Resurgence in quantum field theory

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Resurgent structure in integral

Resurgent structure in integral

In integral, original contour decomposes into **steepest decent contours** (Lefschetz thimbles) associated with **complex saddles**

⇨ **Thimbles associated with distinct saddles have nontrivial relation via Stokes phenomena**

· **Airy integral**

$$\mathrm{Ai}(g^{-2}) = \int_{-\infty}^{\infty} d\phi \exp\left[-i\left(\frac{\phi^3}{3} + \frac{\phi}{g^2}\right)\right]$$

Resurgent structure in integral

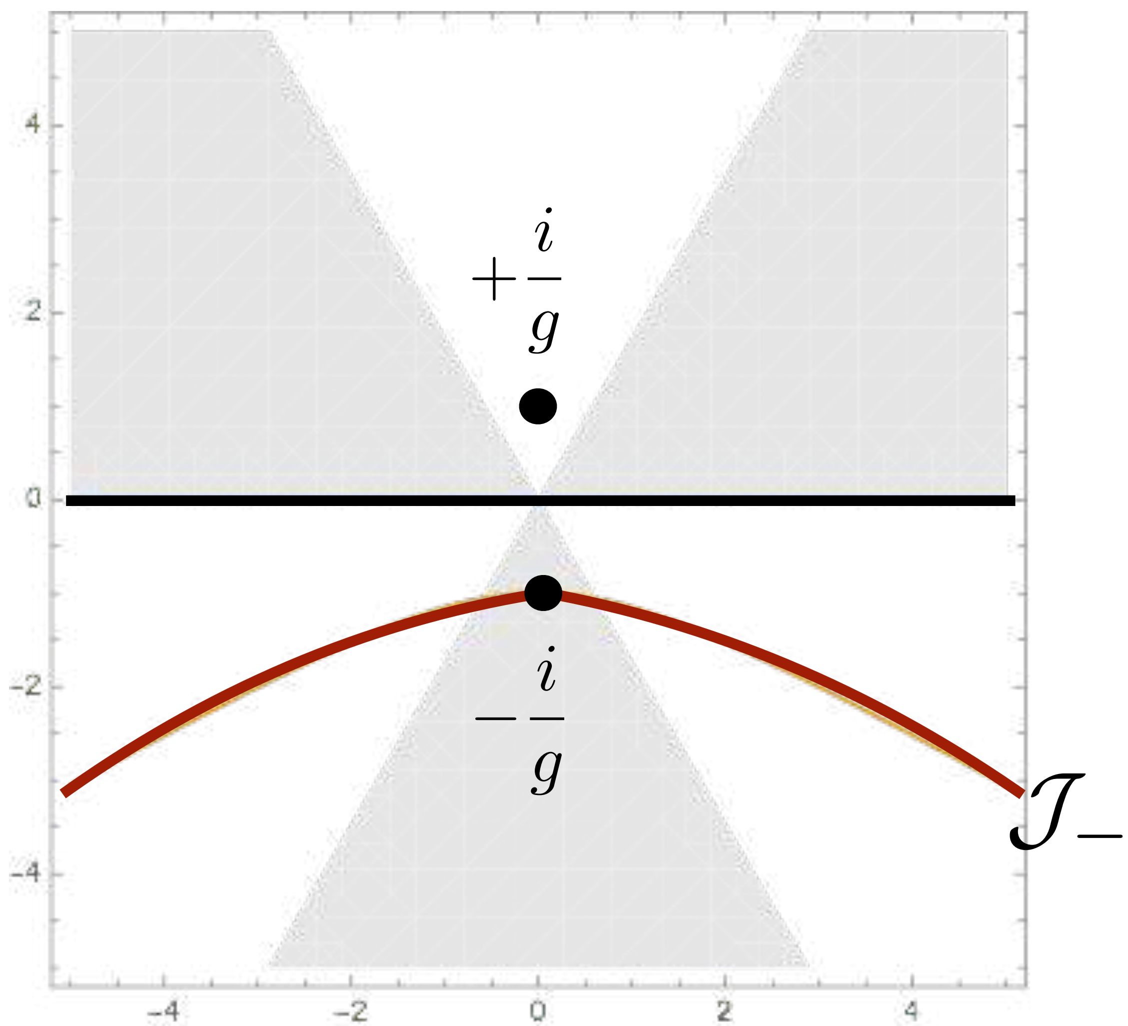
Complex saddle contributions in thimble decomposition
(Steepest descent method)

- **Airy integral**

$$\mathrm{Ai}(g^{-2}) = \int_{-\infty}^{\infty} d\phi \exp\left[-i\left(\frac{\phi^3}{3} + \frac{\phi}{g^2}\right)\right]$$

- $\mathcal{J}_\sigma$ $\quad \mathrm{Im}[S] = \mathrm{Im}[S_0]$, $\mathrm{Re}[S] \leq \mathrm{Re}[S_0]$ **Thimble**
- $n_\sigma = \langle \mathcal{K}_\sigma, \mathcal{C} \rangle$ **Intersection number of dual thimble $\mathcal{K}$ and original contour**

$$\Rightarrow \quad \mathcal{C} = \sum_\sigma n_\sigma \mathcal{J}_\sigma$$



$\arg[g^2] = 0+$

Resurgent structure in integral

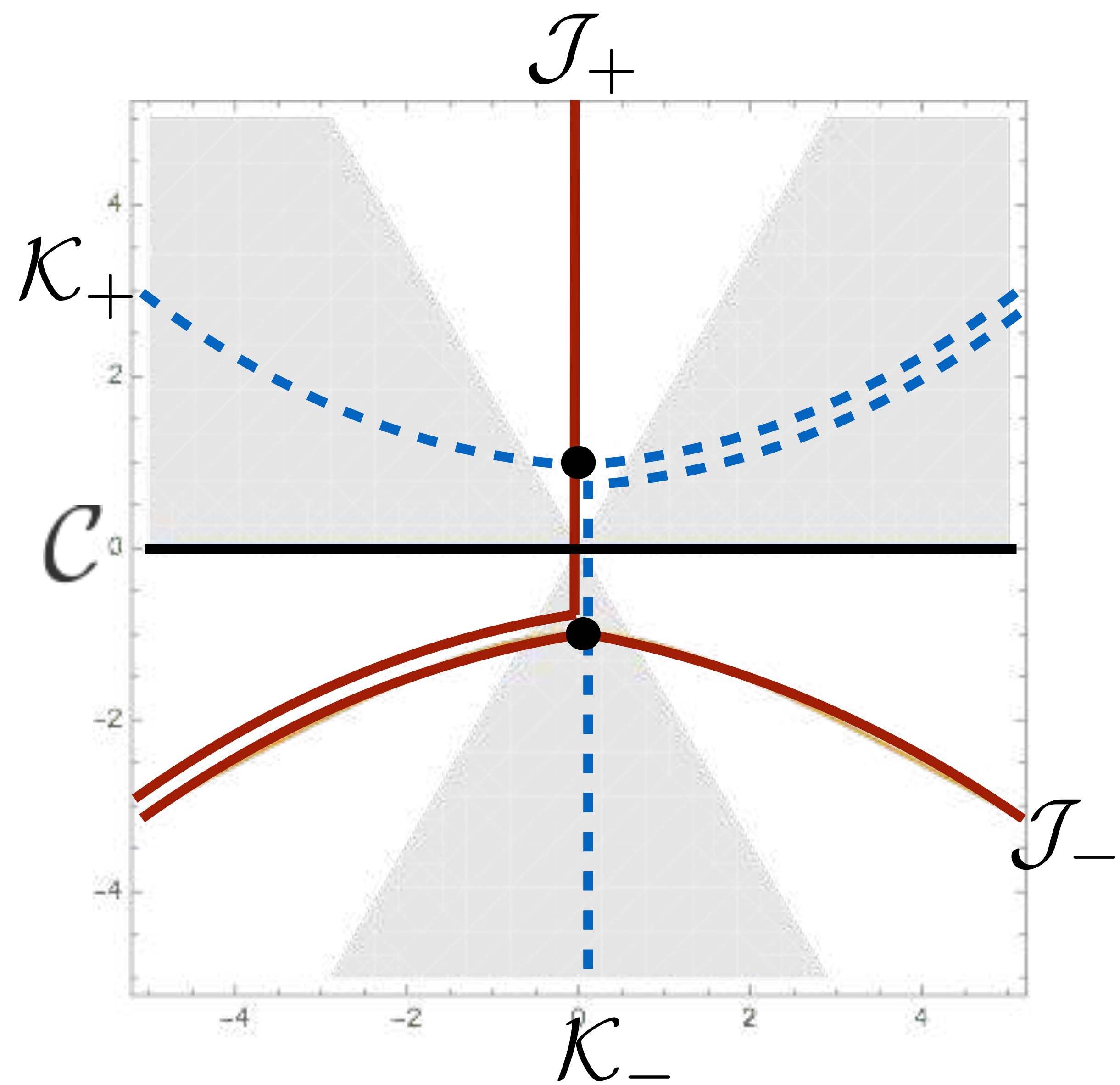
Complex saddle contributions in thimble decomposition (Steepest descent method)

- **Airy integral** $\arg[g^2] = 0+$

$$n_+ = \langle \mathcal{K}_+, \mathcal{C} \rangle = 0$$
$$n_- = \langle \mathcal{K}_-, \mathcal{C} \rangle = 1$$

$$\mathcal{C} = \sum_\sigma n_\sigma \mathcal{J}_\sigma \Rightarrow \boxed{\mathcal{C} = \mathcal{J}_-}$$

valid decomposition till $\arg[g^2] = \frac{2\pi}{3}-$



$\arg[g^2] = 0+$

Resurgent structure in integral

Complex saddle contributions in thimble decomposition
(Steepest descent method)

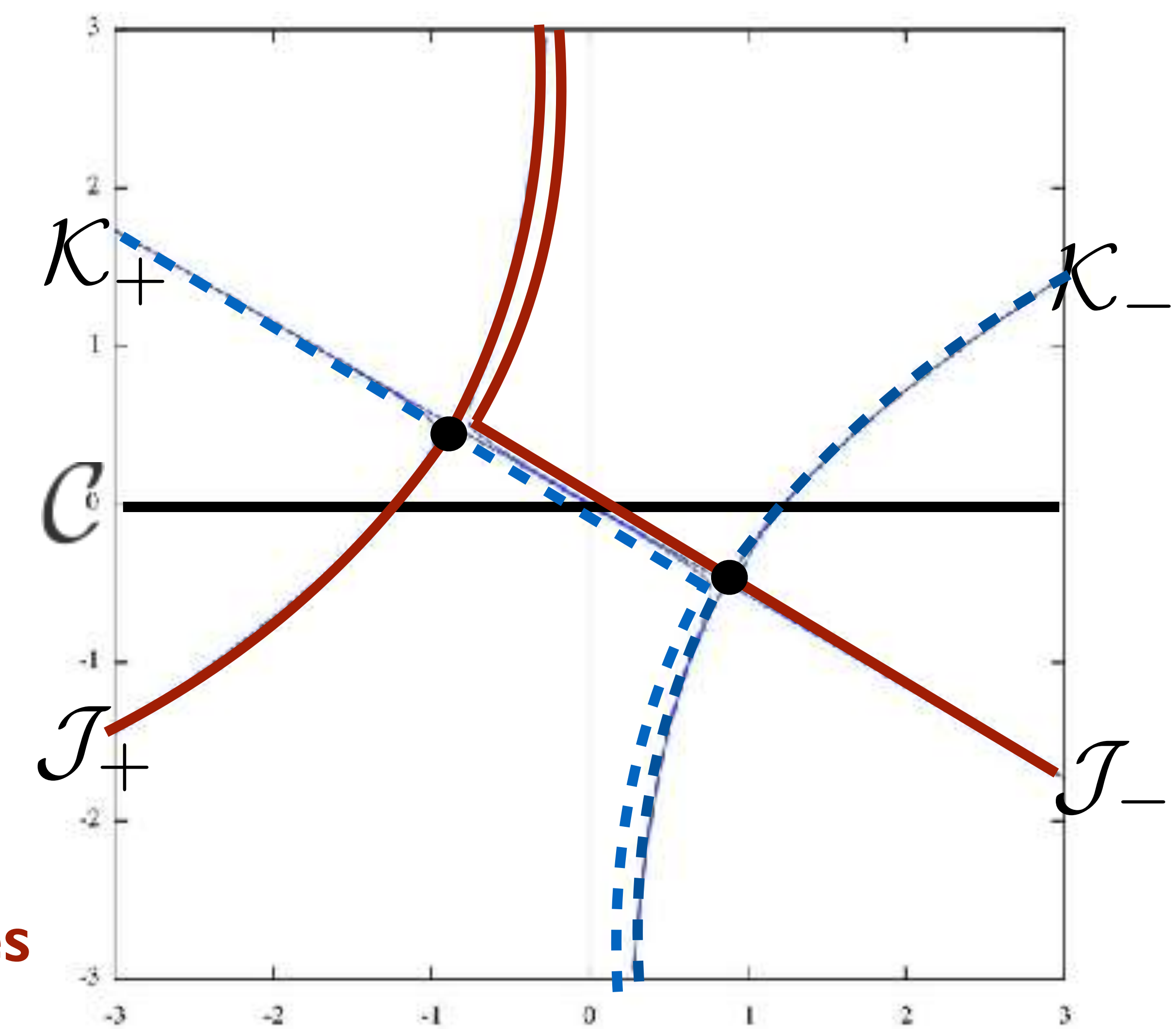
- **Airy integral** $\qquad \arg[g^2] = \frac{2\pi}{3} +$

$$n_+ = \langle \mathcal{K}_+, \mathcal{C} \rangle = 1$$

$$n_- = \langle \mathcal{K}_-, \mathcal{C} \rangle = 1$$

$$\mathcal{C} = \sum_\sigma n_\sigma \mathcal{J}_\sigma \Rightarrow \boxed{\mathcal{C} = \mathcal{J}_- + \mathcal{J}_+}$$

