gravity and intersection theory
   - Ciliated maps and Witten's class
4. Ciliated maps and free probabilities

[BCEG '21]: jw R. Belliard, B. Eynard, E. Garcia-Failde

[BCG '21]: jw G. Borot, E. Garcia-Failde

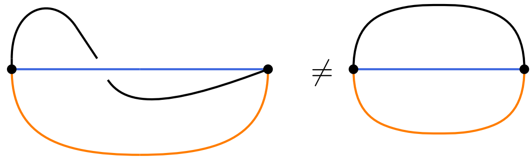
[BCGLS '21]: jw G. Borot, E. Garcia-Failde, F. Leid, S. Shadrin

[CCGG '22]: jw N. Chidambaram, E. Garcia-Failde, A. Giacchetto

Table of contents

Definition (Map)

A map is a graph G where each vertex is endowed with a cyclic ordering of the incident half-edges.



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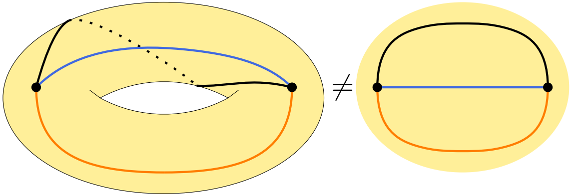

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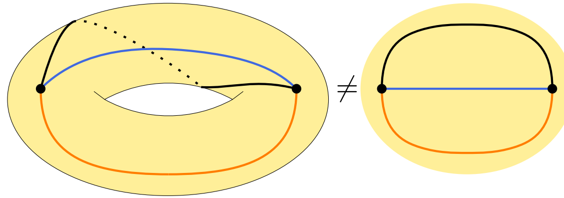

1 face, genus 1 — 3 faces, genus 0

Genus g of a connected map (Euler's formula): $\#\text{vertices} - \#\text{edges} + \#\text{faces} = 2 - 2g$

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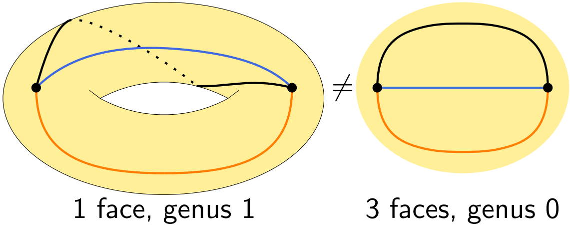

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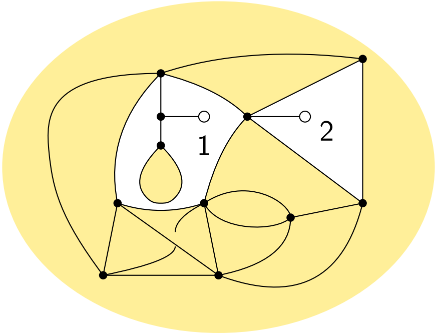
Model of maps: specify constraints



Ciliated maps

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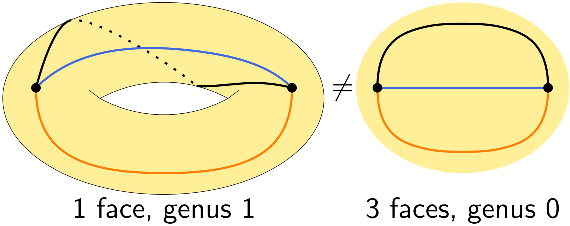

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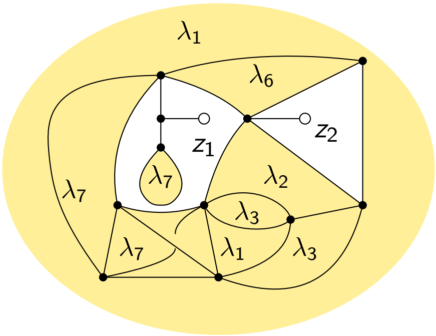
Model of maps: specify constraints, decorations



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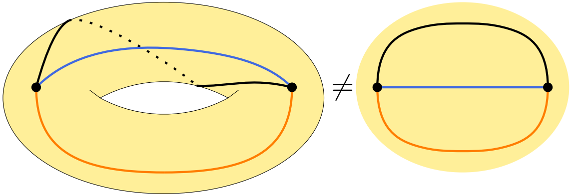

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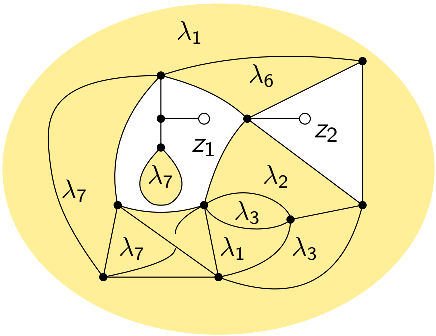
Model of maps: specify constraints, decorations, way of counting.



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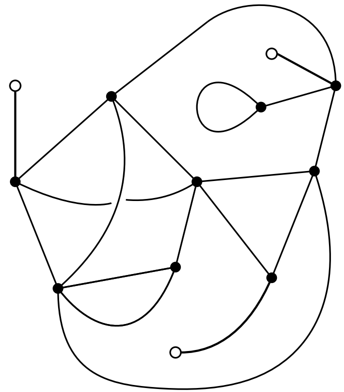
Let $r \geq 2$. A ciliated map is a map with 2 kinds of vertices:



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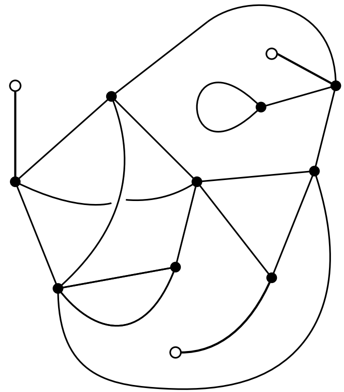

