Invitation to Recursion, Resurgence and Combinatorics: Program

School (4/04-07/04)

• Elba Garcia-Failde:
"An invitation to topological recursion and its ramifications"

Topological recursion associates to some initial data, called spectral curve, a doubly indexed family of multi-differentials on the curve, which often encode important enumerative geometric information. It is a universal procedure since it appears in many different contexts in physics and mathematics, and helps establishing bridges among them.

We will introduce this general method while solving a classical problem that exhibits many of the useful tools around topological recursion: the enumeration of certain graphs embedded on surfaces (=maps), which are related to gaussian integrals over hermitian matrices (see mini-course of Sabine Harribey). Our strategy will be to first study the base topologies: the generating series of disks, that is of planar rooted maps, defines an algebraic curve, called the spectral curve of the problem; the generating series of cylinders is universal in the sense that it is independent of the type of maps and is related to a geometric object attached to the spectral curve, namely the fundamental differential of the second kind. Later we will study higher topologies recursively, which will yield topological recursion in the simplified setting of maps without internal faces. We will finally deduce the topological recursion in the presence of internal faces making use of the powerful deformation formulas of topological recursion.

After this introductory example, we will present spectral curves and topological recursion in general, and show some its most interesting properties and applications. We will finish by explaining two of the most fruitful and powerful connections of topological recursion to other domains. First, we will present the quantum curve conjecture and discuss its constructive solution in some classical contexts, like the Airy differential equation, whose solution encodes Witten—Kontsevich intersection numbers over the moduli space of curves, and the Painlevé differential equations. The quantum curve conjecture claims that one can associate to a spectral curve a differential equation whose solution can be reconstructed by the topological recursion applied to the original spectral curve. This is the starting point for an active and promising connection to the world of resurgence (see mini-course of Nikita Nikolaev). Finally, and depending of time constraints, I will introduce more general structures which help organising the intersection theory of the moduli space of curves: cohomological field theories; I will relate them to topological recursion in general and present several emblematic instances.

• Sabine Harribey:
"An introduction to tensor models: from random geometry to melonic CFTs"

Tensor models are particularly interesting due to their melonic large-N limit which is richer than the large-N limit of vector models but simpler than the planar limit of matrix models. Tensor models were first introduced in zero dimension in the context of random geometry and quantum gravity. They were then extended to quantum mechanical models in one dimension as an alternative to the Sachdev-Ye-Kitaev model without disorder. Finally, they were generalised in higher dimensions as toy models for strongly-coupled QFTs. In this context, they give rise in the infrared to a new kind of conformal field theories analytically accessible, called melonic CFTs.

In these lectures, after reviewing the large-N expansion of vector and matrix models, I will introduce tensor models and derive their melonic large-N limit. I will also present some applications to random geometry and quantum gravity. The second part of the lectures will focus on the bosonic O(N)^3 model. We will derive its large-N limit and study criticality of the

Schwinger-Dyson equation in zero dimension. In the last lecture, I will present the generalisation of this model in higher dimensions and compute fixed points in the infrared for both short-range and long-range interactions.

• Nikita Nikolaev:
"Invitation to Resurgence With a View Towards Geometry"

Divergent series expansions naturally arise in many parts of mathematics and physics from dynamical systems, to gauge theory, to quantisation of Poisson structures, to mirror symmetry and stability in algebraic geometry, to QFT and string theory, to name a few. This course is an introduction to an amazing emerging subject of Exact Perturbation Theory, also known as the theory of Resurgence. This theory provides powerful methods to upgrade divergent expansions to some meaningful analytic information. One such method is called Borel resummation: it involves a detailed analysis (via the Borel transform) of a given divergent expansion in order to recover exponentially small terms (a.k.a., nonperturbative corrections) that otherwise cannot be captured by the ordinary perturbation theory. The astounding big-picture upshot of this theory is that — contrary to Freeman Dyson's conclusion that perturbation theory is incomplete — the divergent sector of perturbation theory actually already encodes all the necessary nonperturbative information, and it is therefore only a matter of applying suitable methods to extract it.

