(8)

Let: $\alpha_k(x) := 1 - e^{-b_k/x^\beta}$ for suitable $b_k, \beta > 0$

depends on a_k — deps on $|A|$

(details omitted.) $\square$

Q: What can we say about the kernel $\mathcal{F}_A = \ker(\text{æ})$?

i.e. Under what conditions on A is $\mathcal{F}_A$ trivial so that æ inj?

A: equally staggering: æ is <u>never</u> injective!

Prop: (non-uniqueness of fns with prescribed asymptotics)

For <u>any</u> A, $\mathcal{F}_A \neq \{0\}$, so $\text{æ}: \mathcal{A}_A \to \mathbb{C}[[x]]$ is not injective.

Pf: if $A = (-\alpha, +\alpha)$, then $e^{-1/x^\beta} \in \mathcal{F}_A$ for any $\beta < \frac{\pi}{2\alpha}$. $\square$

$\Rightarrow$ $\mathcal{F}_A$ is truly huge! $\Rightarrow$ explains why there are so many proofs of the Borel-Ritt Lemma: each proof constructs a splitting of the BR sequence:

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{F}_A \arrow[r] & \mathcal{A}_A \arrow[r, "\text{æ}"] & \mathbb{C}[[x]] \arrow[r] \arrow[l, bend left, dashed, "\text{resummation method}"] & 0 \end{tikzcd} \qquad (\heartsuit)$$

Very important: any resummation method used in the proof of BR Lemma is a splitting of ($\heartsuit$) <u>only</u> as $\mathbb{C}$-vector spaces

i.e. doesn't intertwine product or differential strs!

- In practical terms: if looking for soln of an equation, the method of perturbation theory then resummation is <u>not</u> valid.

§4 Asymptotics with Factorial Growth

Def: $f \in \mathcal{O}(S)$ is asymptotically smooth with factorial growth along A if $\forall S' \Subset S$ $\exists C, M > 0$ st $\forall k \geq 1$

$$\sup_{x \in S'} \left| \frac{f^{(k)}(x)}{k!} \right| \leq C M^k k!$$

nice

$\hat{f} = \sum_{k=0}^{\infty} a_k x^k \in \mathbb{C}[[x]]$ has factorial growth if $\exists C, M > 0$ st

$$|a_k| \leq C M^k k!$$

Lem: $\mathcal{A}_A^1 := \left\{ \begin{array}{l} \text{sectorial germs} \\ \text{nice along } A \end{array} \right\} \subset \mathcal{A}_A$ diff. subalgebra, local ring

$\downarrow \mathfrak{X}$ diff. alg. hom

$\mathbb{C}^1[[x]] := \{\text{p.s. w/ fact. growth}\} \subset \mathbb{C}[[x]]$ —"— , —"—

- $\mathbb{C}\{x\} \subset \mathbb{C}^1[[x]] \subset \mathbb{C}[[x]]$
- Main asympt. exp. theorem: $\boxed{|R_n(x)| \leq C M^n n! \, |x|^n}$
- Now, we want to understand the kernel of $\mathfrak{X}$.

Def: $f \in \mathcal{O}(S)$ is of <u>exponential decay</u> along A if $\forall S' \Subset S$ $\exists C > 0$ and $\tau > 0$ st

$$|f(x)| \leq C e^{-\tau/|x|} \quad \forall x \in S'$$

- $\mathcal{F}^1_A := \left\{ \begin{array}{c} \text{sectorial germs} \\ \text{of exp. decay} \\ \text{along } A \end{array} \right\}$

Lem: $\forall$ arc A, $\mathcal{F}^1_A \subset \mathcal{A}^1_A$ diff. subalgebra and

$$0 \longrightarrow \mathcal{F}^1_A \longrightarrow \mathcal{A}^1_A \xrightarrow{\ \ae\ } \mathbb{C}^1[[x]]$$

<u>note</u>: not claiming that $\ae$ is surjective!

- Proof skipped (uses Stirling approximation)

- <u>Observe</u>: if $|A| \leq \pi$, then $\mathcal{F}^1_A \neq \{0\}$ b/c of functions like $e^{-1/x}$.

Thm (<u>Watson's Lemma</u>):

If $|A| > \pi$, then $\mathcal{F}^1_A = \{0\}$

<u>Pf idea</u>:
- $f \in \mathcal{F}^1_A$ should be like e^{1/x^β} with $\beta < 1$, but e^{1/x^β} isn't nice.
- uses very interesting extension of maximum modulus principle to unbounded domains. //

Thm (Watson's Lemma):

Consider $\mathscr{Æ} : \mathcal{A}^1_A \longrightarrow \mathbb{C}^1[\![x]\!]$.