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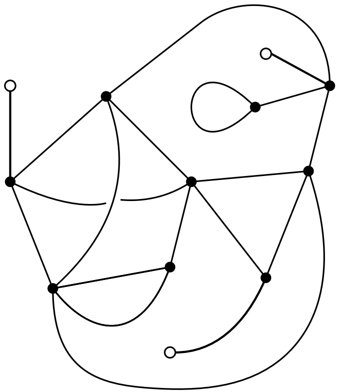

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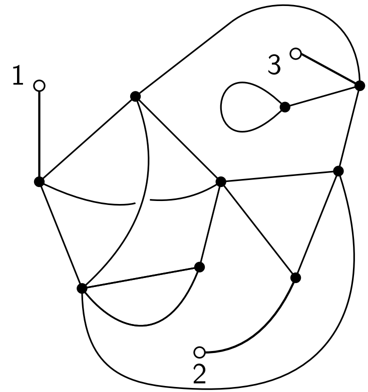

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Ciliated maps of type (g, n)

Let $g \geq 0$, $n \geq 0$. M is a ciliated map of type (g, n) ($M \in \mathfrak{C}_{g,n}$) if it is connected, of genus g, and has n labelled white vertices.

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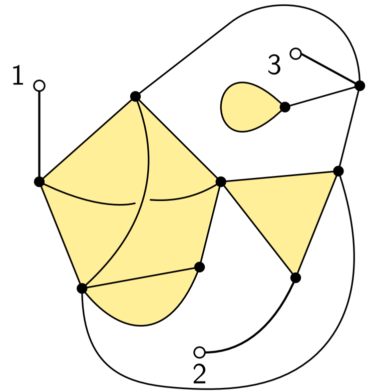

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Decorations

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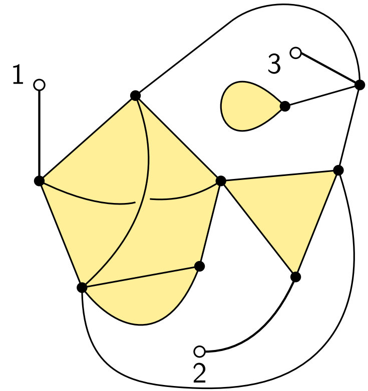

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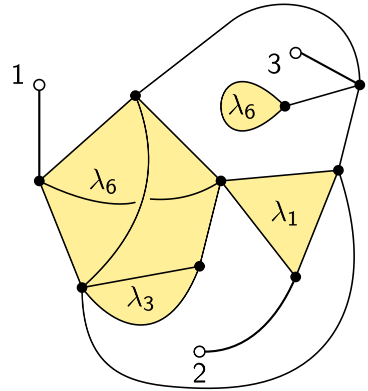

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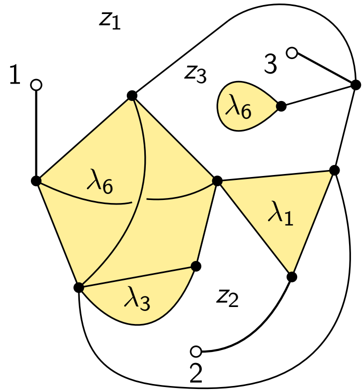

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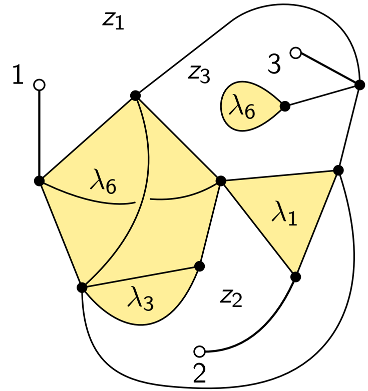

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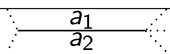
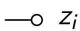
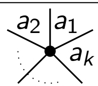
Potential: $V(u) = \sum_{j=1}^{r+1} \frac{v_j}{j} u^j$.

Object	Picture	Weight
Edge		$\mathcal{P}(a_1, a_2) = \frac{a_1 - a_2}{V'(a_1) - V'(a_2)}$
White vertex		1
Black vertex		$\mathcal{V}_k(a_1, \ldots, a_k) = \operatorname{Res}_{u=\infty} \frac{V'(u)\mathrm{d}u}{\prod_{j=1}^{k}(u - a_j)}$

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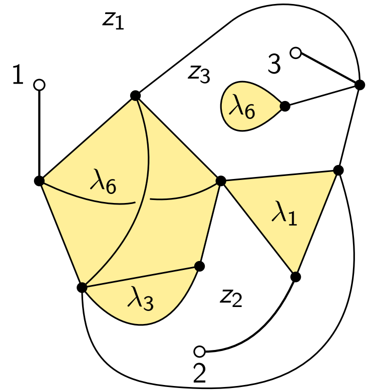

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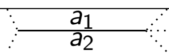
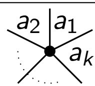
Edge		$\mathcal{P}(a_1, a_2) = \frac{a_1 - a_2}{V'(a_1) - V'(a_2)}$
White vertex	—o z_i	1
Black vertex		$\mathcal{V}_k(a_1, \ldots, a_k) = \operatorname*{Res}_{u=\infty} \frac{V'(u)du}{\prod_{j=1}^{k}(u - a_j)}$

Weight of a ciliated map:

$$\prod_{\text{faces}} t \prod_{\text{edges}} \mathcal{P}(a, b)$$

$$\prod_{\substack{\text{black} \\ \text{vertices}}} \mathcal{V}_k(a_1, \ldots, a_k)$$

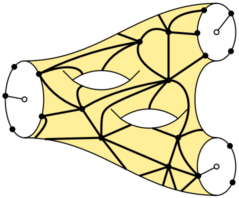
Generating functions

Weighted enumeration of decorated ciliated maps of type (g, n):

$$C_{g,n}(z_1, \ldots, z_n; \underline{\lambda}; \underline{v}) = \sum_{M \in \mathfrak{C}_{g,n}} \frac{\text{weight}(M)}{\#\text{Aut}(M)}$$

weight(M): rational function in z_i, λ_j, v_k.