My aim is to give a systematic treatment of the subject to build a solid foundation. Most of this course therefore will focus on the basic theory of divergent series and Borel resummation. At heart, the subject is just an extension of the ordinary complex analysis, and I will repeatedly stress this sentiment because I think it helps to demystify the subject. We will see, however, that this extension is exceptionally rich and full of remarkable new phenomena. Another feature that I find extraordinary (and that I will emphasise throughout the course) is that, although at first this subject may appear to be confined purely within the realm of analysis, in fact it has as much an algebro-geometric flavour as an analytic one.

I hope to cover the following topics: basic theory of asymptotic expansions, the Borel-Laplace transform, asymptotic expansions with factorial growth, Borel resummation and the Borel-Laplace Method, Stokes phenomenon, endless analytic continuation, algebra of resurgent transseries.

Workshop (10/04-14/04)
Monday

• Nicolas Delporte:
"Tensor models: an overview of non-perturbative results"

I will provide a broad summary of the various attempts that have been pursued to escape the strong melonic universality of tensor models leading to tree-like critical exponents: multiple scaling limits, topological recursion, Borel resummation.

• Sabine Harribey:
"Melonic large-N limit of 5-index irreducible random tensors"

The main feature of tensor models is their melonic large-N limit, leading to applications ranging from random geometry and quantum gravity to many-body quantum mechanics and conformal field theories. However, this melonic limit is lacking for tensor models with ordinary representations of O(N) or Sp(N).
We demonstrate that random tensors with sextic interaction transforming under 5-index irreducible representations of O(N) have a melonic large N limit. This extends the proof obtained for 3-index models with quartic interaction. In this talk, I will present the main ideas of our proof relying on recursive bounds derived from a detailed combinatorial analysis of the Feynman graphs.

• Carlos Perez Sanchez:
"From vector and matrices to tensors: comments and some questions."

This last talk of the 'Combinatorics Team 1' reviews some aspects of vector and matrix models and asks to which extend could these be implemented for tensor models.

• Sergey Shadrin:
"Topological recursion and its interactions with hypergeometric tau functions: a survey of applications"

I want to define topological recursion and give two alternative ways of thinking about it (from loop equations perspective and from algebraic computational perspective). Then I'll define very briefly hypergeometric tau functions, and the goal of my talk is to discuss their interaction.

• Reinier Kramer:
"The spin Gromov-Witten/Hurwitz correspondence for P^1"

Hurwitz theory and Gromov-Witten theory are two different methods of counting maps from curves to a target (also a curve for Hurwitz theory). Okounkov and Pandharipande showed that the Gromov-Witten theory of a target curve is essentially equivalent to Hurwitz theory with completed cycles.
Gromov-Witten theory of target surfaces with a canonical divisor can be localised to that divisor by work of Kiem-Li to give spin Gromov-Witten theory of curves. On the Hurwitz side, spin is introduced by restricting and adding signs to the count. Spin-completed cycles Hurwitz numbers exist and satisfy topological recursion, which is the starting point for proving a spin version of the Gromov-Witten correspondence - up to now only achieved for P^1.
This is based on joint work with Alessandro Giacchetto, Danilo Lewański, and Adrien Sauvaget.

• Elba Garcia-Failde:
"The master relation that simplifies maps and frees cumulants"

In this talk I will present one transformation that appears in very different contexts: combinatorial maps that get simplified, constellations that lose colours, cumulants that get freed, and x and y that get symplectically exchanged in topological recursion. I will explain how to realise all these dualities through a transformation that involves monotone Hurwitz numbers and we call master relation. Expressing the transformation as the action of an operator on the Fock space allows us to find functional relations that relate the generating series of higher order free cumulants and moments, which solves an open problem in free probability and generalises the R transform machinery of Voiculescu. This leads us to a general theory of freeness which takes into account higher genus corrections, which appear naturally in the other two incarnations of the duality. We introduce a notion of surfaced free cumulants from the combinatorics of the poset of surfaced permutations (a generalisation of partitioned permutations) that captures the all-order asymptotic expansion in 1/N of random ensembles of matrices of size N in presence of some unitary invariance.