Stokes phenomenon : at special arg[g^2], thimble decomposition discretely changes



$$\arg[g^2] = \frac{2\pi}{3} +$$

Resurgent structure in integral

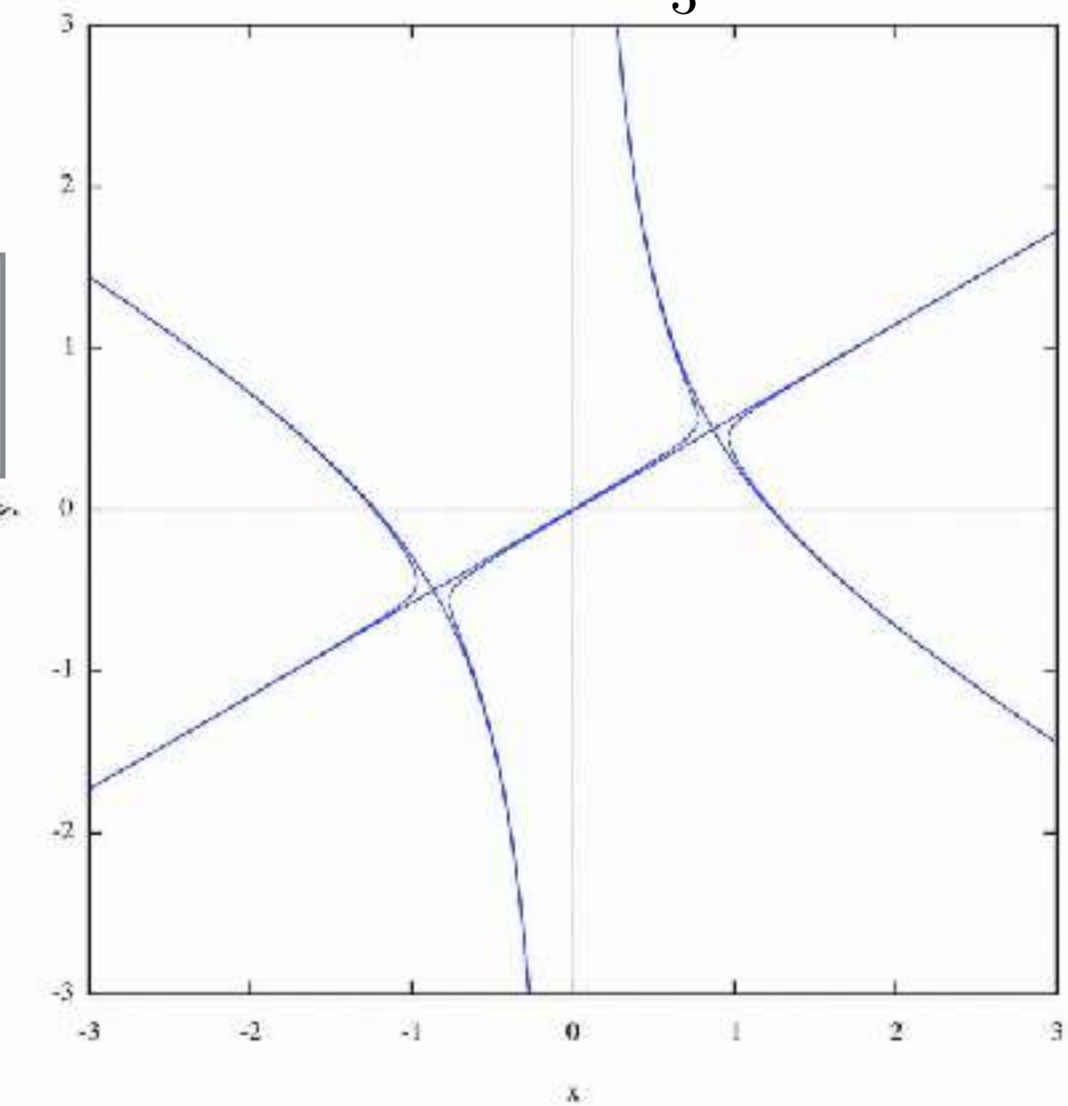
Complex saddle contributions in thimble decomposition (Steepest descent method)

- Airy integral

$$\arg[g^2] = \frac{2\pi}{3}^{-} \qquad \arg[g^2] = \frac{2\pi}{3}^{+}$$

$$\boxed{\mathcal{C} = \mathcal{J}_-} \Longleftrightarrow \boxed{\mathcal{C} = \mathcal{J}_- + \mathcal{J}_+}$$

$\arg[g^2] = -\frac{2\pi}{3} \to \pi$



✻ Thimble decomposition is discretely changed at Stokes line.

✻ Airy function is continuous even at the Stokes line.

$$\mathcal{J}_-\left[\frac{2\pi}{3}^{-}\right] = \mathcal{J}_-\left[\frac{2\pi}{3}^{+}\right] + \mathcal{J}_+$$

Resurgent structure in integral

Complex saddle contributions in thimble decomposition (Steepest descent method)

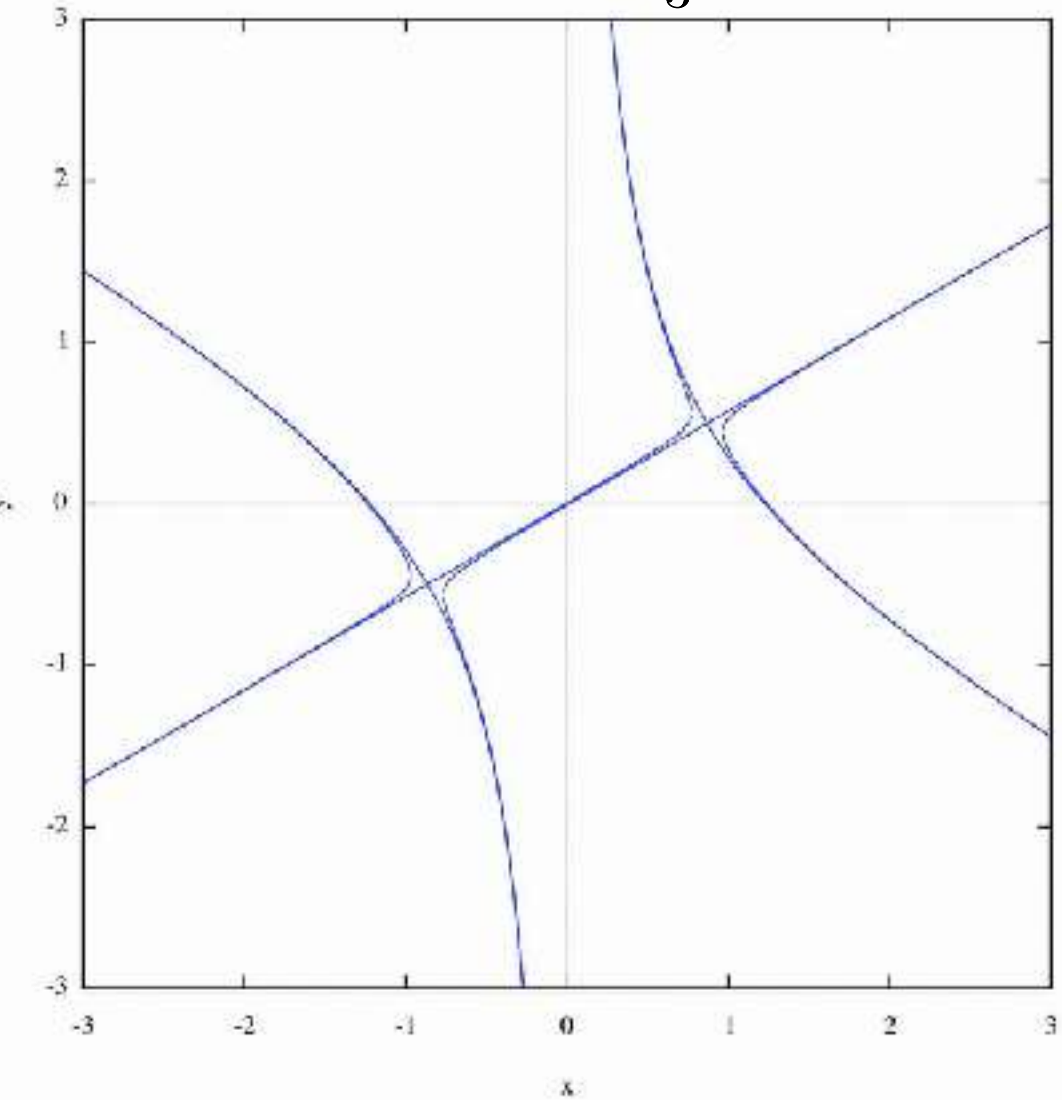
- Airy integral

$$\arg[g^2] = \frac{2\pi}{3}-$$

$$\arg[g^2] = \frac{2\pi}{3}+$$

$$\boxed{\mathcal{C} = \mathcal{J}_-} \Leftrightarrow \boxed{\mathcal{C} = \mathcal{J}_- + \mathcal{J}_+}$$

$$\arg[g^2] = -\frac{2\pi}{3} \to \pi$$



* Thimble decomposition is discretely changed at Stokes line.
* Airy function is continuous even at the Stokes line.

Two thimbles have resurgent relation via ambiguity due to Stokes phenomena !

Resurgent structure in quantum mechanics

Perturbation and Borel resummation

$$\left[H_0 + g^2 H_{\mathrm{pert}}\right]\psi(x) = E\psi(x)$$

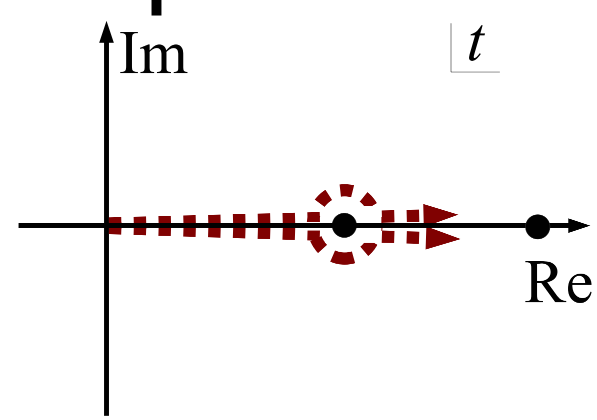
$$P(g^2) = \sum_{q=0}^{\infty} a_q g^{2q}$$

Perturbative series is often divergent factorially

$$a_q \propto q!$$

Borel transform can have singularities on positive real axis

$$BP(t) := \sum_{q=0}^{\infty} \frac{a_q}{q!} t^q$$



$$\Rightarrow \quad \mathbb{B}(g^2 e^{\mp i\epsilon}) = \int_0^{\infty e^{\pm i\epsilon}} \frac{dt}{g^2} e^{-\frac{t}{g^2}} BP(t)$$

Singularities on positive real axis leads to ambiguity

Perturbation and Borel resummation

$$\left[H_0 \;+\; g^2 H_{\rm pert}\right]\psi(x) \;=\; E\psi(x)$$

$$P(g^2) = \sum_{q=0}^{\infty} a_q g^{2q}$$ Perturbative series is often divergent factorially $a_q \propto q!$

$$\mathbb{B}(g^2 e^{\mp i\epsilon}) = \mathrm{Re}[\mathbb{B}(g^2)] \pm i\mathrm{Im}[\mathbb{B}(g^2)]$$

$$\mathrm{Im}[\mathbb{B}(g^2)] \;\approx\; e^{-\frac{A}{g^2}}$$ This should be cancelled by that from non-perturbative contribution!