$M \in$ { ciliated, topology (g, n) }



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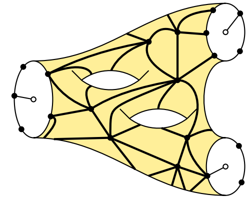

$\}$

Partition function Z (free energy F):

$$Z(\underline{\lambda}, \underline{v}; \hbar) = e^{F(\underline{\lambda}, \underline{v}; \hbar)} = \exp\left(\sum_{g \geq 0} \hbar^{g-1} C_{g,0}\right), \qquad \hbar = \frac{t^2}{N^2}$$

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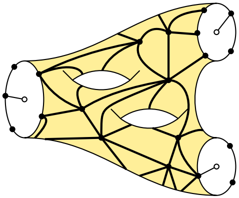

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Goals:

- Compute the $C_{g,n}$'s or the partition function.
- Specialise the parameters $\underline{\lambda}$, $\underline{v}$ to get CFT/Gravity.

Topological Recursion (TR): procedure developed by Chekhov–Eynard–Orantin ('07)

Input		**Output**
Spectral Curve	$\longrightarrow$	Differentials $(\omega_{g,n})_{g\geq 0, n\geq 0}$
$\mathcal{S} = (\Sigma, x, y, \omega_{0,2})$		recursion on $2g - 2 + n$

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Spectral Curve

Σ : Riemann surface;
$x : \Sigma \to \mathbb{P}^1$ branched covering;
$y : \Sigma \to \mathbb{P}^1$; $\omega_{0,2} \in H^0(\Sigma, K^{\boxtimes 2})$.

Topological Recursion

Topological Recursion (TR): procedure developed by Chekhov–Eynard–Orantin ('07)

Input		**Output**
Spectral Curve	$\longrightarrow$	Differentials $(\omega_{g,n})_{g\geq 0, n\geq 0}$
$\mathcal{S} = (\Sigma, x, y, \omega_{0,2})$		recursion on $2g-2+n$

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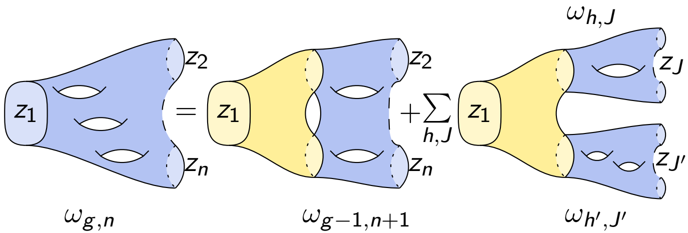

$$\omega_{g,n}(z_1, I) = \sum_{a\in\Sigma, \mathrm{d}x(a)=0} \operatorname*{Res}_{z=a} \frac{\frac{1}{2}\int_{\sigma_a(z)}^{z} \omega_{0,2}(z_1,\cdot)}{(y(z)-y(\sigma_a(z)))\mathrm{d}x(z)} \Bigg(\omega_{g-1,n+1}(z, \sigma_a(z), I) + \sum_{\substack{h+h'=g \\ J\sqcup J'=I}}' \omega_{h,1+J}(z,J)\,\omega_{h',1+J'}(\sigma_a(z), J') \Bigg)$$

$I = \{z_2, \ldots, z_n\}$; $\sigma_a : \Sigma \to \Sigma$ local involution around a.

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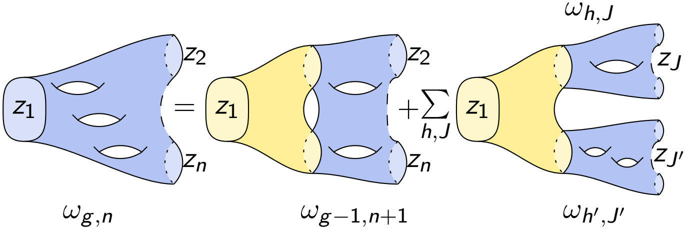

Various applications :

- Matrix models (hermitian, Kontsevich), map enumeration
- Enumerative geometry (Hurwitz numbers)
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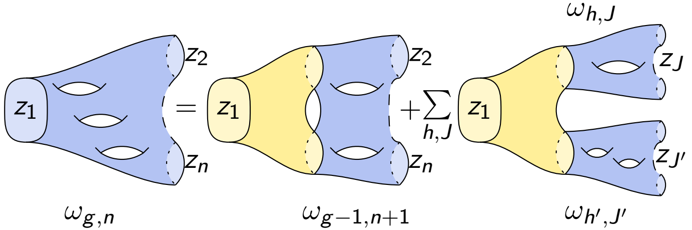

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Goal: prove that ciliated maps satisfy TR.

Topological recursion for ciliated maps

[BCEG '21]: Tutte's equation for ciliated maps.

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$$x(\zeta) = Q(\zeta), \qquad y(\zeta) = \zeta + t \sum_{j=1}^{N} \frac{1}{Q'(\xi_j)(\zeta - \xi_j)}$$

where Q is a degree r polynomial determined by: $Q(\zeta) \underset{\zeta \to \infty}{=} V'(y(\zeta)) + O(1/\zeta)$, and $\xi_j \in \mathbb{P}^1$ s.t. $Q(\xi_j) = V'(\lambda_j)$.

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- the differentials $\left(C_{g,n}(z_1, \ldots, z_n)\mathrm{d}x(\zeta_1)\ldots\mathrm{d}x(\zeta_n)\right)_{g,n}$ can be analytically continued to meromorphic n-forms on $\mathbb{P}^1$ $\omega_{g,n}(\zeta_1, \ldots, \zeta_n)$;

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- for $2g-2+n>0$, they satisfy topological recursion for the spectral curve $\mathcal{S} = (\mathbb{P}^1, x(\zeta), y(\zeta), \frac{\mathrm{d}\zeta_1\mathrm{d}\zeta_2}{(\zeta_1-\zeta_2)^2})$.