This is based on joint work with Gaëtan Borot, Severin Charbonnier, Felix Leid and Sergey Shadrin: https://arxiv.org/abs/2112.12184

Tuesday

• Kohei Iwaki:
"Resurgence structure in exact WKB analysis and topological recursion"

In the first part of the talk, I'll briefly introduce the idea of Borel summation, resurgent analysis and how they are useful to study the Stokes phenomenon for divergent series. I'll treat several examples arising from the exact WKB analysis of Schrödinger-type ODEs.In the latter part, I'll discuss about the resurgence property of the partition function of the topological recursion with the aid of its relation to the Painlevé equations.

• Gergő Nemes:
"Realistic error bounds for asymptotic expansions arising from integrals via resurgence"

In this talk, we shall consider the problem of deriving realistic error bounds for asymptotic expansions resulting from integrals. W. G. C. Boyd demonstrated in the early 1990s that Cauchy–Heine-type representations for remainder terms are highly effective for determining such bounds. I will illustrate how the Borel transform yields a more globally valid expression for remainder terms, utilizing R. B. Dingle's terminant function as a kernel. Through examples, we shall see that this representation is, in some sense, optimal, resulting in error bounds valid in extensive sectors.
Building upon an exact resurgence formula by Sir M. V. Berry and C. J. Howls, I will present analogous results for asymptotic expansions arising from integrals with saddles. Lastly, I will demonstrate the applicability of the Cauchy–Heine-type approach to implicit problems by outlining the recent proof of F. W. J. Olver's conjecture regarding the large negative zeros of the Airy function.

• Nikita Nikolaev:
"Geometry and Borel Summability of Exact WKB Solutions"

One of the most classical settings for Exact Perturbation Theory is the exact WKB method for solving singularly perturbed linear ODEs such as the Schrödinger equation. Such ODEs can be easily solved in exponential power series: this is the famous WKB ansatz. The resulting formal WKB solutions, however, are always divergent and therefore have no direct analytic meaning. Attempting to apply the Borel resummation to get true analytic solutions turns out to be the correct approach, but a difficult mathematical problem, especially for equations of higher-order. I will describe a solution to this problem that I have developed through a series of recent works. Another outcome of this solution is that the constructions involved in the proof can be made completely geometrically invariant. So I will also describe an algebro-geometric formulation of the WKB method for meromorphic connections in terms of invariant splittings of bundle extensions.

• Taro Kimura:
"Instanton counting and q-deformation of Virasoro/W-constraint"

Abstract: Virasoro/W-constraint has been playing a central role in the context of enumerative geometry, which provides algebraic characterization of the partition function and the correlation functions (generating function; tau function). Instanton counting is one of such enumerative problems originally motivated by 4d supersymmetric gauge theory, which shares a lot of concepts with other enumerative problems, including spectral curve and its quantization, WKB expansion, integrability, etc. In this talk, I will show that a discrete analog of the Virasoro/W-constraint, which is in fact the q-deformation of the ordinary Virasoro/W-constraint, would emerge in the context of instanton counting through double quantization of the gauge theory moduli space. I would in particular discuss a possible interplay between geometric representation theory and enumerative geometry.