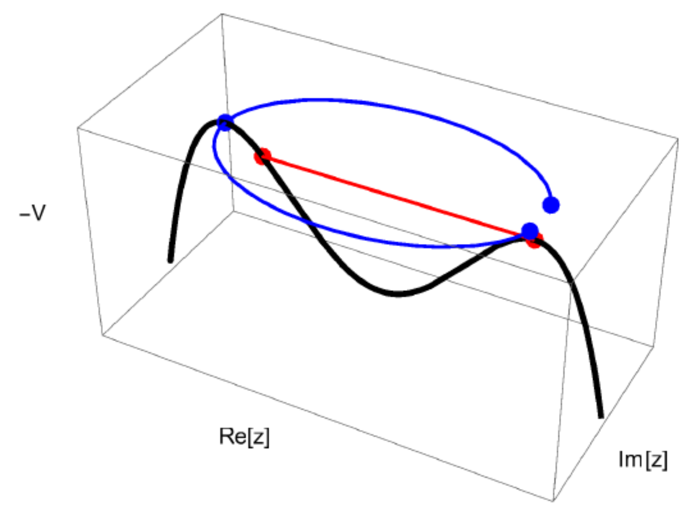
We can study non-perturbative effect in terms of perturbative Borel resummation and resurgent structure !

Complex bion solution as non-pert. contribution

Behtash, et.al. (15) Fujimori, et.al. (16)(17)

ex.) double-well QM

$$x \rightarrow z = x + iy$$



$$\frac{d^2 z}{d\tau^2} = \frac{\partial V}{\partial z}$$

- Complex bion solutions

$$z_{cb}(\tau) = z_1 - \frac{(z_1 - z_T)}{2} \coth \frac{\omega \tau_0}{2} \left[\tanh \frac{\omega(\tau + \tau_0)}{2} - \tanh \frac{\omega(\tau - \tau_0)}{2} \right] \qquad z_T,\ \tau_0 \in \mathbb{C}$$

- Contribution from complex bion to E_0

$$\Rightarrow \quad E_{cb} = \frac{e^{-\frac{1}{3g^2}}}{\pi g^2} \left(\frac{g^2}{2} \right)^{\epsilon} \left[-\cos(\epsilon\pi)\Gamma(\epsilon) \pm \frac{i\pi}{\Gamma(1-\epsilon)} \right]$$

The imaginary ambiguity from bion cancels that from perturbative series

Exact-WKB tells us complete resurgent structure

Sueishi, Kamata, TM, Unsal, [arXiv:2103.06586]

- Exact-WKB leads to exact quantization condition.
- Fredholm det. & resolvent leads to Gutzwiller formula and partition function
- Maslow index is identified as intersection #
- We end up with complete trans-series including both pert. & non-pert.

$\left(-\frac{\hbar^2}{2}\frac{d^2}{dx^2}+V(x)\right)\psi(x)=E\psi(x)$
Schrödinger eq.

$\downarrow$ exact-WKB

$D(E)=0$
exact quantization condition

$\rightarrow$ $\frac{1}{2\pi i}\oint p dx \simeq (N+\frac{1}{2})+O(e^{-\frac{S}{\hbar}})$
generalized Bohr-Sommerfeld

$\downarrow$

$G(E)=-\frac{\partial}{\partial E}\log D(E)$

$\xrightarrow{\text{expand}}$ $G(E)\simeq \sum_n \sum_{p.p.o.} (-1)^n e^{in\oint p dx}$
Gutzwiller trace formula

$\downarrow$ inverse Laplace transform

$Z(\beta)=\frac{1}{2\pi i}\int_{\epsilon-i\infty}^{\epsilon+i\infty} G(E)e^{-\beta E}dE$

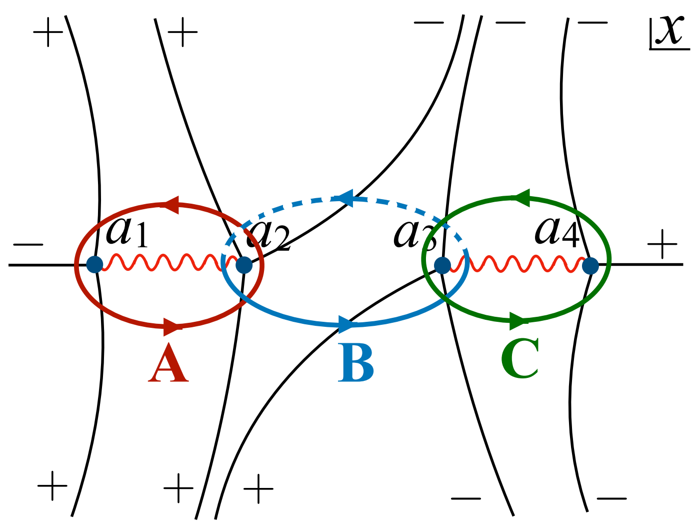
$\xrightarrow{\text{calculate the residues}}$ $Z(\beta)=\sum_n e^{-\beta E_n}$
spectral summation form

$\downarrow$ integral by parts and expand $\log D(E)$

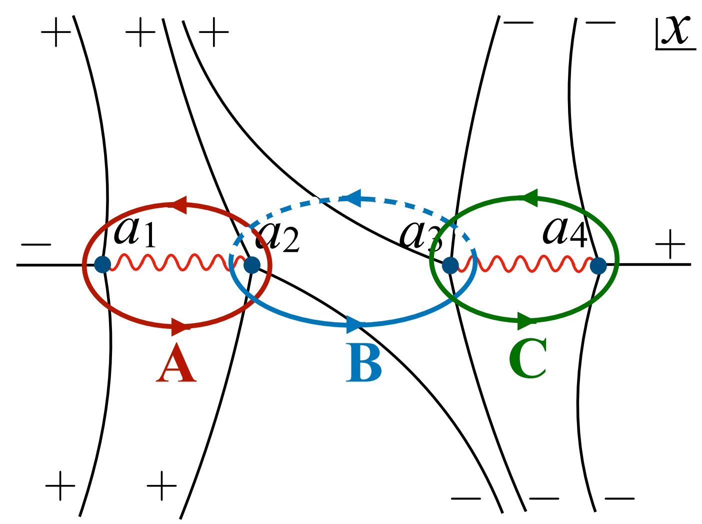
$Z(\beta)=\sum_n a_n\hbar^n+e^{-\frac{S_1}{\hbar}}\sum_n b_n\hbar^n+e^{-\frac{S_2}{\hbar}}\sum_n c_n\hbar^n+\ldots$
path integral(trans-series) form

Exact-WKB tells us complete resurgent structure

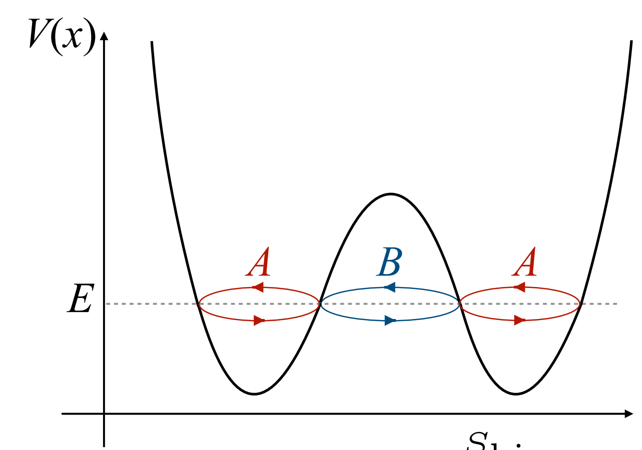
Sueishi, Kamata, TM, Unsal, [arXiv:2008.00379]



$\mathrm{Im}\,\hbar > 0$



$\mathrm{Im}\,\hbar < 0$



$$B \propto e^{-\frac{S_{\mathrm{bion}}}{\hbar}}$$

$$A = e^{\oint_A S_{\mathrm{odd}}}, \qquad B = e^{\oint_B S_{\mathrm{odd}}}, \qquad C = e^{\oint_C S_{\mathrm{odd}}} = 1/A$$

- Normalization condition in $x \longrightarrow -\infty$ gives quantization condition among cycles

$$D \propto \begin{cases} (1+A^+)(1+C^+) + A^+B^+ = 0 & \text{for } \ \mathrm{Im}\,\hbar > 0 \\ (1+A^-)(1+C^-) + C^-B^- = 0 & \text{for } \ \mathrm{Im}\,\hbar < 0 \end{cases}$$

corresponding to Stokes phenomena in perturbative & semiclassical study

- leads to trans-series-form partition function, and resurgent relation among ***A*(pert.) and *B*(bion non-pert.)** : DDP formula Delabaere, Dillinger, Pham (97)

$$Z_{\mathrm{p}}(\beta) = \frac{1}{2\pi i}\int_{\epsilon-i\infty}^{\epsilon+i\infty}\left[-\frac{\partial}{\partial E}\log(1+A)\right]e^{-\beta E}dE + \ (A \rightarrow A^{-1})$$

$$Z_{\mathrm{np}}(\beta) = \ \beta\frac{1}{2\pi i}\int_{\epsilon-i\infty}^{\epsilon+i\infty}\sum_{n=1}^{\infty}\frac{1}{n}\left(-\frac{B}{D_A^2}\right)^n e^{-\beta E}dE$$

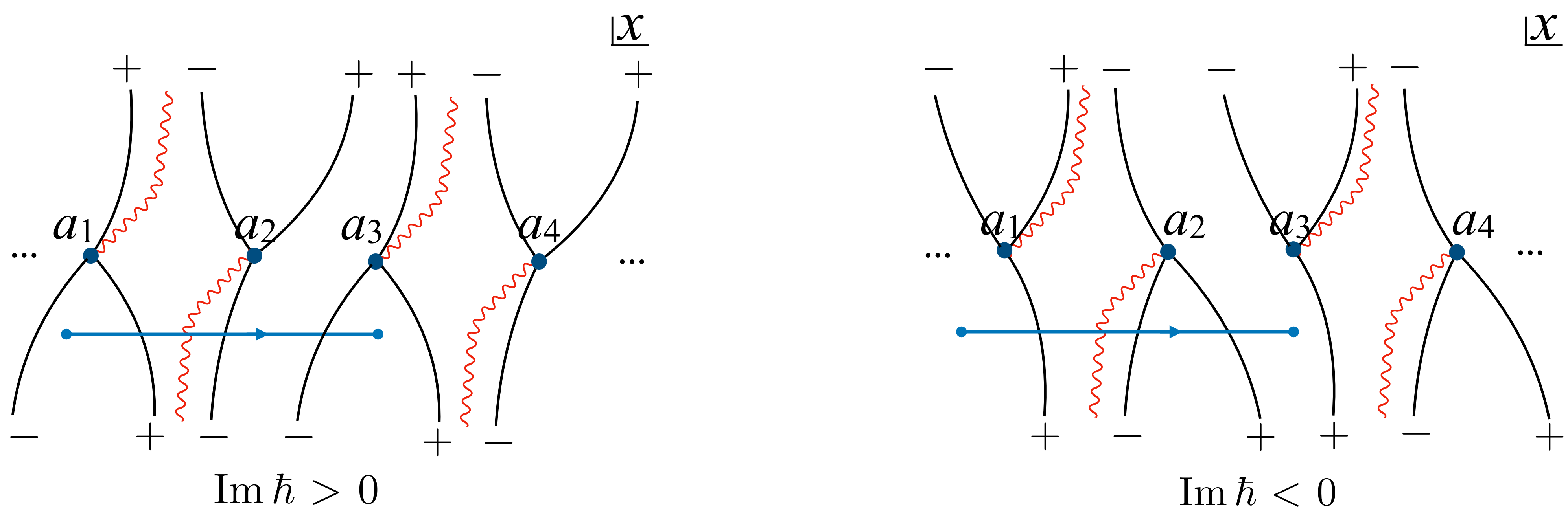
n-bion contributions

Complete resurgent structure

Exact-WKB analysis for S¹ quantum mechanics

Sueishi, Kamata, TM, Unsal, [2103.06586]