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- $\lambda_1 = \lambda_2 = \cdots = \lambda_N = 0$.
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Question: how to access the large order behaviours?

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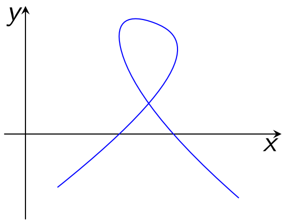

Large maps and singularity of spectral curve

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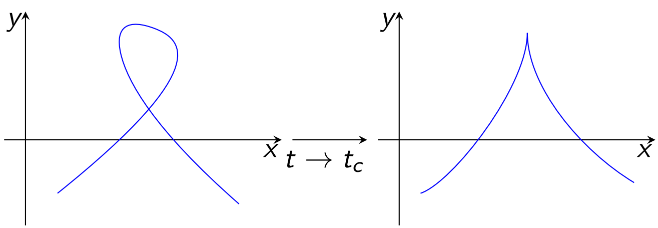

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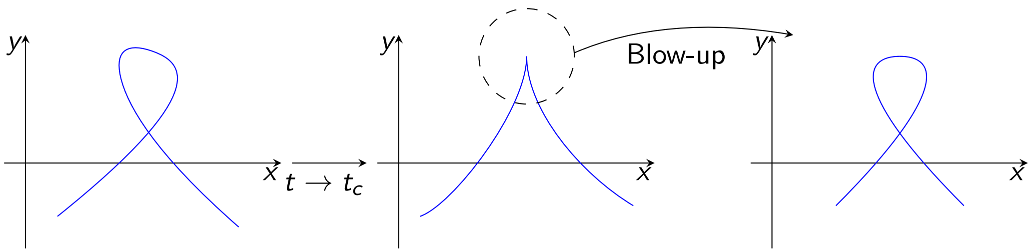

Large maps and (2,2m+1) minimal model

For maps, critical spectral curves of the form

$$\begin{cases} x(\zeta) = A_c \epsilon^{m+\frac{1}{2}} \left[(\zeta^2 - 2u)^{m+\frac{1}{2}} \right]_{\geq 0} + O(\epsilon^{m+\frac{3}{2}}) \\ y(\zeta) = a_c + c_c \epsilon (\zeta^2 - 2u) + O(\epsilon^2) \end{cases} \longleftrightarrow \quad (2, 2m+1) - \text{minimal model}$$

where $\epsilon^2 = t_c - t$ and ζ special parametrisation of the spectral curve.

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Pure gravity

$V(u) = \frac{u^2}{2} - t_3 \frac{u^3}{3}$
Critical point: $tt_3^2 = \frac{1}{12\sqrt{3}}$
Near criticality:
$tt_3^2 = \frac{1}{12\sqrt{3}}(1 - \frac{3}{4}\epsilon^2)^2$

$$\begin{cases} x \sim \frac{\sqrt{t}}{3^{1/4}} \epsilon^{\frac{3}{2}} (\zeta^2 - 2)^{\frac{3}{2}}_{\geq 0} + O(\epsilon^{\frac{5}{2}}) \\ y \sim 3^{\frac{1}{4}} \sqrt{t} (y_c + \epsilon(\zeta^2 - 2)) + O(\epsilon^2) \end{cases}$$

(2,3) minimal model: pure gravity.
[Kontsevich–Witten]

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Lee–Yang singularity

$V(u) = \frac{u^2}{2} - t_4 \frac{u^4}{4} - t_6 \frac{u^6}{6}$
Critical point: $tt_4 = \frac{1}{9}$, $t^2 t_6 = -\frac{1}{270}$
Near criticality:
$tt_6 = -\frac{1}{270}(1 + (2u_0\epsilon)^3)$

$$\begin{cases} x \sim -\frac{8}{5} \sqrt{\frac{t}{3}} \epsilon^{\frac{5}{2}} (\zeta^2 - 2u_0)^{\frac{5}{2}}_{\geq 0} + O(\epsilon^{\frac{7}{2}}) \\ y \sim \sqrt{3t}(2 + \epsilon(\zeta^2 - 2u_0)) + O(\epsilon^2) \end{cases}$$

(2,5) minimal model: Lee–Yang singularity.

Critical exponents and KPZ formula

Topological recursion for the critical spectral curve of $(2, 2m+1)$ minimal model:

$$C_{g,0} \underset{t \to t_c}{\sim} (1 - t/t_c)^{(2-2g)\frac{2m+3}{2m+2}} t_c^{2-2g} \widetilde{C}_{g,0}(1 + o(1 - t/t_c))$$

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Remark: central charge of (p, q) minimal model $c = 1 - 6\frac{(p-q)^2}{pq}$.
Ex: pure gravity $c = 0$, Lee–Yang $c = -\frac{22}{5}$.

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Correlation functions: intersection numbers with Chern classes $\psi_i \in H^2(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$

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Partition function

$$Z^{\mathrm{W}}(\underline{t}; \hbar) := \exp\Big(\sum_{g \geq 0, n \geq 1} \frac{\hbar^{g-1}}{n!} \sum_{a_i=0}^{r-2} \sum_{d_i \geq 0} \prod_{i=1}^n t_{d_i,a_i} \langle \tau_{d_1,a_1} \ldots \tau_{d_n,a_n} \rangle_g \Big)$$

Specialize the parameters

- $v_i = \delta_{i,r+1}$: $V(u) = \frac{u^{r+1}}{r+1}$
- $\sum_{k=1}^{N} \frac{1}{\lambda^j} = 0 \ \forall j \in \{1, \ldots, r+1\}$.

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- [CCGG '22] Associate a Cohomological Field Theory to spectral curve, identified with W^r [Pandharipande–Pixton–Zvonkine '19].

Table of contents

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Ciliated maps and matrix model with external field

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Remarks:

- $\langle \cdot \rangle_c$: classical cumulant of the matrix model;
- the matrix model was actually the inspiration for the study of ciliated maps;
- the particular form of the weights $\mathcal{V}_k$ come from Taylor expansion of $V(M+\lambda)$ (divided difference).