• Yuto Moriwaki:
“Sewing of conformal blocks of vertex operator algebra and braided tensor category”

I will talk about how the representation category of a vertex operator algebra is a braided tensor category from the operad structure of the moduli space of punctured Riemann surfaces of genus 0. For representations of a vertex operator algebra, one can define a D-module on a configuration space. We show that the solution sheaves of these D-modules, called conformal blocks, can be glued together in a manner consistent with the sewing operations on the configuration spaces, which defines a braided tensor category structure on the representation category of a vertex operator algebra.

• Hajime Nagoya:
“On q-isomonodromic deformations and q-Nekrasov functions”

Gamayun, Iorgov and Lisovyy discovered that a Fourier transform of the 4-point Virasoro conformal block with the central charge c=1 expresses the tau function of the sixth Painlevé equation. I will talk on its q-difference case. I explain how to construct fundamental solutions of connection preserving deformations in terms of q-Nekrasov functions. As a result, I give tau functions of q-isomonodromic deformations and determinant formulas for them. The talk is partly based on the joint work with M. Jimbo and H. Sakai.

Wednesday

• Olivier Marchal:
"Quantization of classical spectral curves and isomonodromic deformations"

In this talk, I will review the quantization of classical spectral curves using topological recursion. I will also explain why this quantization procedure is deeply related to isomonodromic deformations of meromorphic connections for which explicit expressions have been recently obtained in the $ \mathfral{g}l_2$ case.

• Alessandro Giacchetto:
"Resurgence and large genus asymptotics of intersection numbers"

Abstract: In this talk, I will present a new approach to the computation of the large genus asymptotics of Witten–Kontsevich intersection numbers. Our technique is based on a resurgent analysis of the n-point function of such intersection numbers, which are computed via determinantal formulae, and relies heavily on the presence of an quantum curve. With this approach, we are able to extend the recent results of Aggarwal with the computation of subleading corrections, and to obtain completely new results on r-spin and Theta-class intersection numbers. Based on a joint work in progress with B. Eynard, E. Garcia-Failde, P. Gregori, D. Lewański.

• Paolo Gregori:
"Non-perturbative Topological Recursion in Jackiw-Teitelboim Gravity"

The connection between matrix models and two-dimensional gravitational theories (including JT gravity) is a very well-established one. For this reason, Topological Recursion has proved to be an invaluable tool for computing observables perturbatively in such theories. In this talk, I will introduce a non-perturbative generalization of Topological Recursion which is particularly well-suited for the study of instanton effects (more specifically, of the ZZ brane type), and apply it to the case of JT gravity. Moreover, by making use of resurgent large order relations, I will show how our results lead to large genus asymptotics of Weil-Petersson volumes.

• Nezhla Aghaei: TBA

• Davide Lettera: TBA

• Omar Kidwai:
"Voros coefficients of quantum curves and the (uncoupled) BPS Riemann-Hilbert problem"

The notion of BPS structure formalizes some of the output of the study of four-dimensional N=2 QFTs, as well as the Donaldson-Thomas theory of CY3 triangulated categories. Bridgeland formulated a certain Riemann-Hilbert-like problem associated to such a structure, seeking jumping functions in the $\hbar$ plane with given asymptotics --- these appear in the description of complex hyperkahler metrics, among other things.
Starting from a "spectral curve", we recall how to associate a BPS structure to it, and review the construction of its corresponding quantum curve (an $\hbar$-dependent Schr\"odinger operator) via topological recursion. To such an operator, one may associate formal series of "quantum periods" known as Voros coefficients. We describe how, in the simplest examples, their Borel sums provide a canonical solution to the corresponding Riemann-Hilbert problem. Based on joint work with K. Iwaki.