ex.) $V(x) = 1 - \cos(x)$ $\quad\quad \psi(x + 2\pi) = e^{-i\theta}\psi(x)$



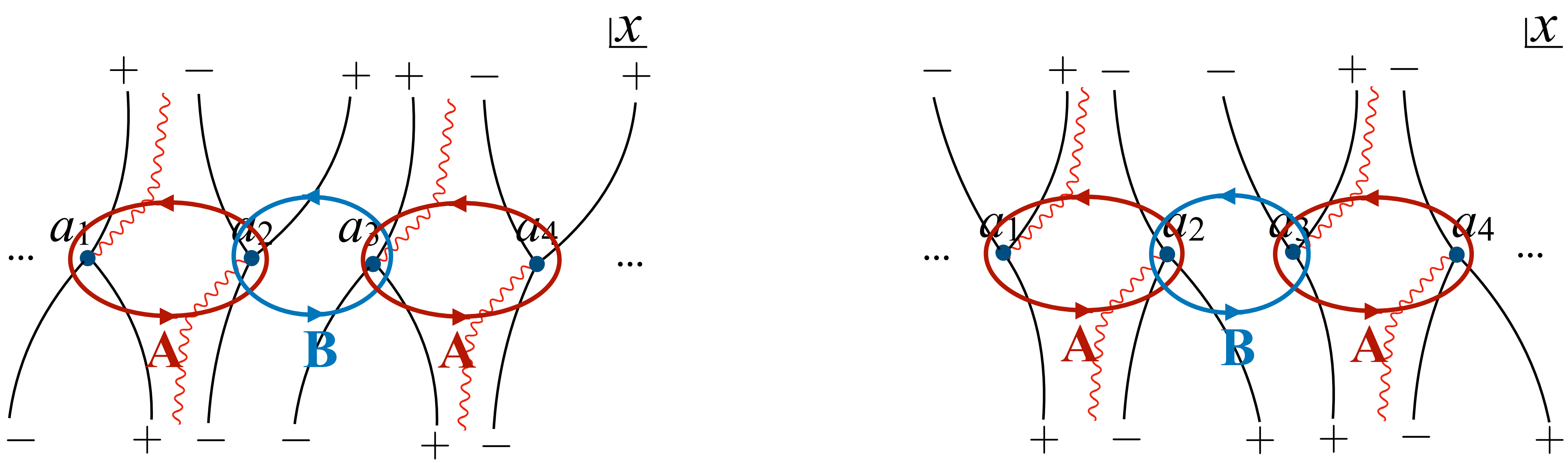
Quantization condition from periodicity of wave function

$$\mathcal{M}^{\pm}\begin{pmatrix}\psi^{+}_{a_1}(x)\\ \psi^{-}_{a_1}(x)\end{pmatrix} = e^{i\theta}\begin{pmatrix}\psi^{+}_{a_1}(x)\\ \psi^{-}_{a_1}(x)\end{pmatrix} \quad \Rightarrow \quad \det\left(\mathcal{M}^{\pm} - e^{i\theta} I\right) = 0$$

Exact-WKB analysis for S^1 quantum mechanics

Sueishi, Kamata, TM, Unsal, [2103.06586]

ex.) $V(x) = 1 - \cos(x)$ $\psi(x+2\pi) = e^{-i\theta}\psi(x)$



$\mathrm{Im}\,\hbar > 0$ $\mathrm{Im}\,\hbar < 0$

Quantization condition from periodicity of wave function

$$\mathcal{M}^{\pm}\begin{pmatrix}\psi^{+}_{a_1}(x)\\ \psi^{-}_{a_1}(x)\end{pmatrix} = e^{i\theta}\begin{pmatrix}\psi^{+}_{a_1}(x)\\ \psi^{-}_{a_1}(x)\end{pmatrix}$$

$$\det\left(\mathcal{M}^{\pm} - e^{i\theta}I\right) = 0$$

$$D^{\pm} \propto 1 + A^{\mp 1} + A^{\mp}B - 2(\sqrt{A})^{\mp 1}\sqrt{B}\cos\theta$$

$$= (1 + A^{\mp 1})\left(1 + \frac{B}{1+A^{\pm 1}} - \frac{\sqrt{B}}{\sqrt{A} + \frac{1}{\sqrt{A}}}(e^{i\theta} + e^{-i\theta})\right) = 0$$

exact agreement with Zinn-Justin-Jentschura's result Zinn-Justin, Jentschura (04)

Exact-WKB analysis for S¹ quantum mechanics

Sueishi, Kamata, TM, Unsal, [2103.06586]

Partition function clearly shows resurgent structure

$$Z(\hbar,\beta)=Z_{\rm pt}(\hbar,\beta)+Z_{\rm np}(\hbar,\beta)$$

$$Z_{\rm np}(\hbar,\beta) \;=\; \sum_{\substack{(Q,K)\in\mathbb{Z}\otimes\mathbb{N}_0\\ |Q|+K>0}} Z_{\rm np}(\hbar,\beta;\{Q,K\})$$

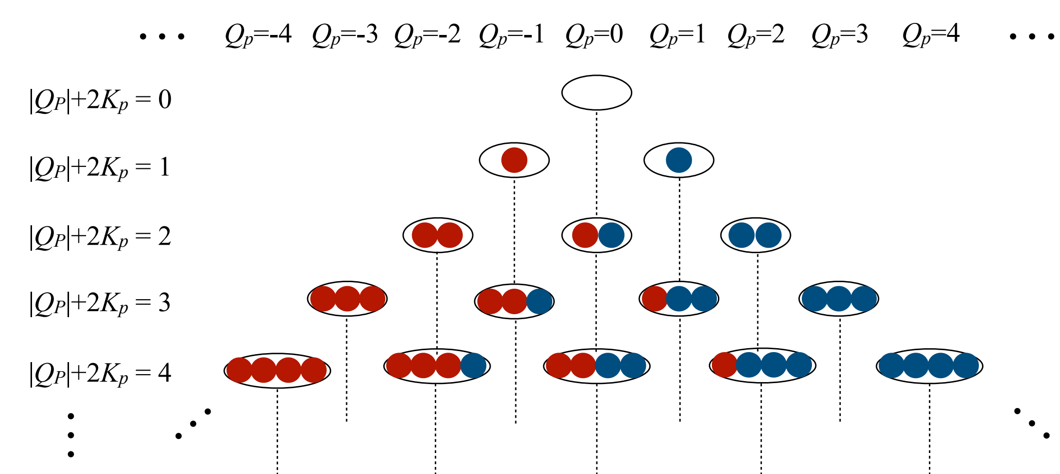
Q : topological charge
K : number of bions

$$Z_{\rm np}(\hbar,\beta;\{Q,K\}) = \frac{\beta}{2\pi i}\int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{(-1)^K}{|Q|+K}\begin{pmatrix}|Q|+K\\ K\end{pmatrix}\left[\frac{e^{-\frac{S_B}{\hbar}}}{2\pi}\Gamma\left(\frac{1}{2}-\frac{E}{\omega_{\mathcal{A}}}\right)^2\left(\frac{\hbar}{32}\right)^{-\frac{2E}{\omega_{\mathcal{A}}}}\right]^{|Q|/2+K}$$
$$\cdot\; {}_2F_1\left(1-K,-K;|Q|+1;-e^{\mp 2\pi i\frac{E}{\omega_{\mathcal{A}}}}\right)\left(e^{\pm 2\pi i\frac{E}{\omega_{\mathcal{A}}}}\right)^K e^{-\beta E+iQ\theta}dE.$$

trans-series including bion contributions

$$Z=\sum_{Q\in\mathbb{Z}} e^{i\theta Q}Z(\hbar,\beta;Q)$$

$$\mathcal{S}_+[Z(\hbar,\beta;Q)]=\mathcal{S}_-[Z(\hbar,\beta;Q)]$$



Resurgent structure is closed in each Q sector : resurgence triangle

1. Resurgent structure in asymptotically free QFT

Nishimura, Fujimori, TM, Nitta, Sakai, JHEP06(2022)151 [arXiv:2112.13999].

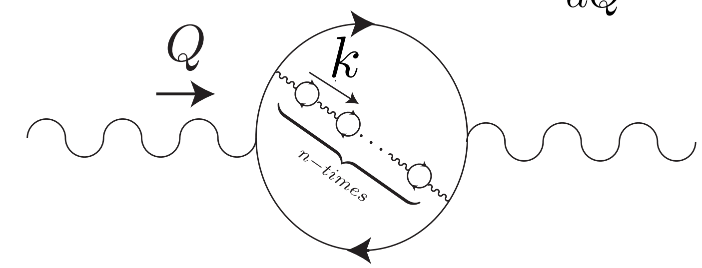
Infrared renormalon in QCD

't Hooft(79)

In asymptotically free QFT, a specific type of ambiguity exists.

◆Adler function (UV & IR convergent)

$$D(Q^2) = 4\pi^2 \frac{d\Pi(Q^2)}{dQ^2}$$

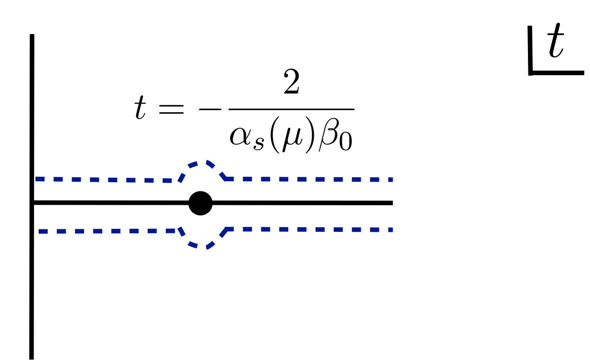


$$D(Q^2) = \alpha_s \sum_{n=0}^{\infty} \int dk^2 \frac{F(k^2/Q^2)}{k^2} \left[\beta_0 \alpha_s \log \frac{k^2}{\mu^2}\right]^n \left(= \sum_{n=0}^{\infty} \int dk^2 \frac{F(k^2/Q^2)}{k^2} \alpha_s(k)\right)$$

$$\approx \alpha_s \sum_{n=0}^{\infty} \left(\frac{\mu^4}{Q^4}\right) \left(-\frac{\alpha_s \beta_0}{2}\right)^n n! \; + \; \text{UV contr.}$$

$\mu = |Q|$

$$\Rightarrow \quad BP(t) = \alpha_s(\mu) \sum_n \left(-\frac{\alpha_s(\mu)\beta_0 t}{2}\right)^n = \frac{\alpha_s(\mu)}{1 + \alpha_s(\mu)\beta_0 t/2}$$



$$\Rightarrow \quad t = -\frac{2}{\alpha_s(\mu)\beta_0}$$

Singularity on positive real axis

$$\Rightarrow \quad \mathbf{B}(\alpha_s) = \mathrm{Re}\mathbf{B} \pm \frac{i\pi}{\beta_0} e^{\frac{2}{\alpha_s \beta_0}} \approx \left(\frac{\Lambda_{QCD}}{Q}\right)^4$$

: Renormalon (surviving in large N)
related to low-energy physics

How is the renormalon ambiguity cancelled?

Essence of our main result

$$\mathrm{Im}\langle \delta D^2 \rangle = \pm\pi \left[\underbrace{\left(\mu^2 e^{-\frac{4\pi}{\lambda_\mu}}\right)}_{\Lambda^4}{}^{2} \Lambda^0 - 2\Lambda^4 + \underbrace{\left(\mu^2 e^{-\frac{4\pi}{\lambda_\mu}}\right)}_{\Lambda^{-4}}{}^{-2} \Lambda^8 \right] \theta(\Lambda - a) = 0$$

known IR renormalon

a: IR cutoff

Λ: Dynamical scale

(1) Renormalon ambiguity is cancelled by combination of ambiguities at two nonpert. orders $\Lambda^4 \propto \exp(-8\pi/\lambda_\mu)$ and $\Lambda^8 \propto \exp(-16\pi/\lambda_\mu)$!

