The master relation that simplifies maps and frees cumulants

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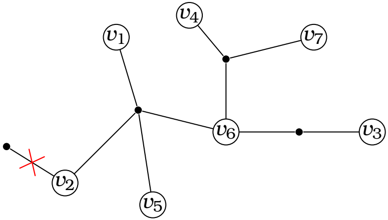
(Based on joint work with G. Borot, S. Charbonnier, F. Leid, S. Shadrin: arXiv:2112.12184)



Workshop: An Invitation to Recursion, Resurgence and Combinatorics (OIST)

April 10, 2023, Okinawa

Outline

1. A triple duality: symplectic, simple and free
2. Master relation: a universal duality?
   - Monotone Hurwitz numbers
3. Origins of the master relation
   - Combinatorial maps and matrix models
   - From maps to free probability via matrix models
   - The origin of the master relation
   - Topological recursion and symplectic invariance
4. Surfaced free probability
   - Higher order free cumulants
   - Open question
   - First and second orders
   - Surfaced free cumulants (of topology (g, n))
5. Moment-free cumulant relations: $M = G_{0,n} \leftrightarrow G^{\vee}_{0,n} = C$
   - Main result
6. Future and ongoing work
7. Bonus: tower of constellations
   - Constellations

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3 contexts:

- **Free probability:**

  Moments φ $\leftrightarrow$ Free cumulants κ

- **Combinatorics:**

  Maps $\leftrightarrow$ Fully simple maps

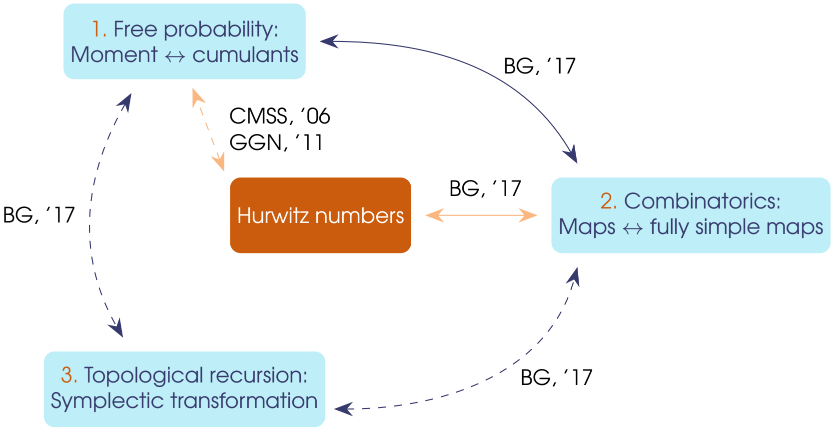
- **Topological recursion (TR):**

$$\begin{cases} \Sigma \text{ Riemann surface} \\ x\colon \Sigma \to \mathbb{C}\mathrm{P}^1,\ y\colon \Sigma \to \mathbb{C}\mathrm{P}^1 \\ \omega_{0,1} = y\,\mathrm{d}x \text{ 1-form} \\ \omega_{0,2} \quad \text{bidifferential} \end{cases} \xrightarrow[]{\textbf{TR}} \begin{array}{l} \text{Multi-differentials} \\ \omega_{g,n}(z_1,\ldots,z_n),\, z_i \in \Sigma, \\ \forall g, n \geq 0. \end{array}$$

$$(x,y) \overset{\text{TR}}{\rightsquigarrow} \omega_{g,n} \quad \leftrightarrow \quad (\check{x},\check{y}) \overset{\text{TR}}{\rightsquigarrow} \check{\omega}_{g,n},$$

with $\mathrm{d}x \wedge \mathrm{d}y = \mathrm{d}\check{x} \wedge \mathrm{d}\check{y}$ (symplectic transformation).

3 incarnations of the master relation: symplectic, simple and free



1. Free probability:
Moment ↔ cumulants

BG, ’17

CMSS, ’06
GGN, ’11

Hurwitz numbers

BG, ’17

2. Combinatorics:
Maps ↔ fully simple maps

BG, ’17

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Symplectic transformation

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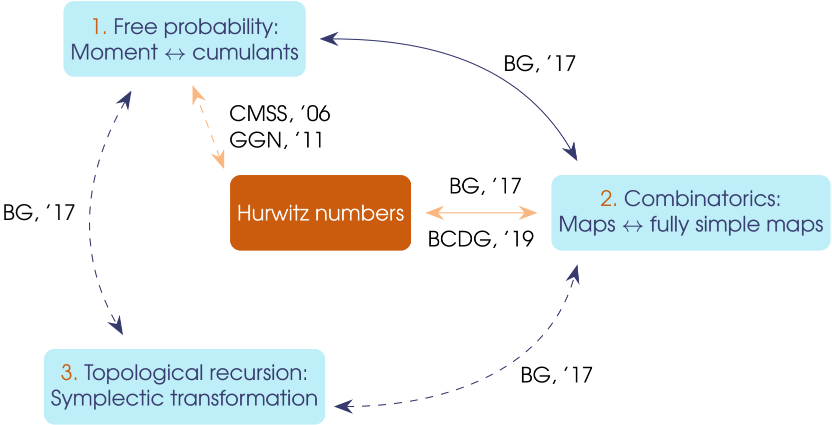

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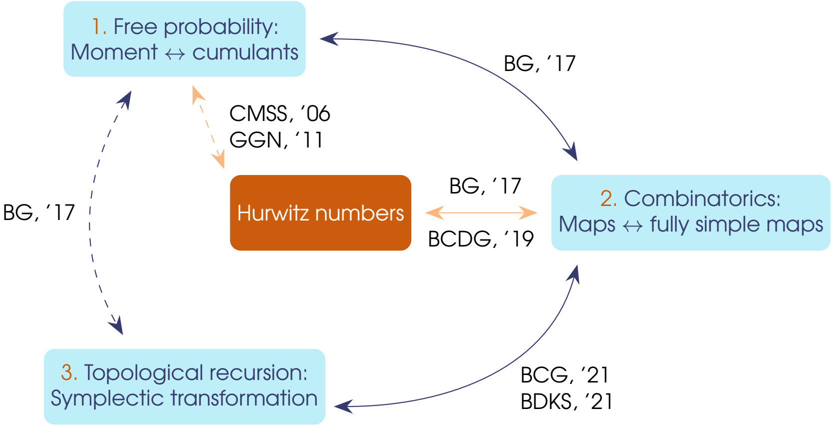