Free probability of the matrix model

Let $\gamma_1, \ldots, \gamma_n \in \mathfrak{S}_n$ be disjoint cycles, of total length L; monomials $\mathcal{P}_{\gamma_i} = \prod_{j=1}^{\ell(\gamma_i)} M_{(\gamma_i)_j, (\gamma_i)_{j+1}}$.

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Theorem [Eynard–Orantin '08]

$\left\langle \mathrm{Tr} M^{\ell_1} \ldots \mathrm{Tr} M^{\ell_n} \right\rangle_c$ are computed via topological recursion on the spectral curve

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Coefficient of $\left(\frac{N}{t}\right)^{2-2g-n}$ in $\left\langle \mathrm{Tr} M^{\ell_1} \ldots \mathrm{Tr} M^{\ell_n} \right\rangle_c$: moments of the surfaced probability space.

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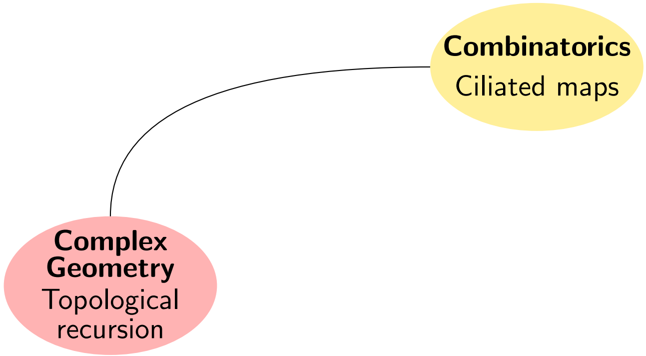
Combinatorics
Ciliated maps

Conclusion


Combinatorics
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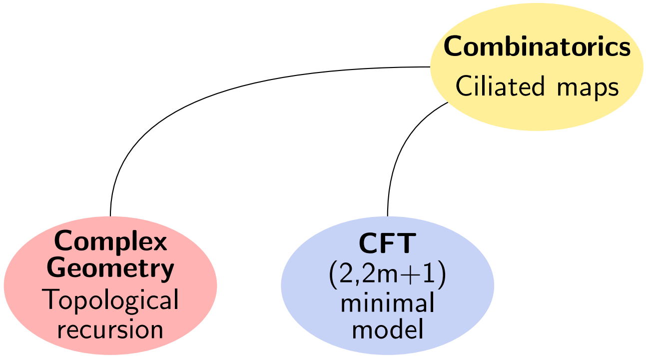
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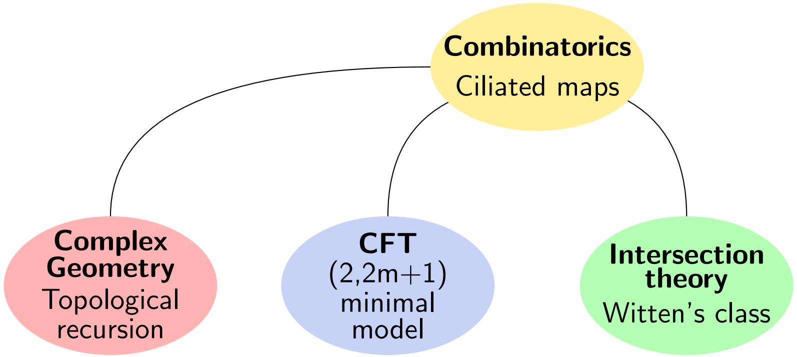

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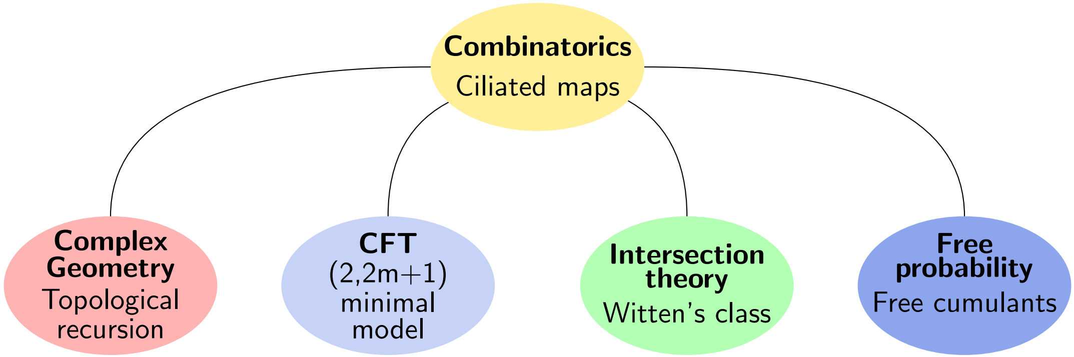

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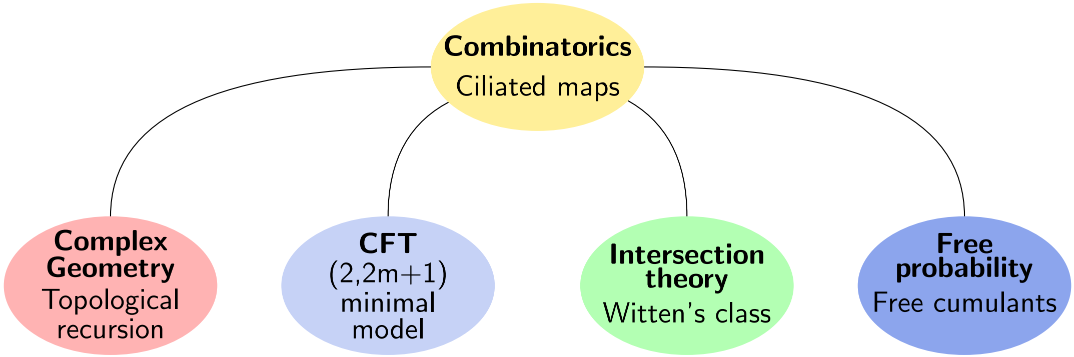

Thank you for your attention!