(2) The ambiguities emerge only for $a<\Lambda$, originating in analytic continuation from $a>\Lambda$ to $a<\Lambda$ ($|p|>\Lambda$ to $|p|<\Lambda$).

(3) There is binomial-expansion-type resurgent structure.

(4) The resurgent structure and the renormlon are drastically changed by infinitely many Stokes phenomena during Z_N-compactification.

Large-N $O(N)$ sigma model on R^2

- Action of $O(N)$ model

$$S = \frac{1}{2g^2}\int d^2x \left[(\partial_i \phi^a)^2 + D\left\{(\phi^a)^2 - 1\right\}\right] \qquad a = 1\ldots N \qquad (\phi^a)^2 = 1$$

- Effective potential in large N

$$V_{\mathrm{eff}}(D) = \frac{N}{2}\left[\int \frac{d^2p}{(2\pi)^2}\log\left(p^2 + D\right) - \frac{D}{\lambda}\right]$$

't Hooft coupling : $\lambda = g^2 N$

UV subtraction with renormalized coupling

$$\Rightarrow \quad V_{\mathrm{eff}}(D) = -\frac{N}{8\pi}D\left(\log\frac{D}{\Lambda^2} - 1\right)$$

Dynamical scale : $\Lambda = \mu\exp\left(-\frac{2\pi}{\lambda_\mu}\right)$

$$\Rightarrow \quad \langle D\rangle = \Lambda^2$$

it works as a dynamical mass

Large-N $O(N)$ sigma model on R^2

- Fluctuation of D $\quad D(x) = \Lambda^2 + \frac{\delta D(x)}{\sqrt{N}}$

- 2-point function of fluctuation of D

condensate

$$\langle \delta D(x) \delta D(0) \rangle = \int \frac{d^2p}{(2\pi)^2} e^{ip\cdot x} \Delta(p) \quad \xrightarrow{x \to 0} \quad \langle \delta D^2 \rangle_{\tilde{a}} \equiv \int_{|p|<\tilde{a}} \frac{d^2p}{(2\pi)^2} \Delta(p)$$

$\boxed{\text{UV cutoff } \tilde{a}}$

$$\Delta(p) \equiv \frac{8\pi \sqrt{p^2 (p^2 + 4\Lambda^2)}}{s_p} \qquad s_p = 4 \log \left(\sqrt{\frac{p^2}{4\Lambda^2} + 1} + \sqrt{\frac{p^2}{4\Lambda^2}} \right)$$

- Exact result of this condensate $\quad$ Novikov, Shifman, Vainshtein, Zakharov (84)

$$\langle \delta D^2 \rangle_{\tilde{a}} = 2\Lambda^4 \int_0^{s_{\tilde{a}}} ds \, \frac{\cosh s - 1}{s} = 2\Lambda^4 \, \mathrm{Chin}(s_{\tilde{a}}) \qquad \mathrm{Chin}(s_{\tilde{a}}) = \mathrm{Chi}(s_{\tilde{a}}) - \log(s_{\tilde{a}}) - \gamma_E$$

Unambiguous and IR convergent

How to derive trans-series

- Expand $\Delta(p)$ w.r.t. Λ^2/p^2 for $|p| \gg \Lambda$ $\rightarrow$ trans-series expression
  (In the end, analytically continue to $|p| < \Lambda$ $\rightarrow$ imaginary ambiguities)

$$s_p = 4\log\left(\sqrt{\frac{p^2}{4\Lambda^2}+1}+\sqrt{\frac{p^2}{4\Lambda^2}}\right) = \frac{8\pi}{\lambda_p} + \frac{4\Lambda^2}{p^2} - \frac{6\Lambda^4}{p^4} + \mathcal{O}(\Lambda^6)$$

$$\boxed{\begin{aligned} &\Lambda^2/p^2 = \exp(-4\pi/\lambda_p) \\ &\lambda_p \equiv \frac{2\pi}{\log(p/\Lambda)} \end{aligned}}$$

$\Rightarrow$ Expansion of $\Delta(p)$ w.r.t. Λ^2/p^2

$$\Delta(p) = p^2 \sum_{l=0}^{\infty} \left(\frac{\Lambda}{p}\right)^{2l} f_l(\lambda_p) \qquad f_l(\lambda_p) = P_l(\Lambda\partial_\Lambda)\,\lambda_p.$$: polynomial of λ_p

$$P_l(t) \equiv \frac{(-1)^l}{l!}\left[(t+l+1)^{(l)} - 4l(t+l)^{(l-1)}\right] \quad \text{with} \quad (a)^{(l)} = \frac{\Gamma(a+l)}{\Gamma(a)}$$

l : order of nonperturbative exponentials

How to derive trans-series

- Trans-series expansion of $<\delta D^2>$

we here introduce $\boxed{\text{IR cutoff } a}$ to regulate IR divergence

$$\left\langle \delta D^2 \right\rangle_{\tilde{a},a} \underset{\text{s.c.}}{=} \sum_{l=0}^{\infty} \Lambda^{2l} C_{2l}, \qquad C_{2l} = \int_{a<|p|<\tilde{a}} \frac{d^2p}{(2\pi)^2}\, p^{2-2l}\, f_l(\lambda_p),$$

$\lambda_{\tilde{a}}$ expansion (formal series) of each coefficient

$$C_{2l} = \sum_{n=0}^{\infty} \lambda_{\tilde{a}}^{n+1} c_{(2l,n)} \qquad \frac{\lambda_p}{4\pi} = \left[\frac{4\pi}{\lambda_{\tilde{a}}} + \log\left(\frac{p^2}{\tilde{a}^2}\right)\right]^{-1} = \sum_{n=0}^{\infty}\left(\frac{\lambda_{\tilde{a}}}{4\pi}\right)^{n+1}\left[-\log\left(\frac{p^2}{\tilde{a}^2}\right)\right]^n$$

- Separate UV and IR contributions

$$C_{2l} = \int_a^{\tilde{a}} \frac{dp}{2\pi}\, p^{3-2l}\, f_l(\lambda_p) = \mathcal{C}_{2l}(p)\big|_a^{\tilde{a}} = \mathcal{C}_{2l}(\tilde{a}) - \mathcal{C}_{2l}(a),$$

ex.) $l=0$ $\quad c_{(0,n)} = \int_{a<|p|<\tilde{a}} \frac{d^2p}{(2\pi)^2}\, p^2 \left(\frac{1}{4\pi}\log\frac{\tilde{a}^2}{p^2}\right)^n \Rightarrow \mathcal{C}_0(p) = \tilde{a}^4 \sum_{n=0}^{\infty}\left(\frac{\lambda_{\tilde{a}}}{8\pi}\right)^{n+1} \Gamma\left(n+1, 2\log\frac{\tilde{a}^2}{p^2}\right)$

Cancellation mechanism

Order Λ^0

$$\mathcal{C}_0(p) = \tilde{a}^4 \sum_{n=0}^{\infty} \left(\frac{\lambda_{\tilde{a}}}{8\pi}\right)^{n+1} \Gamma\left(n+1, 2\log\frac{\tilde{a}^2}{p^2}\right)$$

$$\Lambda^2/p^2 = \exp(-4\pi/\lambda_p)$$

Borel resum.

$$= -p^4 \int_0^\infty dt \, \frac{e^{-t}}{t - \frac{8\pi}{\lambda_p}} = p^4 e^{-8\pi/\lambda_p} \left[\gamma_E + \boxed{\log\left(-\frac{8\pi}{\lambda_p}\right)} - \mathrm{Ein}\left(-\frac{8\pi}{\lambda_p}\right)\right]$$

$$\Rightarrow \quad \mathrm{Im}\, C_0 = \mathrm{Im}\, \mathcal{C}_0(\tilde{a}) - \mathrm{Im}\, \mathcal{C}_0(a) = \pm\{\pi - \pi\theta(a-\Lambda)\}\Lambda^4 = \underline{\pm\pi\Lambda^4\,\theta(\Lambda - a)}$$

The ambiguity emerges only for $a < \Lambda$!

$\lambda_a < 0$

Known IR renormalon !

Order Λ^4

$$\mathcal{C}_4(p) = \boxed{-2\log\left(\frac{4\pi}{\lambda_p}\right)} - \frac{\lambda_p^2 - 2\pi\lambda_p}{8\pi^2}$$

$$\Rightarrow \quad \mathrm{Im}\, C_4 = \mathrm{Im}\, \mathcal{C}_4(\tilde{a}) - \mathrm{Im}\, \mathcal{C}_4(a) = \underline{\mp 2\pi\theta(\Lambda - a)}.$$

The ambiguity emerges only for $a < \Lambda$!

$\lambda_a < 0$

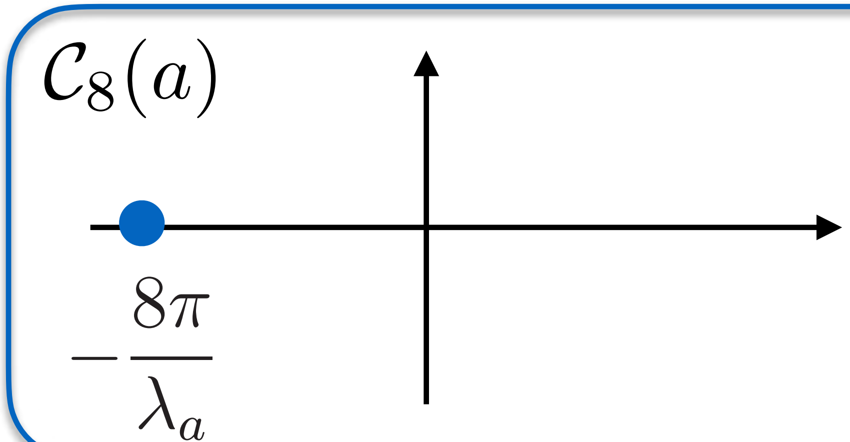
Cancellation mechanism

Order Λ^8

$$\mathcal{C}_8(p) \supset \frac{1}{\tilde{a}^4}\sum_{n=0}^{\infty}\left(-\frac{\lambda_{\tilde{a}}}{8\pi}\right)^{n+1}\Gamma\left(n+1, -2\log\frac{\tilde{a}^2}{p^2}\right) \overset{\text{Borel resum.}}{=} -\frac{1}{p^4}\underline{\int_0^\infty dt\,\frac{e^{-t}}{t+\frac{8\pi}{\lambda_p}}}.$$

$$= \frac{1}{\Lambda^4}\left[-\mathrm{Ein}\left(\frac{8\pi}{\lambda_p}\right) + \boxed{\log\left(\frac{8\pi}{\lambda_p}\right)} + \gamma_E\right]$$

$$\Rightarrow \quad \mathrm{Im}\, C_8 = \pm\theta(\Lambda - a)\frac{\pi}{\Lambda^4}.$$



$a > \Lambda$

$\lambda_a > 0$ $\Rightarrow$ No ambiguity

- The ambiguity emerges only for $a < \Lambda$
- It is accompanied by $\exp(+8\pi/\lambda_a) \propto 1/\Lambda^4$

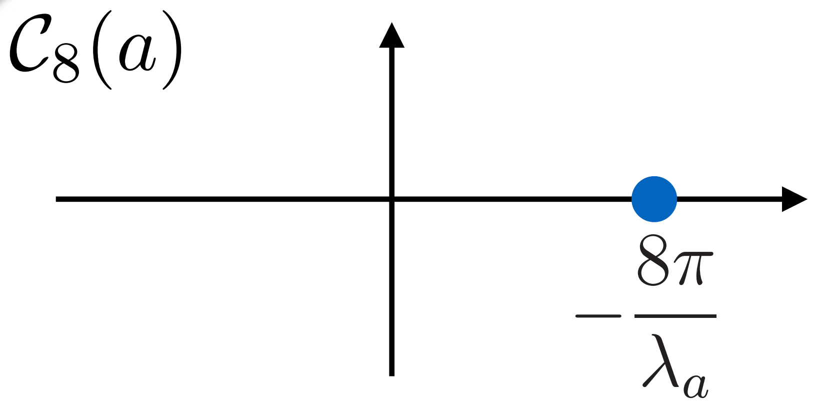
Cancellation mechanism

Order Λ^8

$$\mathcal{C}_8(p) \supset \frac{1}{\tilde{a}^4} \sum_{n=0}^{\infty} \left(-\frac{\lambda_{\tilde{a}}}{8\pi} \right)^{n+1} \Gamma \left(n+1, -2\log \frac{\tilde{a}^2}{p^2} \right) \overset{\substack{\text{Borel} \\ \text{resum.}}}{=} \underline{-\frac{1}{p^4} \int_0^{\infty} dt \, \frac{e^{-t}}{t + \frac{8\pi}{\lambda_p}}}.$$

$$= \frac{1}{\Lambda^4} \left[-\mathrm{Ein}\left(\frac{8\pi}{\lambda_p} \right) + \boxed{\log\left(\frac{8\pi}{\lambda_p} \right)} + \gamma_E \right]$$

$$\Rightarrow \quad \mathrm{Im}\, C_8 = \pm \theta(\Lambda - a) \frac{\pi}{\Lambda^4}.$$



$a < \Lambda$

$\lambda_a < 0$

$$\Rightarrow \quad \pm \exp\left(+\frac{8\pi}{\lambda_a} \right) \propto \pm \frac{1}{\Lambda^4}$$

Ambiguous !