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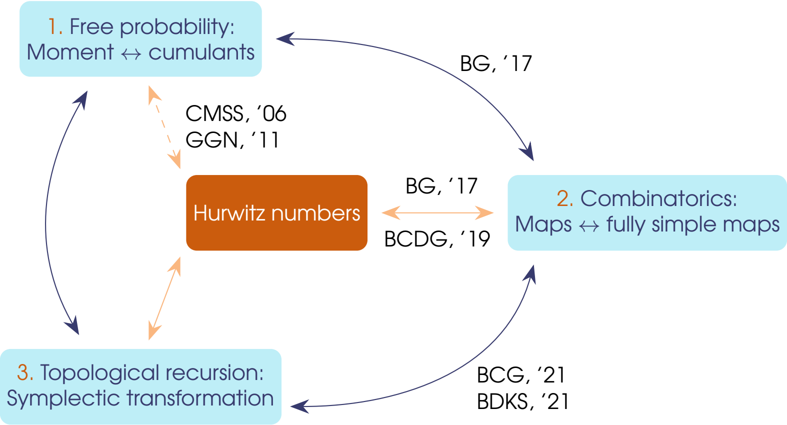

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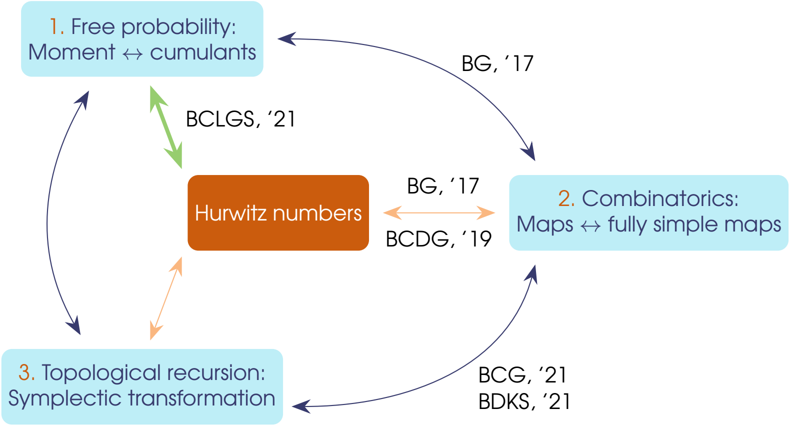

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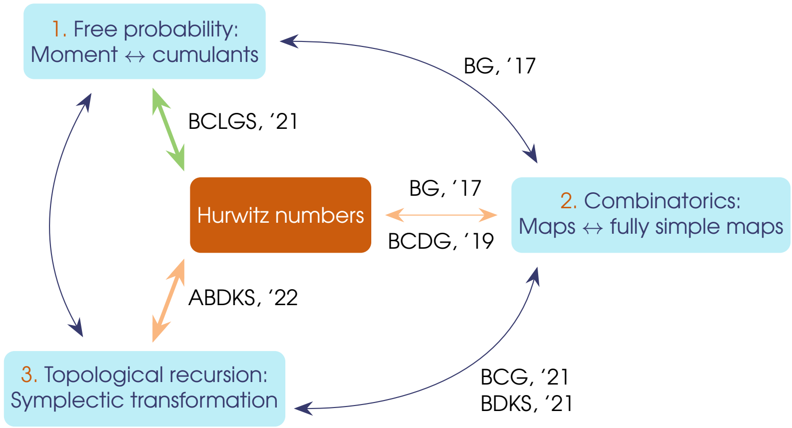

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Double monotone Hurwitz numbers

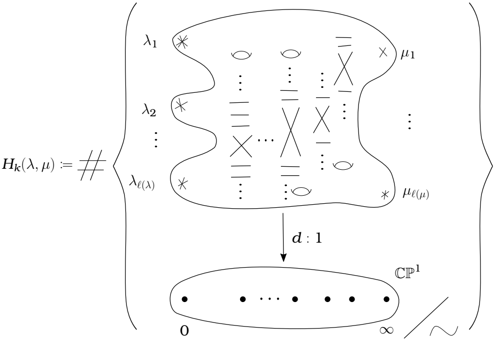
$k, d \in \mathbb{Z}_{\geq 0}, \lambda, \mu \vdash d.$

Definition

Double Hurwitz number $H_k(\lambda, \mu) \rightsquigarrow$ number of possibly disconnected coverings of the sphere with ramification profile

- λ over 0, μ over ∞,
- simply ramified over k points in $\mathbb{P}^1 \setminus \{0, \infty\}$,

weighted by $|\mathrm{Aut}|$.



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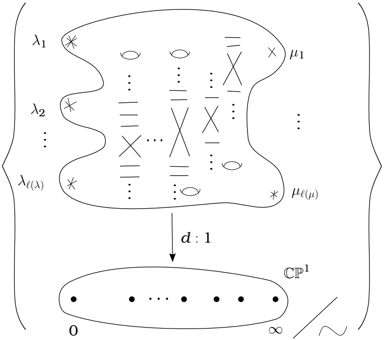
$H_k(\lambda, \mu) \coloneqq \#$



- $C_\lambda \rightsquigarrow$ Conjugacy class in $\mathfrak{S}_d$ of elements of cycle type $\lambda \vdash d$.

$$H_k(\lambda, \mu) = \frac{1}{d!} \left|\{(\sigma, \tau_1, \ldots, \tau_k) \mid \sigma \in C_\lambda,\ \tau_i \in C_{(2,1\ldots,1)},\ \sigma\tau_1 \cdots \tau_k \in C_\mu\}\right| .$$

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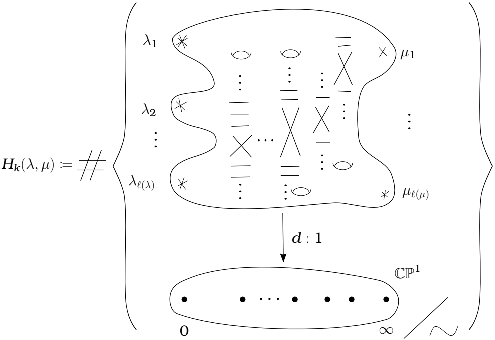

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Transpositions $\tau_i = (a_i\ b_i)$, with $a_i < b_i$, $i = 1,\ldots,k$:

- $b_i \leq b_{i+1} \rightsquigarrow$ *Weakly monotone:* $H_k^{\leq}(\lambda,\mu)$ (Goulden–Guay-Paquet–Novak,'11).
- $b_i < b_{i+1} \rightsquigarrow$ *Strictly monotone:* $H_k^{<}(\lambda,\mu)$.