① if $|A| > \pi$, then $\mathscr{Æ}$ injective

② if $|A| \leq \pi$, then $\mathscr{Æ}$ surjective

Cor (Borel-Ritt-Watson Lemma)

If $|A| \leq \pi$, then

$$0 \longrightarrow \mathcal{F}^1_A \longrightarrow \mathcal{A}^1_A \longrightarrow \mathbb{C}^1[\![x]\!] \longrightarrow 0$$

Pf of Watson's Lemma: Wolog, assume $A = (-\alpha, +\alpha)$, $\alpha < \frac{\pi}{2}$.

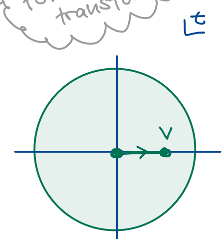
Take $\hat{f}(x) := \sum_{k=0}^{\infty} a_k x^k \in \mathbb{C}^1[\![x]\!]$ so $|a_k| \leq CMk!$.

Need to find some $f \in \mathcal{A}^1_A$ st $\mathscr{Æ}(f) = \hat{f}$.

Let: $\hat{\varphi}(t) := \sum_{k=0}^{\infty} \frac{a_{k+1}}{k!} t^k \in \mathbb{C}\{t\}$

secretely: formal Borel transform

Let: $v \in \mathbb{R}_t$ inside a disc of convergence of $\hat{\varphi}$.



Take: $f_v(x) := a_0 + \int_0^v e^{-t/x} \hat{\varphi}(t)\, dt$

truncated Laplace transform

Claim: $f_v(x) \in \mathcal{A}^1(\mathbb{H}_+)$ and $\mathscr{Æ}(f_v) = \hat{f}$.

↳ proof is similar to proof of Taylor's theorem. □

Shortfalls in Watson's Theorem:

- regimes of injectivity and surjectivity are disjoint
- injectivity does not extend to smaller arcs. (b/c $e^{-1/x}$)
- surjectivity does not extend to larger arcs (we'll see later)
- no practical description of image of $\mathscr{Æ}$ for large arcs.

§ 5. Borel-Laplace Transform

Introduce new complex plane $\mathbb{C}_t$.

Def: The Laplace transform of $\varphi = \varphi(t)$ in the direction θ is

$$\mathcal{L}_\theta[\varphi] := \int_{\mathbb{R}_\theta} e^{-t/x} \varphi(t)\, dt.$$

Given an arc $\Theta \subseteq \mathbb{R}$,

$$\mathcal{L}_\Theta[\varphi] := \left\{ \mathcal{L}_\theta[\varphi] \right\}_{\theta \in \Theta}$$

Def: entire hol. fn $\varphi \in \mathcal{O}(\mathbb{C}_t)$ is of exponential type at ∞ if $\exists\, C, M > 0$

st $$|\varphi(t)| \leq C e^{M|t|} \qquad \forall t \in \mathbb{C}_t.$$

$$\mathcal{E}^1(\mathbb{C}_t) := \left\{ \begin{matrix} \text{entire fns of} \\ \text{exp. type} \end{matrix} \right\} \subset \mathcal{O}(\mathbb{C}_t)$$

Prop: $\forall \varphi \in \mathcal{E}^1(\mathbb{C}_t)$,

$$f(x) := \mathcal{L}[\varphi](x) := \mathcal{L}_\theta[\varphi](x) \qquad \text{where } \theta := \arg(x)$$

is a hol. function in some disc $\mathbb{D}$ around the origin, and $f(0) = 0$

$\Rightarrow$ Laplace transform defines a $\mathbb{C}$-linear map

$$\mathcal{L} : \mathcal{E}^1(\mathbb{C}_t) \longrightarrow x\,\mathbb{C}\{x\}$$

$\hookrightarrow$ not algebra hom! $\qquad \mathcal{L}[\varphi \cdot \psi] \neq \mathcal{L}[\varphi] \cdot \mathcal{L}[\psi].$

Def: convolution product:

$$\varphi * \psi := \int_0^t \varphi(t-u)\,\psi(u)\,du.$$

$\Rightarrow$ $\mathcal{E}^1(\mathbb{C}_t)$ with $*$ is commutative (non-unital) $\mathbb{C}$-algebra

- e.g.: $t^i * t^j = \dfrac{i!\,j!}{(i+j+1)!}\, t^{i+j+1}$ $\Rightarrow$ $\mathbb{C}[\![t]\!]$ also gets $*$.

Prop: $\mathcal{L} : \underset{\text{with } *}{\mathcal{E}^1(\mathbb{C}_t)} \longrightarrow x\,\mathbb{C}\{x\}$ is alg. hom

i.e. $\mathcal{L}[\varphi * \psi] = \mathcal{L}[\varphi] \cdot \mathcal{L}[\psi]$.

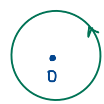
Claim: it is actually an iso!

to prove, we'll construct explicit inverse = Borel transform.