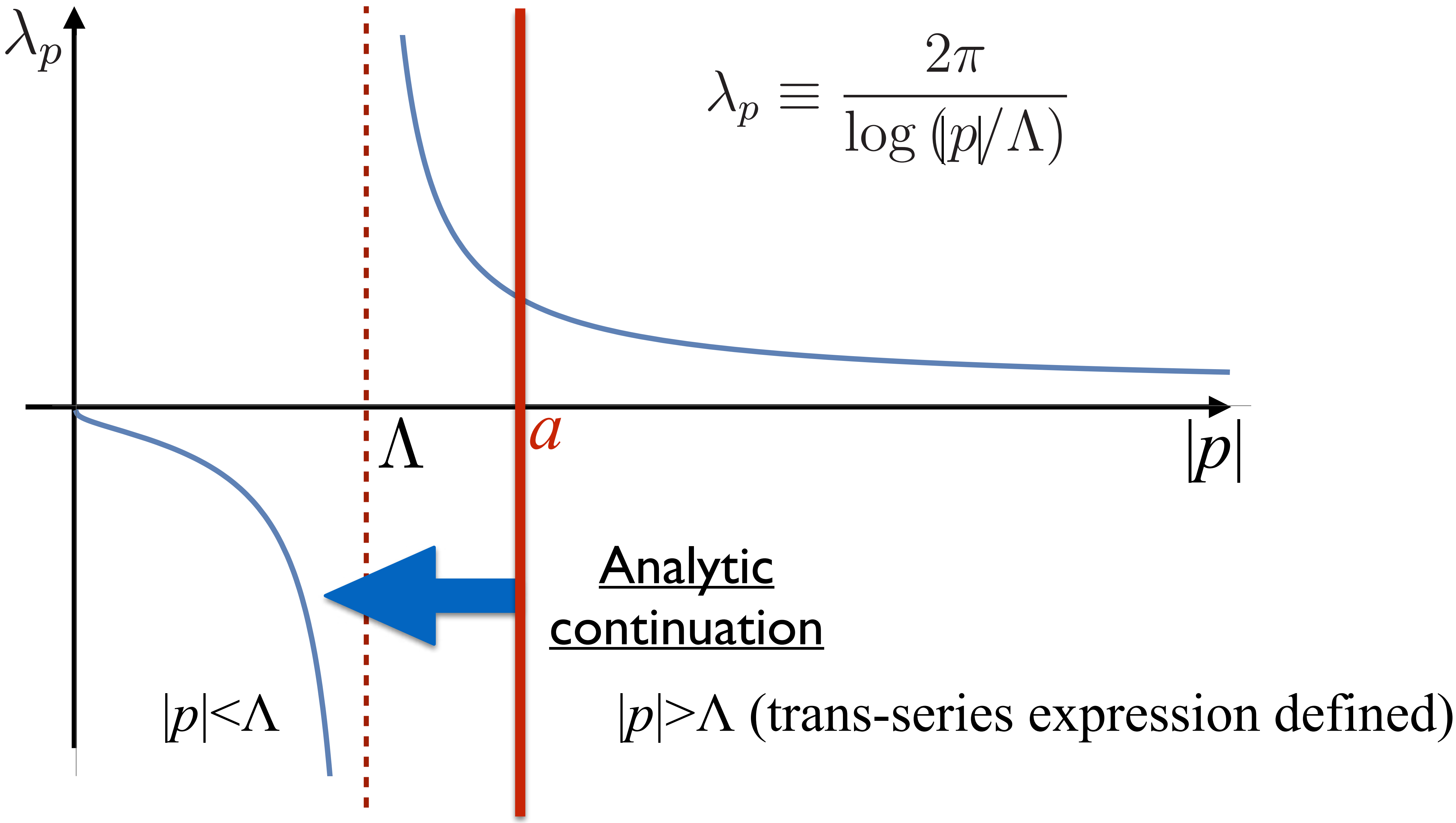
- The ambiguity emerges only for $a < \Lambda$
- It is accompanied by $\exp(+8\pi/\lambda_a) \propto 1/\Lambda^4$

Cancellation mechanism

$$
\begin{aligned}
\langle \delta D^2 \rangle_{\tilde{a},a} \underset{\text{s.c.}}{=} & \sum_{l=0}^{\infty} \Lambda^{2l} \Big[\{\mathcal{C}_{2l}(\tilde{a})\} - \{\mathcal{C}_{2l}(a)\} \Big] \\
= \quad & \Lambda^0 \left[\tilde{a}^4 \left\{ e^{-8\pi/\lambda_{\tilde{a}}} \mathrm{Ei}\left(\frac{8\pi}{\lambda_{\tilde{a}}}\right) \right\} - a^4 \left\{ e^{-8\pi/\lambda_a} \mathrm{Ei}\left(\frac{8\pi}{\lambda_a}\right) \right\} \boxed{\pm\, i\pi\Lambda^4\theta(\Lambda - a)} \right] \\
& + \Lambda^2 \left[\tilde{a}^2 \left\{ \frac{\lambda_{\tilde{a}}}{2\pi} \right\} - a^2 \left\{ \frac{\lambda_a}{2\pi} \right\} \right] \\
& + \Lambda^4 \left[\tilde{a}^0 \left\{ \frac{\lambda_{\tilde{a}}}{4\pi} - \frac{\lambda_{\tilde{a}}^2}{8\pi^2} - 2\log\left(\frac{4\pi}{\lambda_{\tilde{a}}}\right) \right\} - a^0 \left\{ \frac{\lambda_a}{4\pi} - \frac{\lambda_a^2}{8\pi^2} - 2\log\left|\frac{4\pi}{\lambda_a}\right| \right\} \boxed{\mp\, 2\pi i\theta(\Lambda - a)} \right] \\
& + \Lambda^6 \left[\frac{1}{\tilde{a}^2} \left\{ -\frac{\lambda_{\tilde{a}}}{\pi} + \frac{\lambda_{\tilde{a}}^2}{24\pi^2} + \frac{\lambda_{\tilde{a}}^3}{24\pi^3} \right\} - \frac{1}{a^2} \left\{ -\frac{\lambda_a}{\pi} + \frac{\lambda_a^2}{24\pi^2} + \frac{\lambda_a^3}{24\pi^3} \right\} \right] \\
& + \Lambda^8 \left[\frac{1}{\tilde{a}^4} \left\{ e^{8\pi/\lambda_{\tilde{a}}} \mathrm{Ei}\left(-\frac{8\pi}{\lambda_{\tilde{a}}}\right) + \frac{11\lambda_{\tilde{a}}}{8\pi} + \frac{13\lambda_{\tilde{a}}^2}{96\pi^2} - \frac{\lambda_{\tilde{a}}^3}{16\pi^3} - \frac{\lambda_{\tilde{a}}^4}{64\pi^4} \right\} \right. \\
& \qquad \left. - \frac{1}{a^4} \left\{ e^{8\pi/\lambda_a} \mathrm{Ei}\left(-\frac{8\pi}{\lambda_a}\right) + \frac{11\lambda_a}{8\pi} + \frac{13\lambda_a^2}{96\pi^2} - \frac{\lambda_a^3}{16\pi^3} - \frac{\lambda_a^4}{64\pi^4} \right\} \boxed{\pm\, \frac{i\pi}{\Lambda^4}\theta(\Lambda - a)} \right] \\
& + \mathcal{O}(\Lambda^{10}),
\end{aligned}
$$

Analytic continuation $|p|>\Lambda$ to $|p|<\Lambda$ ($a>\Lambda$ to $a<\Lambda$) is responsible for
(1) existence of ambiguities,
(2) cancellation of ambiguities (Λ^{-4} coefficient at the Λ^8 order)

Cancellation mechanism



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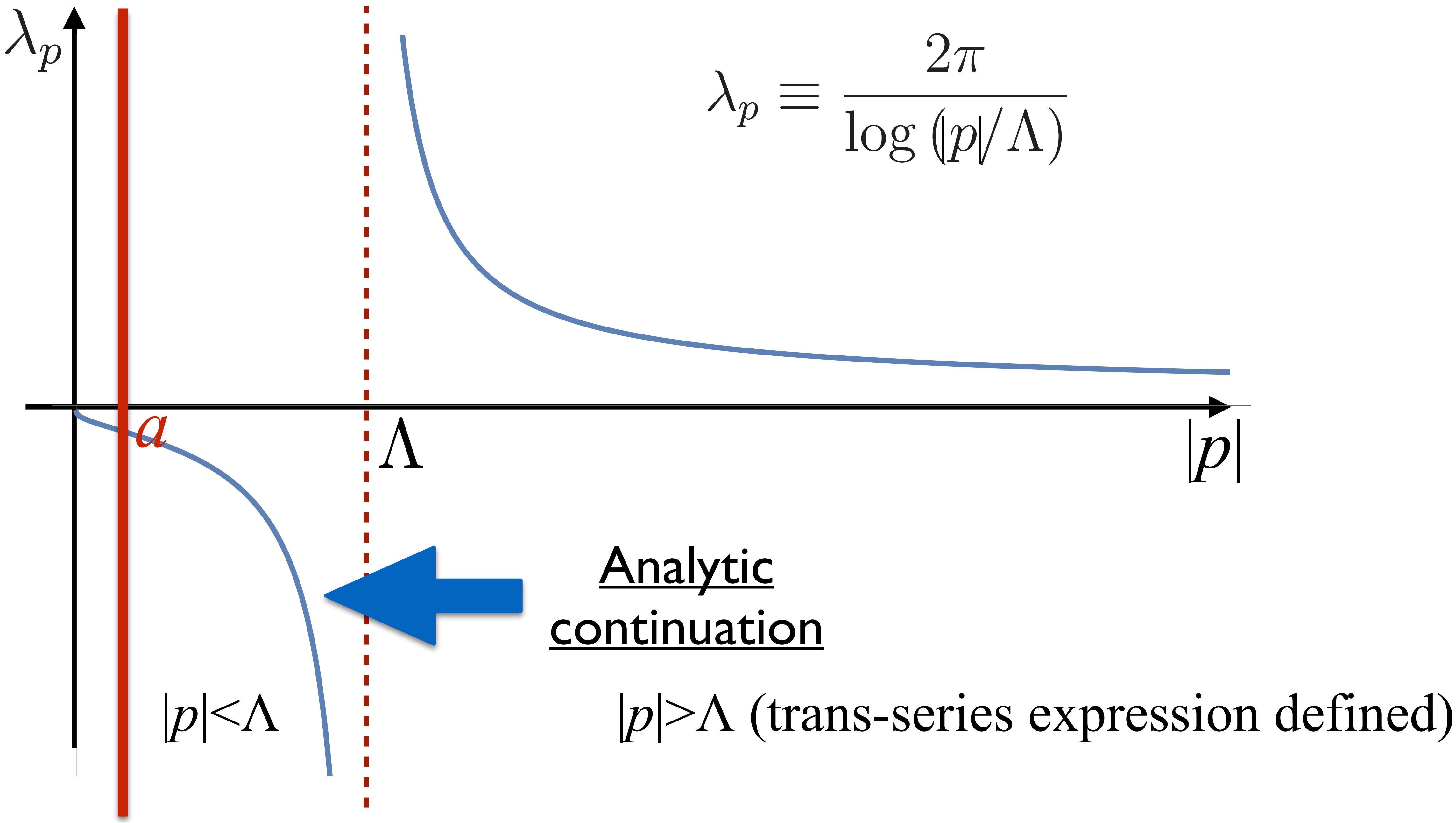
Cancellation mechanism