$$\boxed{H^{<}(\lambda,\mu) = \sum_{k=0}^{d-1} H_k^{<}(\lambda,\mu)\hbar^k \in \mathbb{Q}[\hbar] \quad \text{and} \quad H^{\leq}(\lambda,\mu) = \sum_{k\geq 0} H_k^{\leq}(\lambda,\mu)\,(-\hbar)^k \in \mathbb{Q}[\![\hbar]\!].}$$

Topological partition functions and master relation

Fock space ⇝ completion of the ring of symmetric polynomials with coefficients formal series in $\hbar$:

$$\mathcal{F}_R := R[\![p_1, p_2, p_3, \dots]\!]\,, \qquad \mathcal{F}_{R,\hbar} := \mathcal{F}_R \otimes \mathbb{Q}(\!(\hbar)\!)\,.$$

- $\lambda \in \mathcal{Y} \rightsquigarrow$ Young diagrams. Consider $p_\lambda = p_{\lambda_1} \cdots p_{\lambda_{\ell(\lambda)}}$.
- $z(\lambda) = \prod_{i=1}^{\ell(\lambda)} \lambda_i \prod_{j>1} m_j(\lambda)!$, where $m_j(\lambda)$ is the number of $j's$ in λ.

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Topological partition function: $Z = e^F \in \mathcal{F}_{R,\hbar}$, $F = \sum_{g\geq 0} \hbar^{2g-2} F_g$, $F_g \in \mathcal{F}_R$.

$$Z = \exp\Bigg(\sum_{\substack{g\geq 0 \\ \lambda \in \mathcal{Y}}} \hbar^{2g-2} \frac{F_g(\lambda)}{z(\lambda)} p_\lambda \Bigg) = 1 + \sum_{\lambda \in \mathcal{Y}} \hbar^{-|\lambda|-\ell(\lambda)} Z(\lambda) p_\lambda .$$

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Two topological partition functions Z and $Z^\vee$ satisfy the master relation if

$$\boxed{Z(\lambda) = z(\lambda) \sum_{\mu\vdash|\lambda|} H^{<}(\lambda,\mu) Z^\vee(\mu)} \qquad (\star)$$

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Dual formulation of the master relation:

$$(\star) \Leftrightarrow Z^\vee(\lambda) = z(\lambda) \sum_{\mu\vdash|\lambda|} H^{\leq}(\lambda,\mu) Z(\mu).$$

Multiplicative functions, correlators, open problem and strategy

Topological partition function $Z = e^F \leftrightarrow$ multiplicative function $\Phi_{Z,\hbar} \colon PS \to R[\![\hbar]\!]$, with PS the poset of partitioned permutations.

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$$G_{g,n}(x_1, \ldots, x_n) = \sum_{\ell_1,\ldots,\ell_n>0} F_{g;\ell_1,\ldots,\ell_n} x_1^{\ell_1} \cdots x_n^{\ell_n} + \delta_{g,0}\delta_{n,1}.$$

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$$G_{0,n} \xleftrightarrow{\text{M-C}} G^{\vee}_{0,n}, \quad \text{for } n > 3?$$

Known for $n = 1, 2$ in free probability (and combinatorics) and (for $n = 3$ in topological recursion).

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A map of genus g and n *boundaries* is a connected graph Γ embedded into a closed oriented surface X of genus g such that

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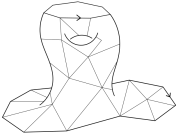

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Maps and fully simple maps

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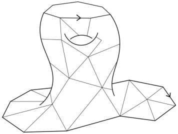

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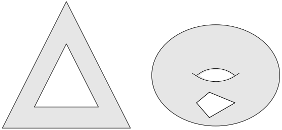

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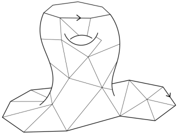

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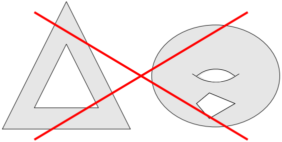

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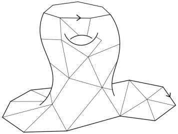

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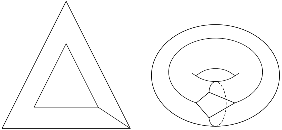

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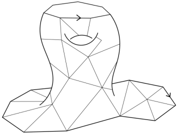

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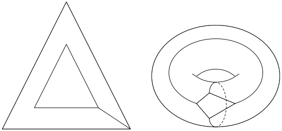
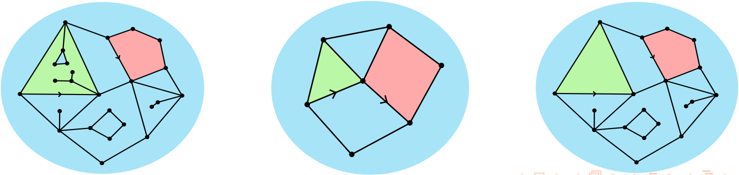


Simple: Boundaries are simple polygons.

Fully simple: Simple and pairwise disjoint boundaries.



Maps and formal hermitian matrix models

Generating series of maps of genus g and n boundaries of lengths $l_1, \dots, l_n$:

$$\mathrm{Map}^{[g]}_{l_1,\dots,l_n} := \sum_{\mathcal{M} \in \mathbb{M}^{[g]}_n(l_1,\dots,l_n)} \prod_{f \in \mathrm{IFaces}(\mathcal{M})} t_{\text{length}(f)} \cdot$$

$\mathrm{FSMap}^{[g]}_{k_1,\dots,k_n} \rightsquigarrow$ Same for fully simple maps.

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Moments and classical cumulants:

$$\left\langle \prod_{i=1}^n \mathrm{Tr}\, M^{\ell_i} \right\rangle \quad \text{and} \quad c_n\left(\mathrm{Tr}\, M^{\ell_1}, \dots, \mathrm{Tr}\, M^{\ell_n}\right).$$

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- $\gamma = (c_1\; c_2\; \dots\; c_{\ell(\gamma)})$ cycle in $\mathfrak{S}_N \rightsquigarrow \mathcal{P}_\gamma(M) := \prod_{i=1}^{\ell(\gamma)} M_{c_i,\gamma(c_i)}$.

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where γ_i are pairwise disjoint cycles of $\mathfrak{S}_N$ ($N \geq \sum_{i=1}^{n} \ell(\gamma_i)$).