Tutte's equation

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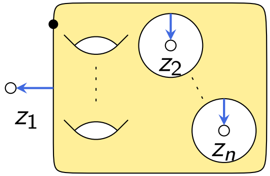

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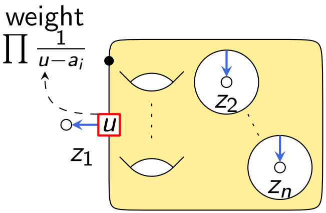

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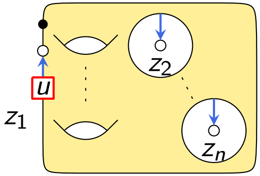

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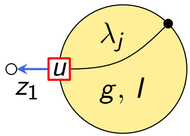
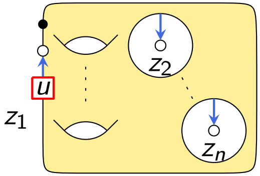
(a) 



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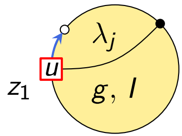

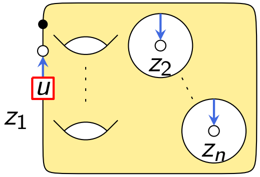

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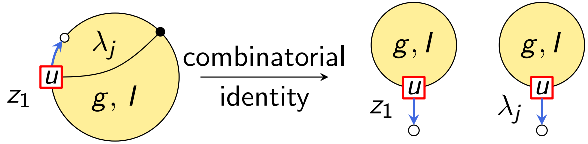

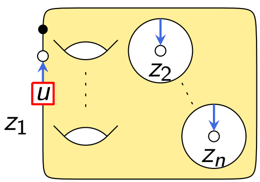

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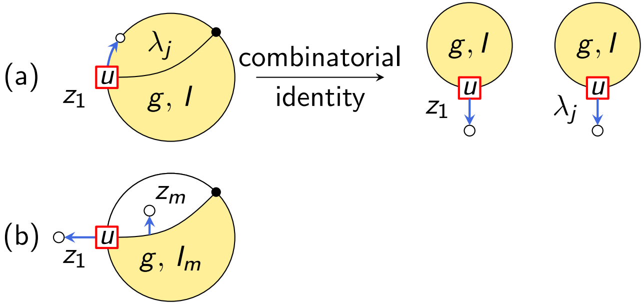

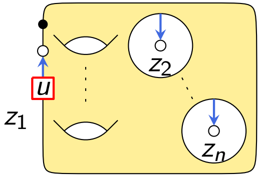

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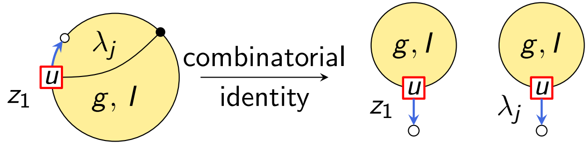

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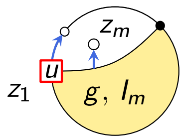

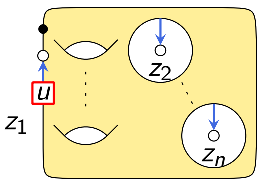

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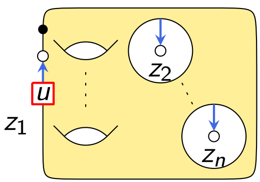

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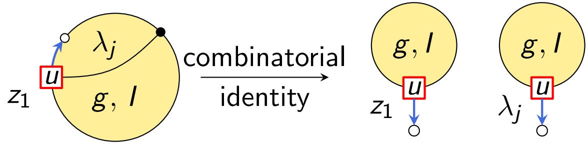

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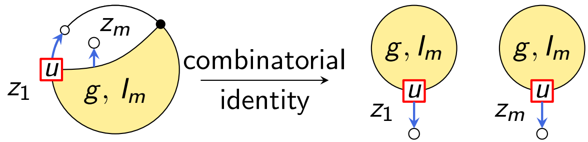

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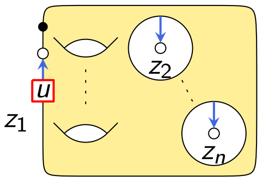

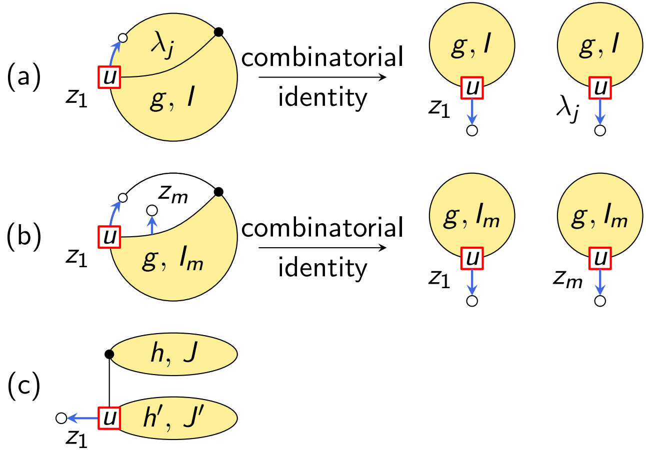

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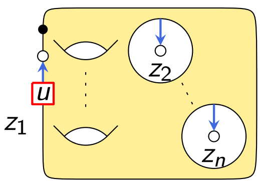

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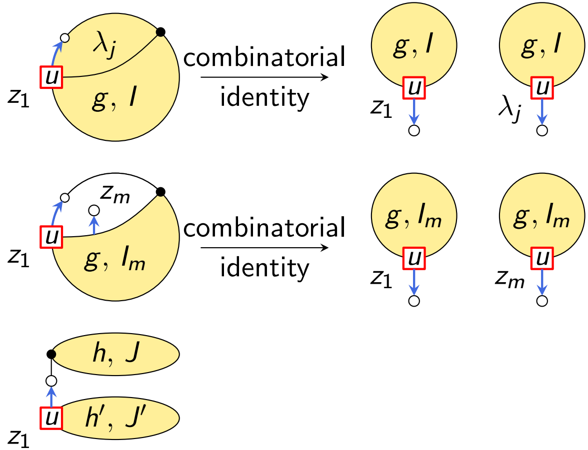