Analytic continuation $|p|>\Lambda$ to $|p|<\Lambda$ ($a>\Lambda$ to $a<\Lambda$) is responsible for
(1) existence of ambiguities,
(2) cancellation of ambiguities (Λ^{-4} coefficient at the Λ^{8} order)

Correlation function in Large-N O(N)

Result of imaginary ambiguities

$$\Lambda^2/p^2 = \exp(-4\pi/\lambda_p)$$

$$\mathrm{Im}\,\langle \delta D(x)\delta D(0)\rangle_a = \pm\pi\Lambda^4 \sum_{l=0}^{\infty}\sum_{\overline{n}=0}^{\infty} A_{l,\overline{n}} \left(\frac{\Lambda^2 x^2}{4}\right)^{\overline{n}}$$

l : order of nonpert. exponentials
$\overline{n}$: power of x^2

- Binomial-expansion-type cancellation

$$A_{l,\overline{n}} = (-1)^{l+\overline{n}} \frac{1}{(\overline{n}!)^2}\left[\binom{2\overline{n}+4}{l} - 4\binom{2\overline{n}+2}{l-1}\right]$$

$\overline{n}\backslash l$	0	1	2	3	4	5	6	7	$\cdots$
0	-1	0	2	0	-1	0	0	0	$\cdots$
1	1	-2	-1	4	-1	-2	1	0	$\cdots$
$\vdots$									

$\Rightarrow$ 0

cancellation occurs for each $x^{2\overline{n}}$ order

Large-N $\mathrm{CP}^{N\text{-}1}$ sigma model

$$\mathcal{L} = \frac{1}{g^2}\left[\sum_{a=1}^{N} |\mathcal{D}_i \phi^a|^2 + D\left(|\phi^a|^2 - 1\right)\right] \qquad \mathcal{D}_i\phi^a = (\partial_i + iA_i)\phi^a$$

$$\langle F_{\mu\nu}^2 \rangle_{\tilde a, a} = -\frac{1}{2N}\sum_{l=0}^{\infty}\Lambda^{2l}\int_0^\infty dt\, \Lambda^t \left[\tilde a^{2\eta_l(t)} - a^{2\eta_l(t)}\right]\frac{\tilde P_l(t)}{\eta_l(t)}$$

condensate of field strength

$$\eta_l(t) = 2 - l - \frac{t}{2}$$

- on R^2

$$\mathrm{Im}\,\langle F_{\mu\nu}^2\rangle_{\tilde a, a} = \frac{\pm\pi}{N}\left[\boxed{}\left(\tilde a e^{-\frac{2\pi}{\lambda_{\tilde a}}}\right)^4 - \boxed{4}\left(\tilde a e^{-\frac{2\pi}{\lambda_{\tilde a}}}\right)^2\Lambda^2 + \boxed{6}\Lambda^4 - \boxed{4}\left(\tilde a e^{-\frac{2\pi}{\lambda_{\tilde a}}}\right)^{-2}\Lambda^6 + \boxed{}\left(\tilde a e^{-\frac{2\pi}{\lambda_{\tilde a}}}\right)^{-4}\Lambda^8\right]\theta(\Lambda - a) = 0$$

- on $\mathrm{R}^1 \times \mathrm{S}^1$ (Z_N-twist) $\quad L\Lambda \ll 1 \quad NL\Lambda \gg 1$

$$\mathrm{Im}\,\langle F_{\mu\nu}^2\rangle_{\tilde a, a}^{\mathbb{R}\times S^1} = \frac{\pm\pi}{NL}\left[\boxed{2}\left(\tilde a e^{-\frac{2\pi}{\lambda_{\tilde a}}}\right)^3 - \boxed{6}\left(\tilde a e^{-\frac{2\pi}{\lambda_{\tilde a}}}\right)\Lambda^2 + \boxed{6}\left(\tilde a e^{-\frac{2\pi}{\lambda_{\tilde a}}}\right)^{-1}\Lambda^4 - \boxed{2}\left(\tilde a e^{-\frac{2\pi}{\lambda_{\tilde a}}}\right)^{-3}\Lambda^6\right]\theta(\Lambda - a) = 0$$

- Both cases have binomial-expansion-type resurgent structures.
- Z_N-twisted compactification drastically changes the structure.

Large-N $\mathrm{CP}^{N\text{-}1}$ sigma model

$$\mathcal{L} = \frac{1}{g^2}\left[\sum_{a=1}^{N} |\mathcal{D}_i \phi^a|^2 + D\left(|\phi^a|^2 - 1\right)\right] \qquad \mathcal{D}_i\phi^a = (\partial_i + iA_i)\phi^a$$

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condensate of field strength

$$\eta_l(t) = 2 - l - \frac{t}{2}$$

- on R^2

$$\mathrm{Im}\,\langle F_{\mu\nu}^2\rangle_{\tilde{a},a} = \frac{\pm\pi}{N}\left[\underbrace{\boxed{\left(\tilde{a}e^{-\frac{2\pi}{\lambda\tilde{a}}}\right)^4}}_{\Lambda^4} - 4\left(\tilde{a}e^{-\frac{2\pi}{\lambda\tilde{a}}}\right)^2\Lambda^2 + 6\Lambda^4 - 4\left(\tilde{a}e^{-\frac{2\pi}{\lambda\tilde{a}}}\right)^{-2}\Lambda^6 + \left(\tilde{a}e^{-\frac{2\pi}{\lambda\tilde{a}}}\right)^{-4}\Lambda^8\right]\theta(\Lambda - a) = 0$$

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- Both cases have binomial-expansion-type resurgent structures.
- Z_N-twisted compactification drastically changes the structure.

What happens in compactification

During compactification, the resurgent structure changes, where Stokes phenomena occur every time one of Kaluza-Klein masses $\tilde{n}/R$ becomes larger than the dynamical scale Λ !

$$\mathrm{Im}\,\langle \delta D(x)\delta D(0)\rangle_a\big|_l = \pm\pi \sum_{\tilde{n}\in\mathbb{Z}} \boxed{\Lambda^{2l}} P_l(\Lambda\partial_\Lambda) \left[\boxed{\frac{\Lambda^{3-2l}}{R}} \frac{e^{-i\frac{\tilde{n}}{R}x}}{\sqrt{1-\frac{\tilde{n}^2}{R^2\Lambda^2}}} \theta\left(\Lambda^2 - \frac{\tilde{n}^2}{R^2}\right) \right]$$