From maps to free probability via matrix models

Free probability from matrix model:

$$\varphi_{\ell_1,\ldots,\ell_n} = \lim_{N\to\infty} N^{n-2} c_n\big(\mathrm{Tr}\, M^{\ell_1}, \ldots, \mathrm{Tr}\, M^{\ell_n}\big),$$

$$\kappa_{\ell_1,\ldots,\ell_n} = \lim_{N\to\infty} N^{n-2+d} c_n\big(\mathcal{P}_{\gamma_1}(M), \ldots, \mathcal{P}_{\gamma_n}(M)\big),\ d = \sum_{i=1}^{n} \ell_i.$$

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$$c_n\big(\operatorname{Tr} M^{\ell_1}, \ldots, \operatorname{Tr} M^{\ell_n}\big) = \sum_{g\geq 0} N^{2-2g-n} \mathrm{Map}^{[g]}_{\ell_1,\ldots,\ell_n},$$

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Remark: For more general multi-tracial hermitian measures, stuffed maps.

From maps to free probability

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From maps to free probability (with genus corrections)

$$\varphi^{[g]}_{\ell_1,\ldots,\ell_n} = \mathrm{Map}^{[g]}_{\ell_1,\ldots,\ell_n}, \quad \kappa^{[g]}_{\ell_1,\ldots,\ell_n} = \mathrm{FSMap}^{[g]}_{\ell_1,\ldots,\ell_n}.$$

The origin of the master relation

$\lambda \vdash d$. $\mathrm{Map}_\lambda^\bullet$ and $\mathrm{FSMap}_\lambda^\bullet$ generating series of possibly disconnected maps with boundary lengths given by λ and with weight $N^{\chi(\mathcal{M})}$.

Theorem (Borot–G-F, '17, Borot–Charbonnier–Do–G-F, '19)

$$\mathrm{FSMap}_\lambda^\bullet = z(\mu) \sum_{\lambda \vdash d} H^{\leq}(\lambda, \mu)\Big|_{\hbar = \frac{1}{N}} \mathrm{Map}_\mu^\bullet, \tag{1}$$

$$\mathrm{Map}_\lambda^\bullet = z(\lambda) \sum_{\mu \vdash d} H^{<}(\lambda, \mu)\Big|_{\hbar = \frac{1}{N}} \mathrm{FSMap}_\mu^\bullet. \tag{2}$$

3 proofs:

- Via matrix models: Express

$$\mathrm{FSMap}_\lambda^\bullet = \Big\langle \mathcal{P}_\lambda(A) \Big\rangle = \Big\langle \prod_{i=1}^{n} \mathcal{P}_{\gamma_i}(A) \Big\rangle = \Big\langle \int_{\mathcal{U}_N} \mathcal{P}_\lambda(UAU^{-1}) \mathrm{d}U \Big\rangle$$

  in terms of the $\Big\langle \prod_{i=1}^{n} \mathrm{Tr}\, M^{\lambda_i} \Big\rangle$, using Weingarten calculus.
- 2 combinatorial proofs $\rightsquigarrow$ 1 via bijective combinatorics.

Proof via bijective combinatorics (joint work with G. Borot, S. Charbonnier and N. Do)

Definition

Dessin d'enfant ⇝ map with each edge adjacent to one boundary face and one internal face. Boundary faces ⇝ blue faces and internal faces ⇝ red faces.

Proof via bijective combinatorics (joint work with G. Borot, S. Charbonnier and N. Do)

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$D_k(\lambda, \mu)$ ⇝ number of (possibly disconnected) dessins d'enfant with blue face degrees by λ and red face degrees by μ, and with k more edges than vertices.

$$D_k(\lambda, \mu) = z(\lambda) H_k^{<}(\lambda, \mu).$$

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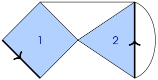
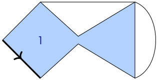
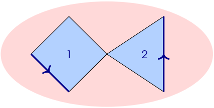


ordinary map

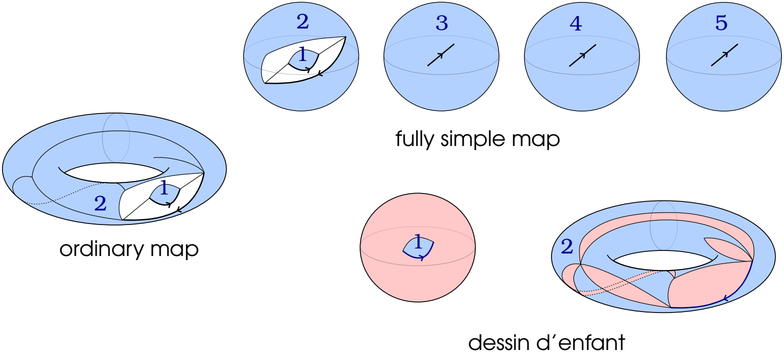


fully simple map



dessin d'enfant

Proof via bijective combinatorics (joint work with G. Borot, S. Charbonnier and N. Do)



Slogan: The fully simple map encodes the internal faces of the map while the dessin encodes how the boundaries of the map intersect.

Symplectic invariance

$$(\Sigma, (x, y)) \overset{\mathsf{TR}}{\rightsquigarrow} \omega_{g,n}(z_1, \ldots, z_n) \;\; (\omega_{g,0} = \mathfrak{F}_g \in \mathbb{C})$$

Symplectic invariance

$$\begin{tikzcd}
(\Sigma, (x, y)) \arrow[r, rightsquigarrow, "\mathrm{TR}"] \arrow[d, mapsto, "{\Phi \text{ preserving } |\mathrm{d}x \wedge \mathrm{d}y|}"'] & \omega_{g,n}(z_1, \ldots, z_n) \ (\omega_{g,0} = \mathfrak{F}_g \in \mathbb{C}) \\
(\Sigma, (\check{x}, \check{y})) &
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$\Phi=\mathcal{E}\colon (x,y)\mapsto(y,x)$

not well understood.

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Let $x(z) = \alpha + \gamma(z + \frac{1}{z})$.