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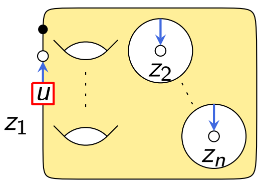

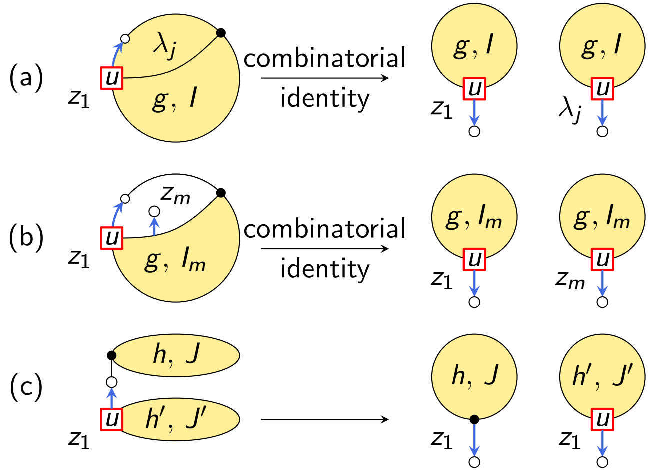

Tutte's equation

- Main combinatorial tool: Tutte's equation (edge removal from the maps).

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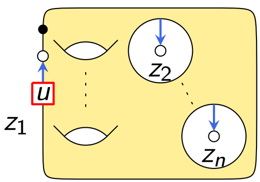

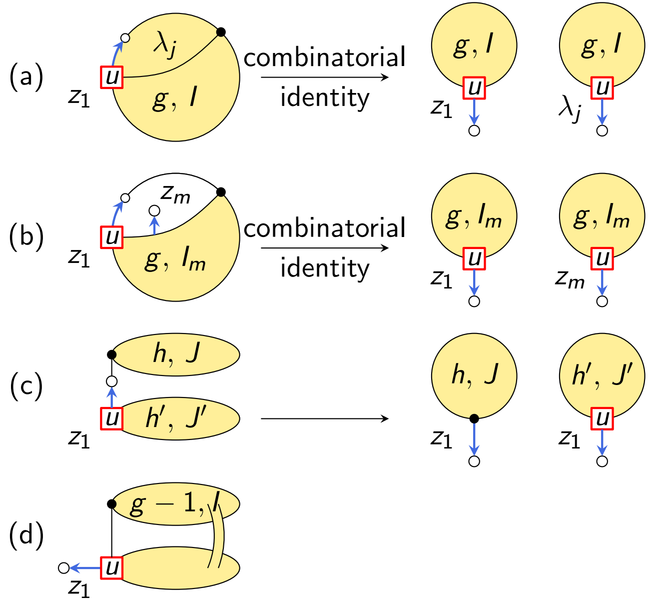

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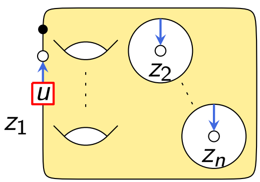

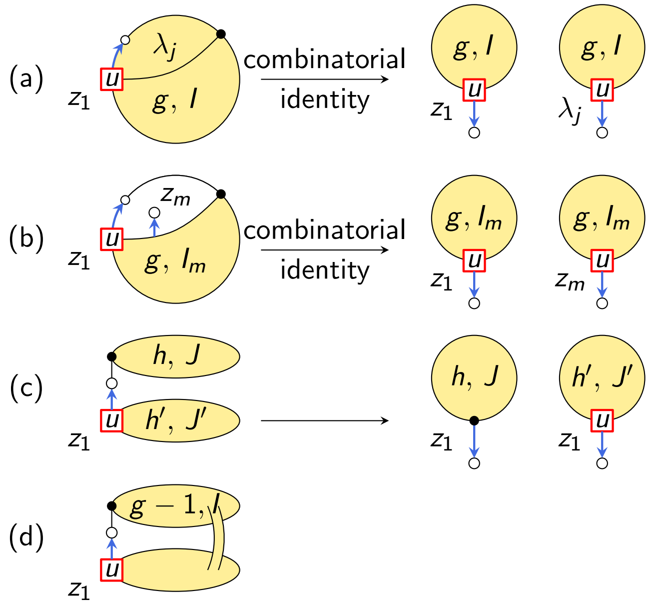

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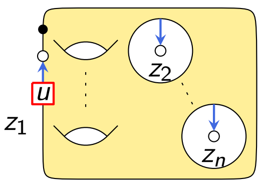

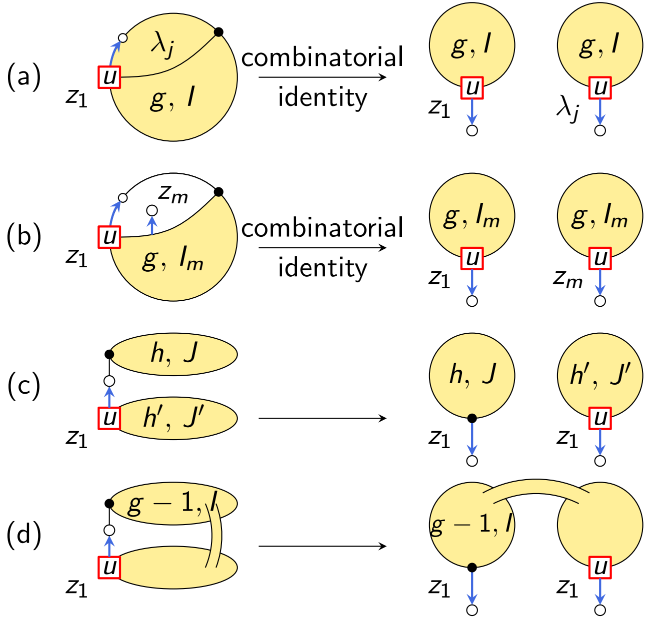

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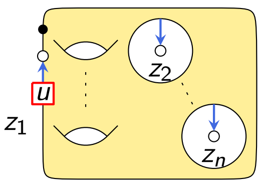