Defn: The Borel transform of $f = f(x)$ is

$$\mathcal{B}[f] := \frac{1}{2\pi i} \oint e^{t/x} f(x) \frac{dx}{x^2}$$

integral over



Lem: $\forall\, \hat{f} \in \mathbb{C}\{x\}$, $\varphi(t) := \mathcal{B}[\hat{f}](t) \in \mathcal{E}^1(\mathbb{C}_t)$.

So the Borel transform defines a $\mathbb{C}$-linear map

$$\mathcal{B} : \mathbb{C}\{x\} \longrightarrow \mathcal{E}^1(\mathbb{C}_t)$$

- $\mathcal{B}[1] = 0 \Rightarrow \ker(\mathcal{B}) = \mathbb{C}$
- $\mathcal{B}[x] = 1$
- $\mathcal{B}[x^n] = \dfrac{1}{(n-1)!}\, t^{n-1}$

$$\left.\begin{array}{l} \\ \\ \end{array}\right\} \quad \mathcal{B}\Big[\sum_{k=0}^{\infty} a_k x^k\Big] = \sum_{k=0}^{\infty} \frac{a_{k+1}}{k!}\, t^k$$

- $\mathcal{B}[f\cdot g] = \mathcal{B}[f] * \mathcal{B}[g] + f(0)\,\mathcal{B}[g] + g(0)\,\mathcal{B}[f]$

  $\Rightarrow$ $\mathcal{B}$ is not an algebra hom unless restricted to the ideal $x\,\mathbb{C}\{x\}$

Cor: $\mathcal{B} : x\,\mathbb{C}\{x\} \longrightarrow \mathcal{E}^1(\mathbb{C}_t)$ alg. hom.

Thm: (Convergent Borel-Laplace Isomorphism)

$$\begin{tikzcd} x\,\mathbb{C}\{x\} \arrow[r, bend left, "\mathcal{B}"] \arrow[r, phantom, "\cong"] & \mathcal{E}^1(\mathbb{C}_t) \arrow[l, bend left, "\mathcal{L}"] \end{tikzcd}$$

as algebras

Pf hint: change order of integration and use Cauchy's integral formula.

Def:

- $\hat{\mathcal{L}} : \mathbb{C}[[t]] \longrightarrow \mathbb{C}[[x]]$ — formal Laplace transform

$$\sum_{k=0}^{\infty} b_k t^k \longmapsto \sum_{k=0}^{\infty} k!\, b_k x^{k+1}$$

- $\hat{\mathcal{B}} : \mathbb{C}[[x]] \longrightarrow \mathbb{C}[[t]]$ — formal Borel transform

$$\sum_{k=0}^{\infty} a_k x^k \longmapsto \sum_{k=0}^{\infty} \frac{a_{k+1}}{k!} t^k$$

Lem: $\hat{f} \in \mathbb{C}^1[[x]] \iff \hat{\mathcal{B}}[\hat{f}] \in \mathbb{C}\{t\}$

and

$$\begin{tikzcd} \mathbb{C}^1[[x]] \arrow[r, "\hat{\mathcal{B}}"] & \mathbb{C}\{t\} \\ \mathbb{C}\{x\} \arrow[u, hook] \arrow[r, "\mathcal{B}"'] & \mathcal{E}^1(\mathbb{C}_t) \arrow[u, "J"'] \end{tikzcd}$$

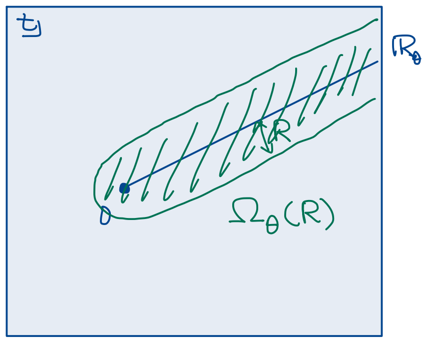
$\mathbb{C}$-linear map

Thm (Formal Borel-Laplace Isomorphism):

$$\begin{tikzcd}[column sep=huge]
{}_{\times}\mathbb{C}[[x]] \arrow[r, bend left, "\hat{\mathcal{B}}"] \arrow[r, phantom, "\cong"] & \mathbb{C}[[t]] \text{ with } * \arrow[l, bend left, "\hat{\mathcal{L}}"] \\
{}_{\times}\mathbb{C}^1[[x]] \arrow[u, hook] \arrow[r, bend left, "\hat{\mathcal{B}}"] & \mathbb{C}\{t\} \arrow[u, hook] \arrow[l, bend left, "\hat{\mathcal{L}}"]
\end{tikzcd}$$

as algebras

- Want to extend this to asymptotics with factorial growth
  Due to Watson, Nevanlinna, and Sokal
- Let θ be a direction.
  Let $\Omega_\theta(R) := \{t \mid \mathrm{dist}(t, \mathbb{R}_\theta) < R\}$
  halfstrip centred at $\mathbb{R}_\theta$



$$\mathcal{E}^1_{(\theta)} := \{$$