Theorem (Eynard, '05)

$$\begin{tikzcd}
(\mathbb{CP}^1,(x,y=W_1^{[0]}(x)),\omega_{0,2}=B) \arrow[d, rightsquigarrow, "\mathrm{TR}"] \\
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\end{tikzcd}$$

$$\forall 2g-2+n>0,\ z_i\to\infty.$$

Maps

$$\xleftrightarrow{\mathcal{E}}$$

Symplectic invariance

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(\Sigma,(x,y)) \arrow[r, rightsquigarrow, "\text{TR}"] \arrow[d, mapsto, "{\begin{array}{r}\Phi\\ \text{preserving}\\ |\mathrm{d}x\wedge\mathrm{d}y|\end{array}}"'] & \omega_{g,n}(z_1,\ldots,z_n)\ (\omega_{g,0}=\mathfrak{F}_g\in\mathbb{C}) \arrow[d, equal, "?"] \\
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Theorem (Borot–Charbonnier–G-F, '21)

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(\mathbb{CP}^1,(x,y=W_1^{[0]}(x)),\omega_{0,2}=B) \arrow[rr, leftrightarrow, "{(0,1)\text{ et }(0,2)\ \checkmark}"] \arrow[d, rightsquigarrow, "\text{TR}"] & & (\mathbb{CP}^1,(y,x),\omega_{0,2}=B) \arrow[d, rightsquigarrow, "\text{TR}"] \\
\begin{array}{c}\frac{\omega_{g,n}(z_1,\ldots,z_n)}{\mathrm{d}x_1\cdots\mathrm{d}x_n}=W_n^{[g]}(x_1,\ldots,x_n),\\ \forall 2g-2+n>0,\ z_i\to\infty.\end{array} \arrow[rr, leftrightarrow, "\mathcal{E}"] & & \begin{array}{c}\frac{\check{\omega}_{g,n}(z_1,\ldots,z_n)}{\mathrm{d}y_1\cdots\mathrm{d}y_n}=X_n^{[g]}(y_1,\ldots,y_n),\\ \forall 2g-2+n>0,\ z_i\to\infty.\end{array}
\end{tikzcd}$$

Maps

Fully simple maps

- Our proof (Borot–Charbonnier–G-F, '21): combinatorial, via ciliated maps.
- Proof by Bychkov–Dunin-Barkowsi–Kazarian–Shadrin, '21: via Fock space formalism (x replaced by $1/x$, as later).

Outline

1. A triple duality: symplectic, simple and free
2. Master relation: a universal duality?
   - Monotone Hurwitz numbers
3. Origins of the master relation
   - Combinatorial maps and matrix models
   - From maps to free probability via matrix models
   - The origin of the master relation
   - Topological recursion and symplectic invariance
4. Surfaced free probability
   - Higher order free cumulants
   - Open question
   - First and second orders
   - Surfaced free cumulants (of topology (g, n))
5. Moment-free cumulant relations: $M = G_{0,n} \leftrightarrow G^{\vee}_{0,n} = C$
   - Main result
6. Future and ongoing work
7. Bonus: tower of constellations
   - Constellations

Partitioned permutations (Collins, Mingo, Śniady, Speicher ,'06)

Partitioned permutations: $(\mathcal{U},\gamma) \in PS(d), \mathcal{U} \in P(d), \gamma \in S(d), \mathcal{U} \geq \mathbf{0}_\gamma$.

$$|(\mathcal{U},\gamma)| := d + \#\mathrm{cyc}(\gamma) - 2\#\mathrm{blocks}(\mathcal{U}) \geq 0, \quad |(\mathbf{0}_{\mathrm{id}},\mathrm{id})| = d + d - 2d = 0.$$

Example: $\mathcal{U} = \{\{1,2,3,4,5\},\{6,7,8,9\}\}, \gamma = (1,2,3)(4,5)(6,7,8)$.

Partitioned permutations (Collins, Mingo, Śniady, Speicher ,'06)

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Convolution: $f, g\colon PS \to \mathbb{C}$

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Delta function:

$$\delta(\mathcal{A},\alpha) = \begin{cases}1 & \text{if } \mathcal{A}=\mathbf{0}_{\mathrm{id}} \text{ and } \alpha=\mathrm{id}\,,\\ 0 & \text{otherwise}\,.\end{cases}$$

Zeta function:

$$\zeta(\mathcal{A},\alpha) := \begin{cases}1 & \text{if } \mathcal{A}=\mathbf{0}_\alpha\,,\\ 0 & \text{otherwise}\,.\end{cases}$$

Möbius function: $\exists!\mu\colon PS(d)\to\mathbb{C}$ such that $\mu*\zeta=\zeta*\mu=\delta$.

The open problem

$f : PS \to \mathbb{C}$ multiplicative function (i.e. $f(1_d, \gamma)$ depends only on the conjugacy class of γ and $f(\mathcal{U}, \gamma) = \prod_{U \in \mathcal{U}} f(1_U, \gamma|_U)$).

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Encode $\varphi_{\ell_1,\dots,\ell_n}$ and $\kappa_{\ell_1,\dots,\ell_n}$ into the generating series:

$$n = 1 : \quad M(x) := 1 + \sum_{\ell \geq 1} \varphi_\ell x^\ell, \quad C(w) := 1 + \sum_{\ell \geq 1} \kappa_\ell w^\ell.$$

Higher order:

$$M_n(x_1, \dots, x_n) := \sum_{\ell_1,\dots,\ell_n \geq 1} \varphi_{\ell_1,\dots,\ell_n} x_1^{\ell_1} \dots x_n^{\ell_n},$$

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Question: Functional relation between $M_n(x_1,\dots,x_n)$ and $C_n(w_1,\dots,w_n)$?

First and second orders

R-transform machinery:

- $n = 1$: (Voiculescu,'86)

  $$C(xM(x)) = M.$$

  Originally: Relation between the R-transform $R(w)$ and the Stieltjes transform $W(x)$, $C(w) = 1 + wR(w)$ and $W(x) = x^{-1}M(x^{-1})$.
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$\mathbf{n} = \mathbf{1}, \mathbf{2}$: (Borot, G-F ,'17) from combinatorics of fully simple maps.
$\mathbf{n} = \mathbf{3}$: (Borot, Charbonnier, G-F ,'21) for specific unitary invariant hermitian matrix models, from topological recursion.

Higher order probability space $(\mathcal{A}, \varphi)$ and free cumulants κ

$\mathcal{A}$ algebra, $\varphi = (\varphi_n)_{n \geq 1}$ moments, with $\varphi_n \colon \mathcal{A}^n \to \mathbb{C}$ linear.
Decorate PS with $\mathcal{A}$: $PS(\mathcal{A}) := \bigcup_{d \geq 0} PS(d) \times \mathcal{A}^d$.
For $1 \leq j \leq n$, set $L_j = \sum_{i=1}^{j} \ell_i$. Moments are multiplicative functions:

$$\varphi(1_{\ell_1+\ldots+\ell_n, \gamma_1 \cdots \gamma_n})[a_1, \ldots, a_{\ell_1+\ldots+\ell_n}] := \varphi_n(a_1 \cdots a_{\ell_1}, \ldots, a_{L_{n-1}+1} \cdots a_{L_n})$$

Free cumulants:

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Definition (Higher order freeness)

$(\mathcal{A}_i)_{i\in I}$ are free if $\kappa(1_n,\pi)[a_1,\ldots,a_d]=0$, $\forall\pi\in S(d)$ whenever $\exists i(p)\neq i(q)$ such that $a_p\in\mathcal{A}_{i(p)}$ and $a_q\in\mathcal{A}_{i(q)}$.