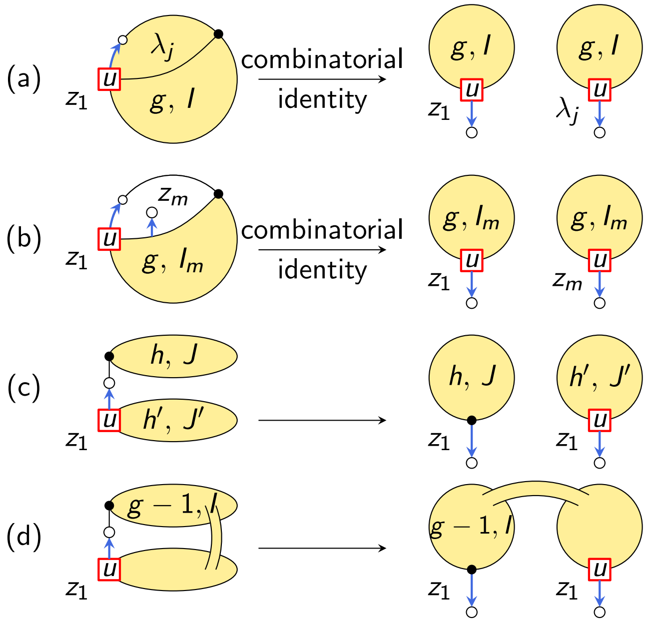

- Analytical treatment (technical): structure of the poles, loop equations.

Back

Spectral curve of fully simple maps

The spectral curve is given by:

$$\begin{cases} x(z) = Q(a+cz) = \left[V'(a+c(z+z^{-1}))\right]_{\geq 0} \\ y(z) = a + c(z+z^{-1}) \end{cases}$$

where a and c satisfy the following:

$$Q(a) = 0 \qquad \text{and} \qquad c = \frac{t}{Q'(a)}$$

Back

Free energies from TR

Formula of TR: for $g \geq 0$, $n \geq 1$ and $2g - 2 + n > 0$.

$$\omega_{g,n}(z_1, I) = \sum_{a \in \Sigma, \mathrm{d}x(a)=0} \operatorname*{Res}_{z=a} \frac{\frac{1}{2}\int_{\sigma_a(z)}^{z} \omega_{0,2}(z_1, \cdot)}{(y(z) - y(\sigma_a(z)))\mathrm{d}x(z)} \Bigg(\omega_{g-1,n+1}(z, \sigma_a(z), I) + \sum_{\substack{h+h'=g \\ J \sqcup J' = I}}' \omega_{h,1+J}(z, J)\omega_{h',1+J'}(\sigma_a(z), J') \Bigg)$$

$I = \{z_2, \ldots, z_n\}$; $\sigma_a : \Sigma \to \Sigma$ local involution around a.

For $g \geq 2$:

$$C_{g,0} = \frac{1}{2-2g} \sum_{a \in \Sigma, \mathrm{d}x(a)=0} \operatorname*{Res}_{z=a} \Phi(z)\omega_{g,1}(z)$$

where $\Phi'(z) = -y(z)x'(z)$.

Back

$\mathcal{A}$: non commutative algebra.
[Voiculescu '80s] :
Non commutative probability space
Moments:

$$\phi : S[\mathcal{A}] \to \mathbb{C}$$

Free Cumulants:

$$\phi(\sigma)[\cdot] = \sum_{\pi \in NC(\sigma)} \kappa(\pi)[\cdot]$$

Freeness of A, $B \subset \mathcal{A}$.

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Freeness of A, $B \subset \mathcal{A}$.

Random matrices
A_N: random hermitian, size N.

$$A_N \underset{N\to\infty}{\to} a.$$

Asymptotic expansion of cumulants:

$$\mathbb{E}_c\left[\mathrm{Tr}(A_N^k)\right] \underset{N\to\infty}{=} N\,\phi(\gamma_k)[a,\ldots,a] + O(N^{-1})$$

$\mathcal{A}$: non commutative algebra.
[Collins–Mingo–Śniady–Speicher '07] :
Higher order probability space
Moments:

$$\phi : PS[\mathcal{A}] \to \mathbb{C}$$

Free cumulants:

$$\phi = \zeta * \kappa$$

Higher order freeness of A, $B \subset \mathcal{A}$.

Random matrices

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Asymptotic expansion of cumulants:

$$\mathbb{E}_c\left[\mathrm{Tr}(A_N^{k_1}),\ldots,\mathrm{Tr}(A_N^{k_n})\right] \underset{N\to\infty}{=} N^{2-n}\phi(1_{\mathbf{k}},\gamma_{k_1,\ldots,k_n})[a\ldots,a] + O(N^{-n})$$

$\mathcal{A}$: non commutative algebra.
[BCGLS '21] :
Surfaced probability space
Moments:

$$\phi : \mathbb{PS}[\mathcal{A}] \to \mathbb{C}$$

Free cumulants:

$$\phi = \zeta \circledast \kappa$$

Surfaced freeness $A, B \subset \mathcal{A}$.
Back

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$$A_N \underset{N\to\infty}{\to} a.$$

Asymptotic expansion of cumulants:

$$\mathbb{E}_c\left[\mathrm{Tr}(A_N^{k_1}), \dots, \mathrm{Tr}(A_N^{k_n})\right] \underset{N\to\infty}{=} \sum_{g\geq 0} N^{2-n-2g}\phi(1_{\mathbf{k}}, \gamma_{k_1,\dots,k_n}, g)[a \dots, a]$$