• Giacomo Umer:
"Topological recursion in the F-world"

In 1996 Dubrovin showed that the potential given by the intersection indices of a CohFT can be associated to a certain Frobenius manifold. Later on, thanks to Givental and Teleman, a full reconstruction of a CohFT was possibile in the semi-simple case from the degree 0 part. In 2014 Dunin-Barkowski, Orantin, Shadrin and Spitz made sense of the potential of a semi-simple

Frobenius manifold by means of Eynard-Orantin topological recursion, furthermore leading to its re-expression as the logarithm of the partition function of an Airy structure.
Dropping the data of a metric, one can generalise the notion of Frobenius manifolds and work with F-manifolds, admitting vector potentials instead of scalar ones. In 2018 Buryak and Rossi introduced the weaker notion of F-CohFTs, which unlike CohFTs do not satisfy the self-gluing axiom, and established the link to flat F-manifolds in the same spirit as it was done before. The aim of this project is to propose a completion of this picture by introducing F-modified versions of topological recursion and Airy structures.
This talk is based on joint work in progress with G. Borot and A. Giacchetto.

• Dimitrios Mitsios:
"ODE and recursion relations for correlators in integrable systems and random matrices”

Thursday

• Tatsuhiro Misumi:
“Resurgence in quantum field theory”

We investigate the resurgent structure of quantum field theory, with emphasis on IR-renormalon and phase transition.
(1)We first study the 2D sigma models at large N. We show that the renormalon imaginary ambiguity is cancelled by the combined imaginary ambiguities at the different orders of the trans-series.
(2)We secondly discuss the nontrivial relation between phase transition and Borel singularities. We show the order of phase transition in some QFT can be understood by studying the details of collision of Borel singularities.
(3)We thirdly discuss the application of Exact-WKB to a variety of quantum-mechanical systems. In particular, a generic quantization condition for QM describing a particle on S1 is derived by use of Exact-WKB.

• Ioana Coman:
“False-mock pairs as Z-invariants”

A new family of topological 3-manifold invariants which are q-series with certain integrality properties that allow their categorification has been proposed recently. They have a definition based on the combinatorial data specifying the associated 3-manifold, though this is of restricted applicability and limited only to manifolds which satisfy a certain condition. From a number theory perspective, they are interesting because they provide examples of holomorphic quantum modular forms. Here I will discuss an underlying hidden symmetry of these invariants and how this helps to predict, through number theory considerations, what these invariants should be for manifolds not covered by the original definition. Since these invariants are part of a web of relations, being related to WRT invariants and to partition functions of certain corresponding 3-dimensional SQFTs, I will also comment on other possible computational approaches, including through resurgence.

• Yasuyuki Hatsuda: TBA

Friday

- Raimar Wulkenhaar: TBA

- Romain Pascalie:

"JT gravity at finite cut-off."

Abstract : After reviewing Jackiw-Teitelboim (JT) gravity in the Schwarzian limit, I will present how to describe the theory at finite cut-off. The main step is to consider more general geometries, namely immersions, rather than a specific class of embeddings as in the Schwarzian limit. Then I will briefly show how this impacts the expansion of the action in powers of the cut-off. Finally I will present how we can numerically generate typical boundary curves in flat JT. Based on work in collaboration with Frank Ferrari and Nicolas Delporte.

- Maciej Dolega:

"Jack-deformed (weighted) Hurwitz numbers and their combinatorial interpretation"

The (weighted) Hurwitz numbers are one of the primary objects in enumerative geometry. From the point of view of integrable hierarchies, one can approach them by studying the associated generating series that is encoded by Schur symmetric functions. The latter has a famous one-parameter deformation called Jack polynomials. It seems that when we replace the Schur functions by the Jack functions in this generating series, the positivity and integrality is mysteriously not affected.

We prove that this Jack-deformed generating series of (weighted) Hurwitz numbers can be understood as the generating series of some geometric objects that we construct. We focus on their combinatorial counterpart that we describe in details, and we explain how this construction extends a combinatorial approach to the classical Hurwitz theory by introducing a deformation parameter that interpolates between the complex (orientable) Hurwitz theory and the real (non-orientable) Hurwitz theory. If time permits, we describe special cases of our construction that provides topological expansion for various matrix models studied previously in the literature. Based on the joint work with Chapuy.