If $\varphi_n=0$ for $n\geq 2$: recover first order freeness.
As classical cumulants linearise adding independent variables, free cumulants linearise adding free variables: If $a,b\in\mathcal{A}$ are free,

$$\kappa(1_{|\lambda|},\gamma)[a+b,\ldots,a+b]=\kappa(1_{|\lambda|},\gamma)[a,\ldots,a]+\kappa(1_{|\lambda|},\gamma)[b,\ldots,b],$$

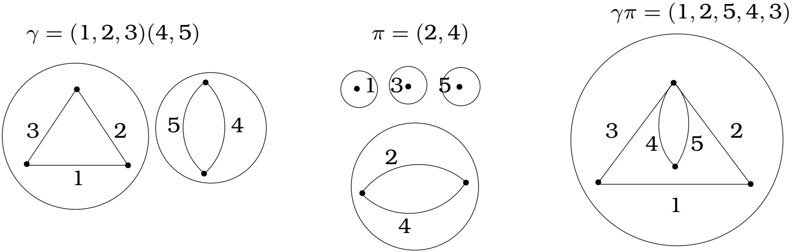
for $\lambda\vdash d$ and $\gamma\in C_\lambda$.

Surfaced free probability

Extended multiplication on partioned permutations:

$$(\mathcal{U},\gamma)\odot(\mathcal{V},\pi) := (\mathcal{U}\vee\mathcal{V},\gamma\circ\pi).$$

(Can also be understood as multiplication on surfaced permutations).



$$|(\mathbf{0}_\gamma,\gamma)|+|(\mathbf{0}_\pi,\pi)| = 5+2-2\cdot 2+5+4-2\cdot 4 = 3+1 = 4 = 5+1-2 = |(\mathbf{0}_{\gamma\pi},\gamma\pi)|.$$

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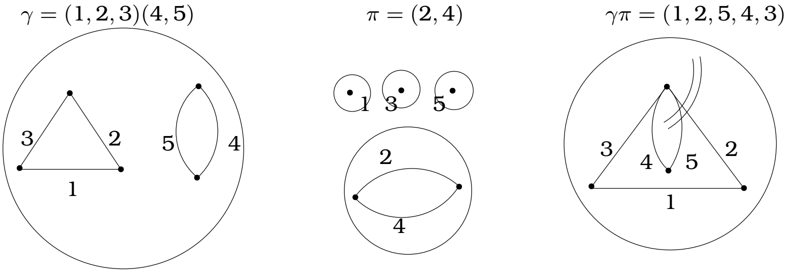

$$|(\mathbf{0}_\gamma,\gamma)|+|(\mathbf{0}_\pi,\pi)| = 5+2-2\cdot 1+5+4-2\cdot 4 = 5+1 = 6 \neq 4 = 5+1-2 = |(\mathbf{0}_{\gamma\pi},\gamma\pi)|.$$

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Extended convolution:

$$(f_1 \circledast f_2)(\mathcal{C},\gamma) := \sum_{(\mathcal{A},\alpha)\odot(\mathcal{B},\beta)=(\mathcal{C},\gamma)} f_1(\mathcal{A},\alpha) f_2(\mathcal{B},\beta)\,.$$

Extended zeta function:

$$\zeta_\hbar(\mathcal{A},\alpha) := \hbar^{|\alpha|}\zeta(\mathcal{A},\alpha), \quad |\alpha| = d - \#\mathbf{0}_\alpha\,.$$

Extended Möbius function $\mu_\hbar \colon PS(d) \to \mathbb{C}[\![\hbar]\!]$ uniquely determined by

$$\mu_\hbar \circledast \zeta_\hbar = \zeta_\hbar \circledast \mu_\hbar = \delta\,.$$

$\Rightarrow$ Notion of (g,n)-freeness.

Theorem (Borot, Charbonnier, Leid, Shadrin, G-F, '21)

$(A_N)_N, (B_N)_N$ ensembles of random matrices of size N, $(A_N)_N$ unitarily invariant, A_N independent of B_N. If $A_N \to a$, $B_N \to b$, when $N \to \infty$, up to order (g_0, n_0), then a and b are (g_0, n_0)-free.

Generalises (Voiculescu,'91) (first order freeness); corrections of order $N^{-2g_0-n_0}$.

Outline

1. A triple duality: symplectic, simple and free
2. Master relation: a universal duality?
   - Monotone Hurwitz numbers
3. Origins of the master relation
   - Combinatorial maps and matrix models
   - From maps to free probability via matrix models
   - The origin of the master relation
   - Topological recursion and symplectic invariance
4. Surfaced free probability
   - Higher order free cumulants
   - Open question
   - First and second orders
   - Surfaced free cumulants (of topology (g, n))
5. Moment-free cumulant relations: $M = G_{0,n} \leftrightarrow G^{\vee}_{0,n} = C$
   - Main result
6. Future and ongoing work
7. Bonus: tower of constellations
   - Constellations

Moment-free cumulant functional relations

- $\mathcal{G}_{0,n}(\mathbf{r}+1)$: set of bicoloured trees with white vertices labeled from 1 to n having valency $r_1+1,\ldots,r_n+1$, and without univalent black vertices.

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Theorem (Borot, Charbonnier, Leid, Shadrin, G-F, '21)

Let $x_i = w_i/C(w_i)$. *For* $n \geq 3$,

$$M_n(x_1,\ldots,x_n) = \sum_{r_1,\ldots,r_n\geq 0}\ \sum_{T\in\mathcal{G}_{0,n}(\mathbf{r}+1)} \left(\prod_{i=1}^{n} \mathrm{O}_{r_i}(w_i)\right) \prod_{I\in\mathcal{I}(T)}^{\prime} C_{\#I}(w_I)\,.$$

- Weight per tree: $\mathcal{W}(T) := \prod'_{I\in\mathcal{I}(T)} C_{\#I}(w_I)$.
- $\prod' \rightsquigarrow C_2(w_i,w_j)$ should be replaced with $C_2(w_i,w_j) + \frac{w_iw_j}{(w_i-w_j)^2}$, if $i\neq j$.

Set of bicolored graphs

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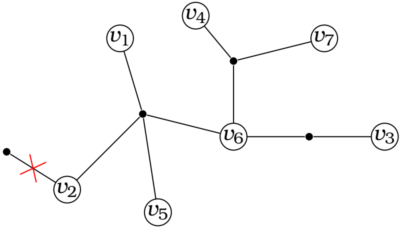
Example: n=7



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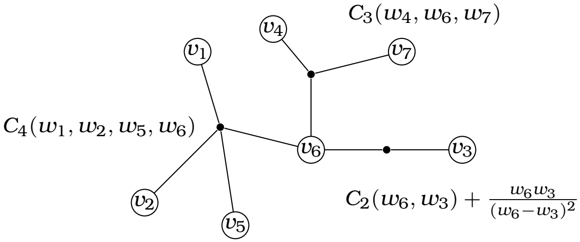
Example: $T\in\mathcal{G}_{0,7}(1,1,1,1,1,3,1)$



$$\mathcal{W}(T)=C_4(w_1,w_2,w_5,w_6)C_3(w_4,w_6,w_7)\Big(C_2(w_6,w_3)+\frac{w_6w_3}{(w_6-w_3)^2}\Big).$$

Finite sums and example

- $\mathcal{G}_{0,n}(\mathbf{r}+1)$: set of bicoloured trees with white vertices labeled from 1 to n having valency $r_1+1,\ldots,r_n+1$, and without univalent black vertices.

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For n fixed, $\mathcal{G}_{0,n}(\mathbf{r}+1)\neq\emptyset$ only for finitely many $\mathbf{r}=(r_1,\ldots,r_n)\in\mathbb{N}^n$.

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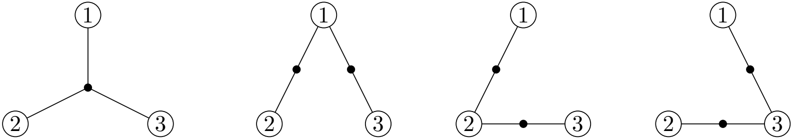
$$T_0=T_{0,0,0},\qquad T^{(1)}=T_{1,0,0},\qquad T^{(2)}=T_{0,1,0},\qquad T^{(3)}=T_{0,0,1}.$$



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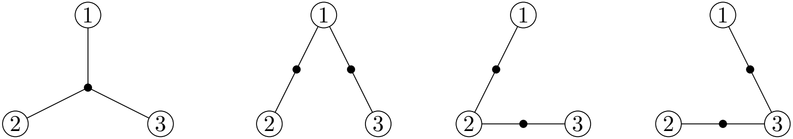

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Finite sums and example

- $\mathcal{G}_{0,n}(\mathbf{r}+1)$: set of bicoloured trees with white vertices labeled from 1 to n having valency $r_1+1,\dots,r_n+1$, and without univalent black vertices.

Remark

For n fixed, $\mathcal{G}_{0,n}(\mathbf{r}+1)\neq\emptyset$ only for finitely many $\mathbf{r}=(r_1,\dots,r_n)\in\mathbb{N}^n$.

Ex: n=3 $\rightsquigarrow$ $\mathcal{G}_{0,3}(\mathbf{r}+1)\neq\emptyset$ only for $\mathbf{r}\in\{(0,0,0),(1,0,0),(0,1,0),(0,0,1)\}$, and there is only one $T_{\mathbf{r}}\in\mathcal{G}_{0,3}(\mathbf{r}+1)$ for each of these $\mathbf{r}$:

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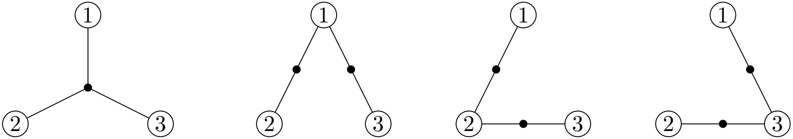

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Only terms with $m\leq r$ give contribution $\neq 0$ to $\mathrm{O}_r(w)$.

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Remarks ⇒ The sums of the RHS of

$$M_n(x_1,\ldots,x_n) = \sum_{r_1,\ldots,r_n\geq 0}\ \sum_{T\in\mathcal{G}_{0,n}(\mathbf{r}+1)} \left(\prod_{i=1}^{n}\mathrm{O}_{r_i}(w_i)\right) \prod_{I\in\mathcal{I}(T)}{}^{\prime}\, C_{\#I}(w_I)$$

are finite.

Example: $n = 3$

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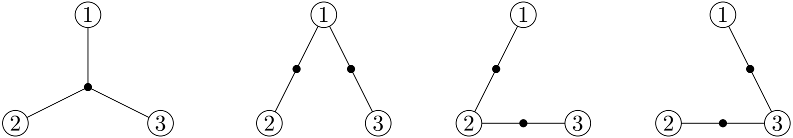

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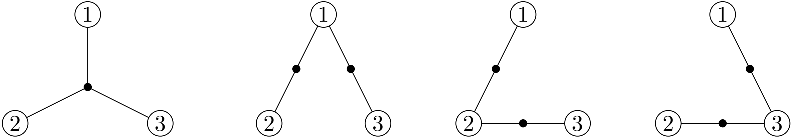

$$\begin{aligned}
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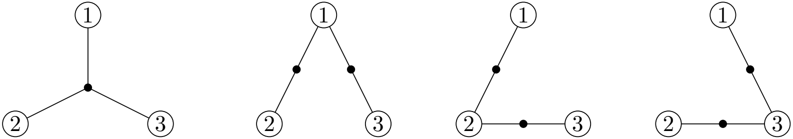

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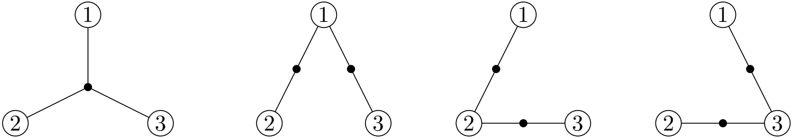

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$$M_3(x_1, x_2, x_3) = \frac{1}{\prod_{i=1}^{3} C(w_i)x'(w_i)}\left(\mathcal{W}(T_0) + \sum_{i=1}^{3} w_i \partial_{w_i} \frac{\mathcal{W}(T^{(i)})}{C(w_i)^2 x'(w_i)}\right),$$

with $\mathcal{W}(T^{(i)}) = \prod_{j \neq i}\left(C_2(w_i, w_j) + \frac{w_i w_j}{(w_i - w_j)^2}\right)$.

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To prove

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Idea of proof:

$$\begin{tikzcd} & \Phi_{Z,\hbar} = \zeta_\hbar \circledast \Phi_{Z^\vee,\hbar} \\ Z(\lambda) = z(\lambda)\sum_{\nu\vdash d} H^{<}(\lambda,\nu)Z^\vee(\nu) \arrow[ur, Leftrightarrow, "①"] & \end{tikzcd}$$

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Outline

1. A triple duality: symplectic, simple and free
2. Master relation: a universal duality?
    - Monotone Hurwitz numbers
3. Origins of the master relation
    - Combinatorial maps and matrix models
    - From maps to free probability via matrix models
    - The origin of the master relation
    - Topological recursion and symplectic invariance
4. Surfaced free probability
    - Higher order free cumulants
    - Open question
    - First and second orders
    - Surfaced free cumulants (of topology (g, n))
5. Moment-free cumulant relations: $M = G_{0,n} \leftrightarrow G^{\vee}_{0,n} = C$
    - Main result
6. Future and ongoing work
7. Bonus: tower of constellations
    - Constellations

Questions: future and ongoing work

- Master relation simplifies maps; for constellations it forgets one color (from $(m+1)$-constellations to m**-constellations**). Studying these towers of problems related by the master relation (also from TR and free probability). Other **meaningful towers**?

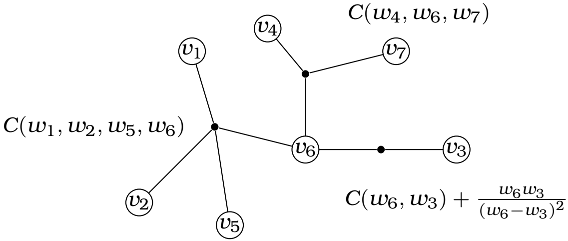
- Further consequences in **free probability**? From the work of Arizmendi, Leid, Speicher, in free probability the master relation can be realised by conjugating with a free circular element c. This explains the tower of constellations in that context. Is that phenomenon still true for higher genus moments and free cumulants (moments of a are cumulants of cac^*, if a and c are free of all orders)?

- **Symplectic invariance** of TR? **Theorem:** (Alexandrov, Bychkov, Dunin-Barkowski, Kazarian, Shadrin) If we have TR for $G_{g,n}$, we have TR for $G^{\vee}_{g,n}$ with a **symplectically transformed spectral curve**. (Hock) Laplace transform of the duality relation (suitable in the **quantum curves** setting).

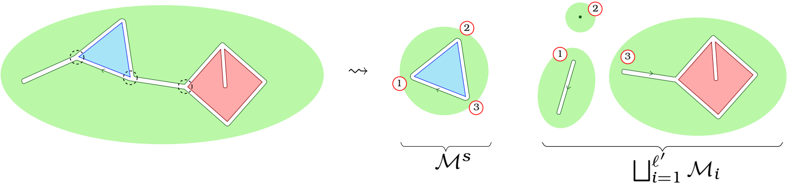
- Extend to the **orthogonal/real** symmetric setting.

- **Combinatorial proof** of the functional relations? Ongoing work of Lionni.

- Relation to ongoing work of Zuber on counting **partitions of genus** g?



ご清聴ありがとうございました

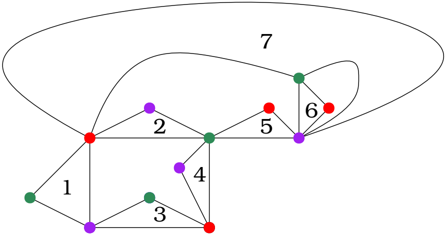


Outline

Constellations

m-constellation ($m \geq 2$):

1. faces coloured in black and white and only faces of different colour can be adjacent;
2. black faces are of degree m (hyperedges) and white faces are or degree multiple of m;
3. $\exists$ a coloring of the vertices in $\{1, \ldots, m\}$ such that around every black face the vertices are of colours $1, 2, \ldots, m$ clockwise.



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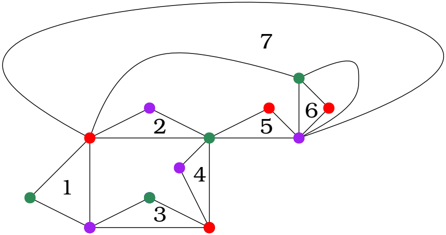

Can be encoded by $m+1$ permutations $\sigma_0, \ldots, \sigma_m$ (acting on hyperedges) such that $\sigma_0 = \sigma_1 \cdots \sigma_m$, where

- σ_i, $i = 1, \ldots, m \rightsquigarrow$ hyperedges around the vertices of colour i;
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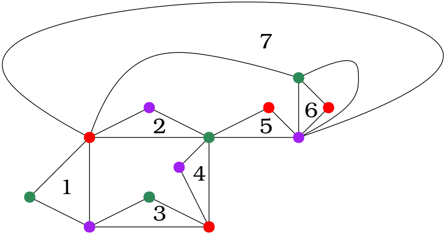

$$\sigma_1 = (13)(2)(4)(576)$$
$$\sigma_2 = (1)(245)(3)(67)$$
$$\sigma_3 = (127)(34)(5)(6)$$
$$\sigma_0 = \sigma_1\sigma_2\sigma_3 = (14)(25)(73)(6)$$

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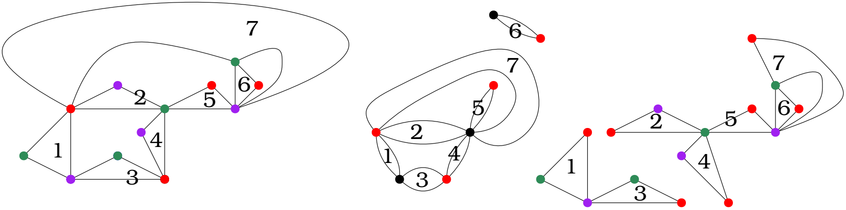
$\circledast\mu_\hbar$ = simplifying one constellation = forgetting one colour

Master relation for constellations ⇝ bijection:

$(m+1)$-constellation $\mapsto$ (dessin, “simple”$(m+1)$-constellation $= m$-constellation)

$\sigma_0, \sigma_1, \dots, \sigma_{m+1}$ s.th. $\sigma_0 = \sigma_1 \cdots \sigma_{m+1}$ $\mapsto$ $((\sigma_0, \sigma_1 \cdots \sigma_m, \sigma_{m+1}),$ s.th. $\sigma_0 = (\sigma_1 \cdots \sigma_m)\sigma_{m+1}$ $(\sigma_0\sigma_{m+1}^{-1}, \sigma_1, \dots, \sigma_m))$ s.th. $\sigma_0\sigma_{m+1}^{-1} = \sigma_1 \cdots \sigma_m$



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$\sigma_0\sigma_3^{-1} = \sigma_1\sigma_2 = (13)(2475)(6)$

Simplify the last colour of the $(m+1)$-constellation (red). Dessin ⇝ information about the colour $m+1$; m-constellation ⇝ the other m colours.