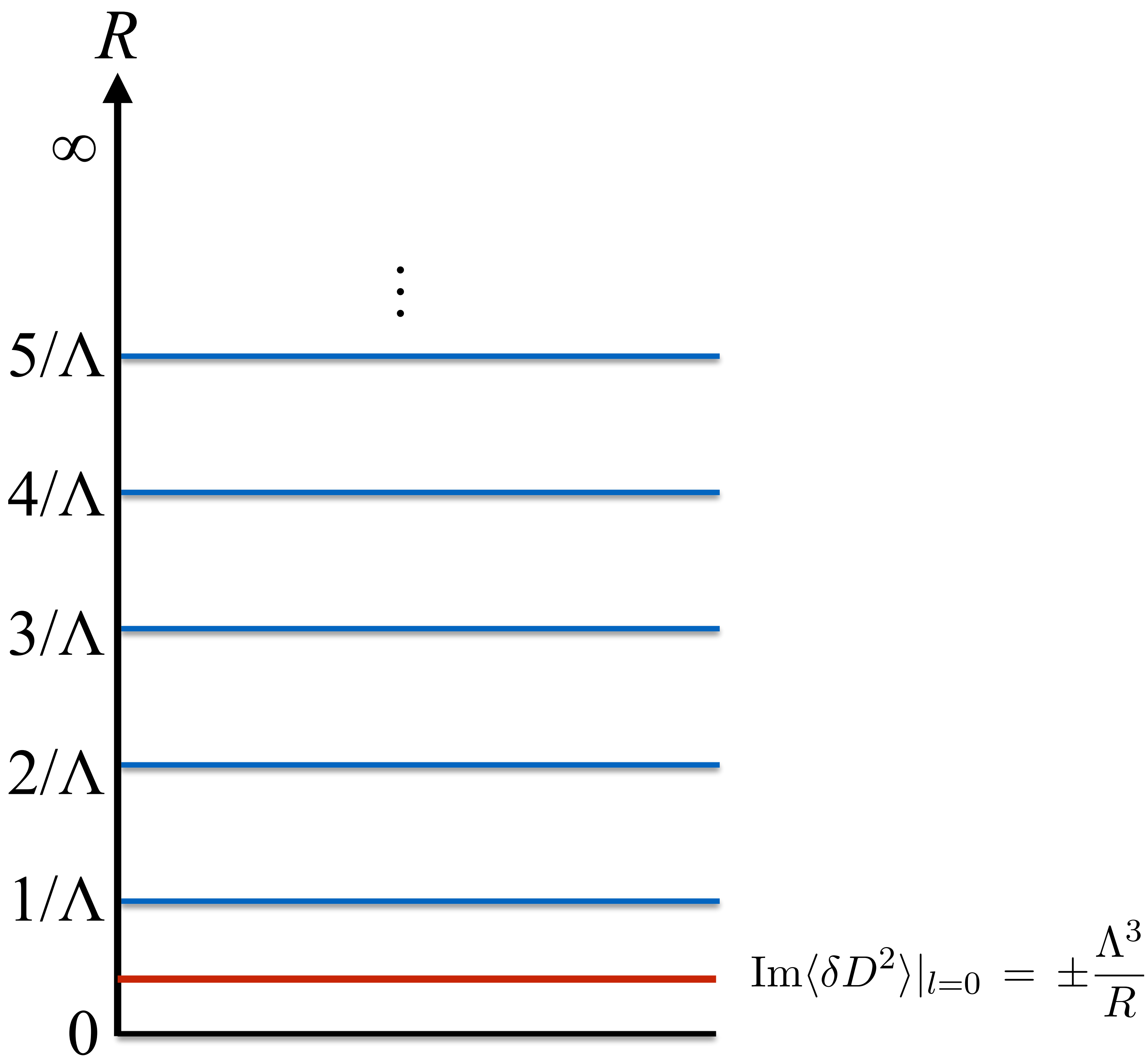
l : order of nonpert. exponentials

$\tilde{n}$: KK mode

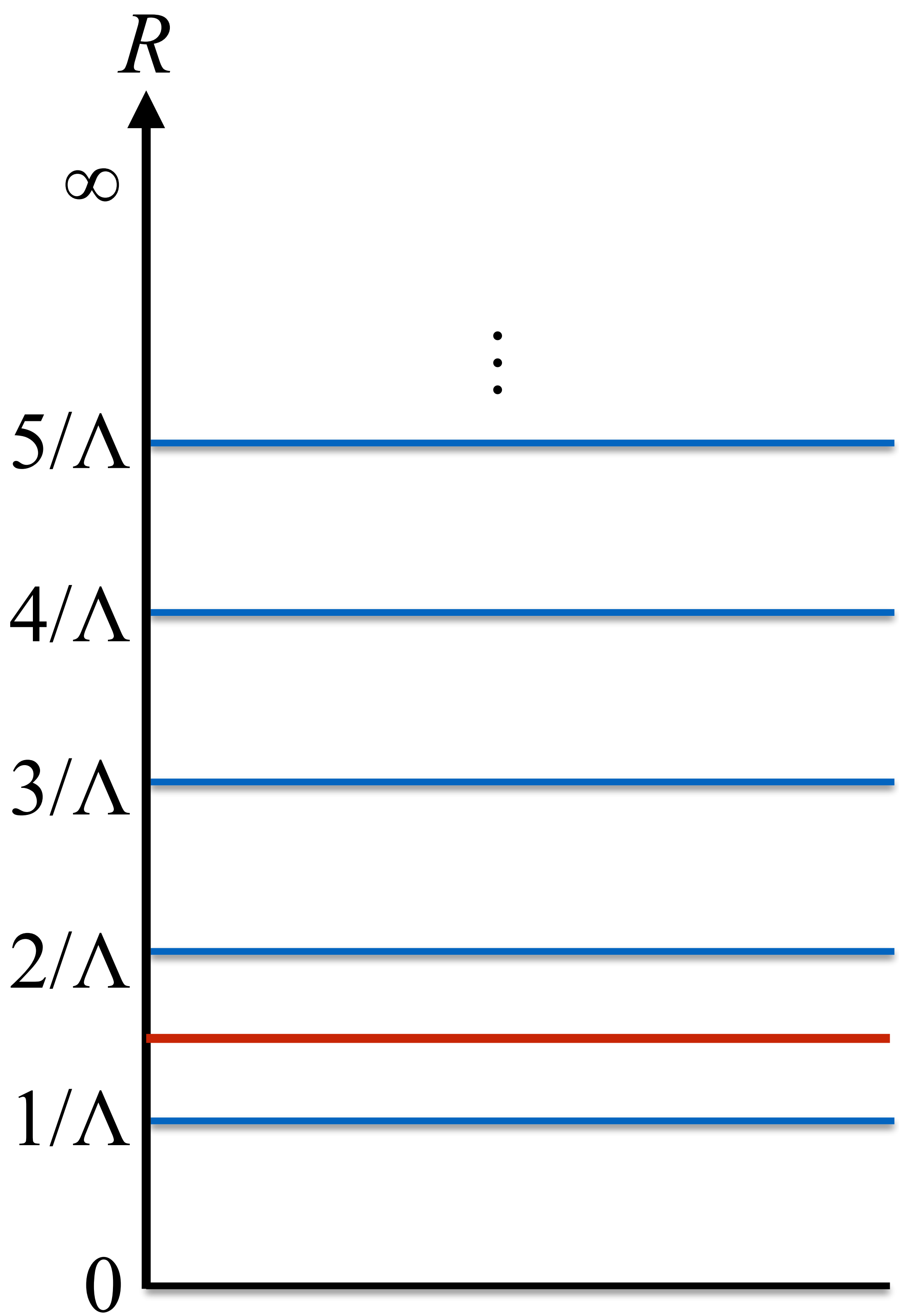
Infinitely many Stokes phenomena during compactification change renormalon ambiguity from $\mathrm{O}(\Lambda^4)$ to $\mathrm{O}(\Lambda^3/R)$.

$$\mathrm{Im}\,\langle \delta D(x)\delta D(0)\rangle_a\big|_{l=0} = \pm \begin{cases} \Lambda^3/R & \text{for } R < \Lambda^{-1} \\ \Lambda^4 + \cdots & \text{for } R \to \infty \end{cases}$$

What happens in compactification

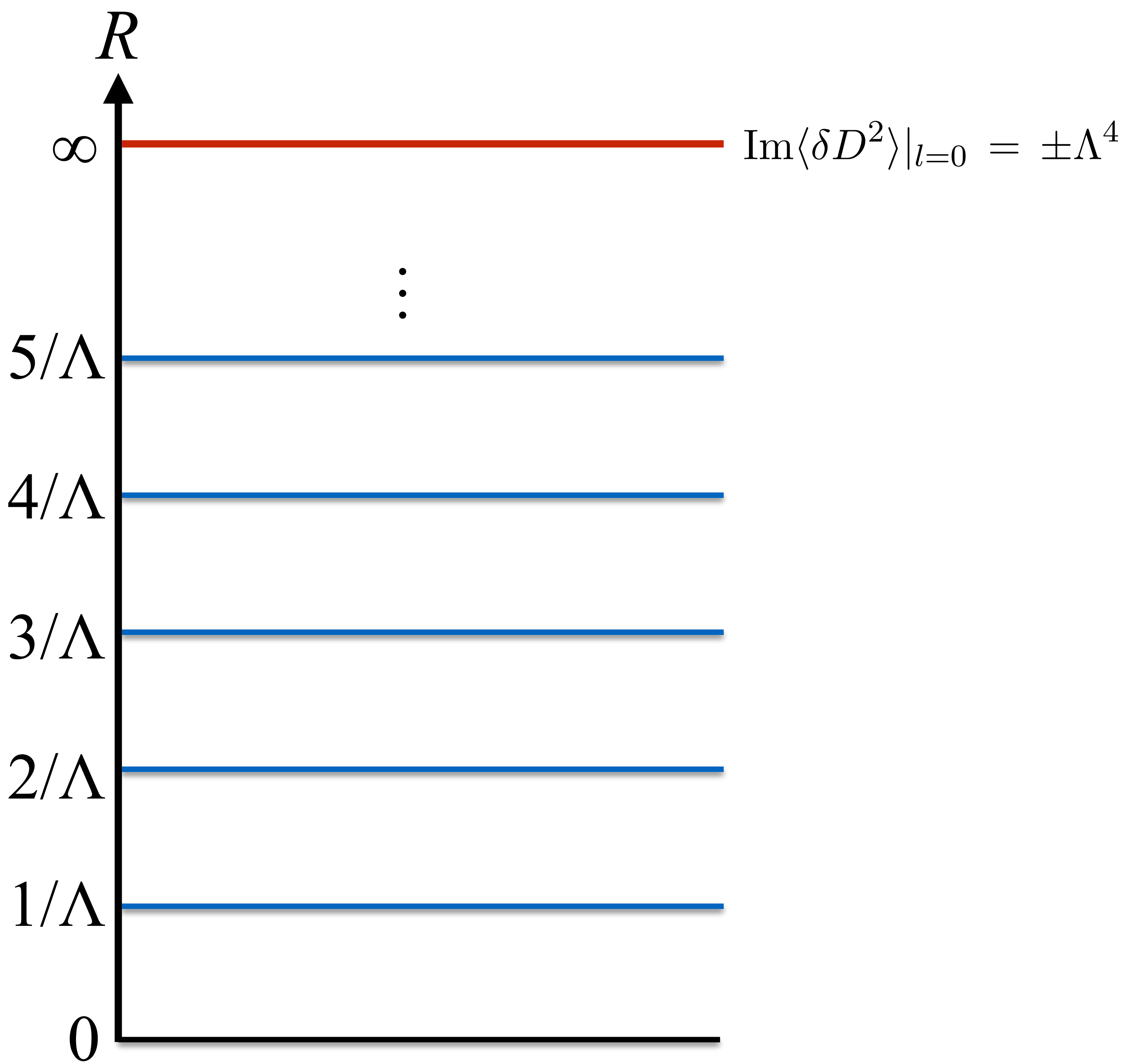


What happens in compactification



$$\mathrm{Im}\langle \delta D^2 \rangle|_{l=0} = \pm \frac{\Lambda^3}{R}\left(1 + \frac{2}{\sqrt{1 - \frac{1}{(R\Lambda)^2}}}\right)$$

What happens in compactification



2. Phase transition and Resurgence

Fujimori, Honda, Kamata, TM, Sakai, Yoda, PTEP10(2021)103B04, [arXiv:2103.13654].

Phase transition and resurgence

1st order phase transition is understood as Anti-Stokes phenomenon : change of dominant saddles (stationary points)

- Anti-Stokes phenomenon is encoded in perturbative series
- The picture is consistent with Lee-Yang zero picture.
- Recently 2nd-order phase transition is discussed in localized SQED

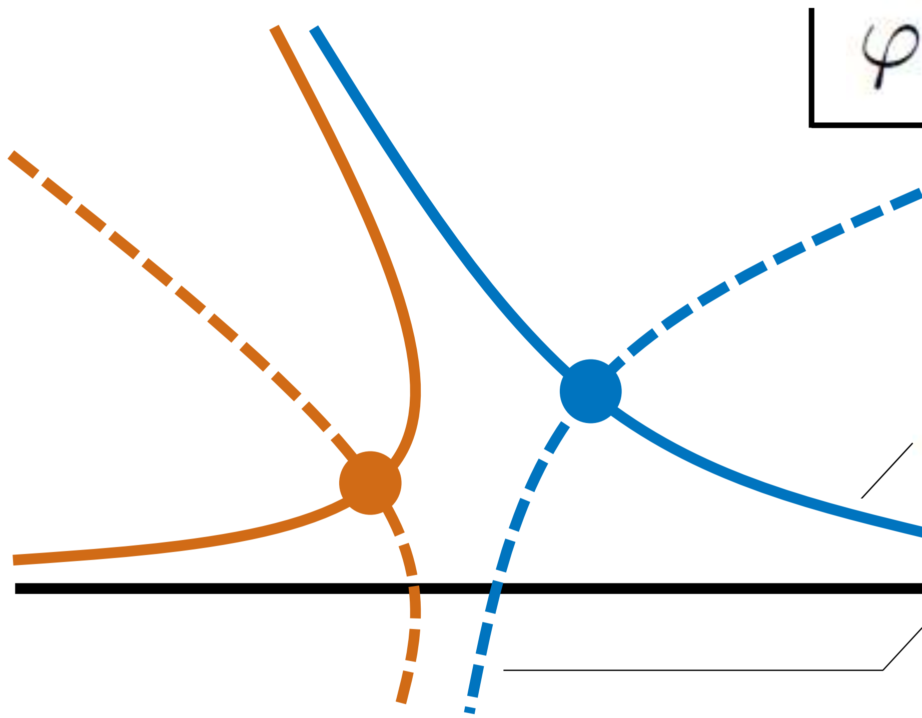
Kanazawa, Tanizaki (15), Dunne, et.al. (16)(17)(18)

Russo, Tierz(17)

Can 2nd and higher order phase transitions be understood in terms of thimble decomposition and resurgence theory?

Fujimori, Honda, Kamata, TM, Sakai, Yoda (21)

Stokes and anti-stokes phenomena



$$\mathcal{J}_i : \frac{\mathrm{d}\varphi(t)}{\mathrm{d}t} = +\frac{\overline{\delta S[\varphi]}}{\delta \varphi}, \quad \varphi(-\infty) = \varphi_i$$

$$\mathcal{K}_i : \frac{\mathrm{d}\varphi(t)}{\mathrm{d}t} = -\frac{\overline{\delta S[\varphi]}}{\delta \varphi}, \quad \varphi(-\infty) = \varphi_i$$

$$\left.\frac{\delta S[\varphi]}{\delta \varphi}\right|_{\varphi=\varphi_i} = 0$$

- Stokes phenomenon : Change of intersection numbers

$$\mathrm{Im}[S[\varphi_i]] = \mathrm{Im}[S[\varphi_j]]$$

Resurgent structure

- Anti-Stokes phenomenon : Change of dominant saddles

$$\mathrm{Re}[S[\varphi_i]] = \mathrm{Re}[S[\varphi_j]]$$

1st order phase transition

3D N=4 U(1) SUSY gauge theories on S^3

Parameters

- FI parameter η
- # of hypermultiplets $2N_f$
- mass m

Variables after localization

- Coulomb branch parameter σ

Partition function via localization

$$Z = \int d\sigma\, e^{-S(\sigma)} \qquad S(\sigma) = N_f \left[-i\lambda\sigma + \ln(\cosh\sigma + \cosh m) \right]$$

effective action

Saddle-point approx. in 't Hooft-like limit $N_f \to \infty, \quad \lambda \equiv \frac{\eta}{N_f} = \text{fixed}$ Russo, Tierz(17)

$$\sigma_n^{\pm} = \log\left(\frac{-\lambda\cosh m \pm i\Delta(\lambda, m)}{i+\lambda} \right) + 2\pi i n \quad (n \in \mathbb{Z})$$

saddles (stationary points)

$$\Delta(\lambda, m) = \sqrt{1 - \lambda^2 \sinh^2 m}$$

3D N=4 U(1) SUSY gauge theories on S³

Parameters

- FI parameter η
- # of hypermultiplets $2N_f$
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Variables after localization

- Coulomb branch parameter σ

Partition function via localization

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effective action

Saddle-point approx. in 't Hooft-like limit

$N_f \to \infty, \quad \lambda \equiv \frac{\eta}{N_f} = \text{fixed}$ Russo, Tierz(17)

$$\frac{d^2F}{d\lambda^2} = \begin{cases} \frac{N_f}{1+\lambda^2}\left(1 + \frac{\cosh m}{\sqrt{1-\lambda^2\sinh^2 m}}\right) & \lambda < \lambda_c \\ \frac{N_f}{1+\lambda^2} & \lambda \geq \lambda_c \end{cases}$$

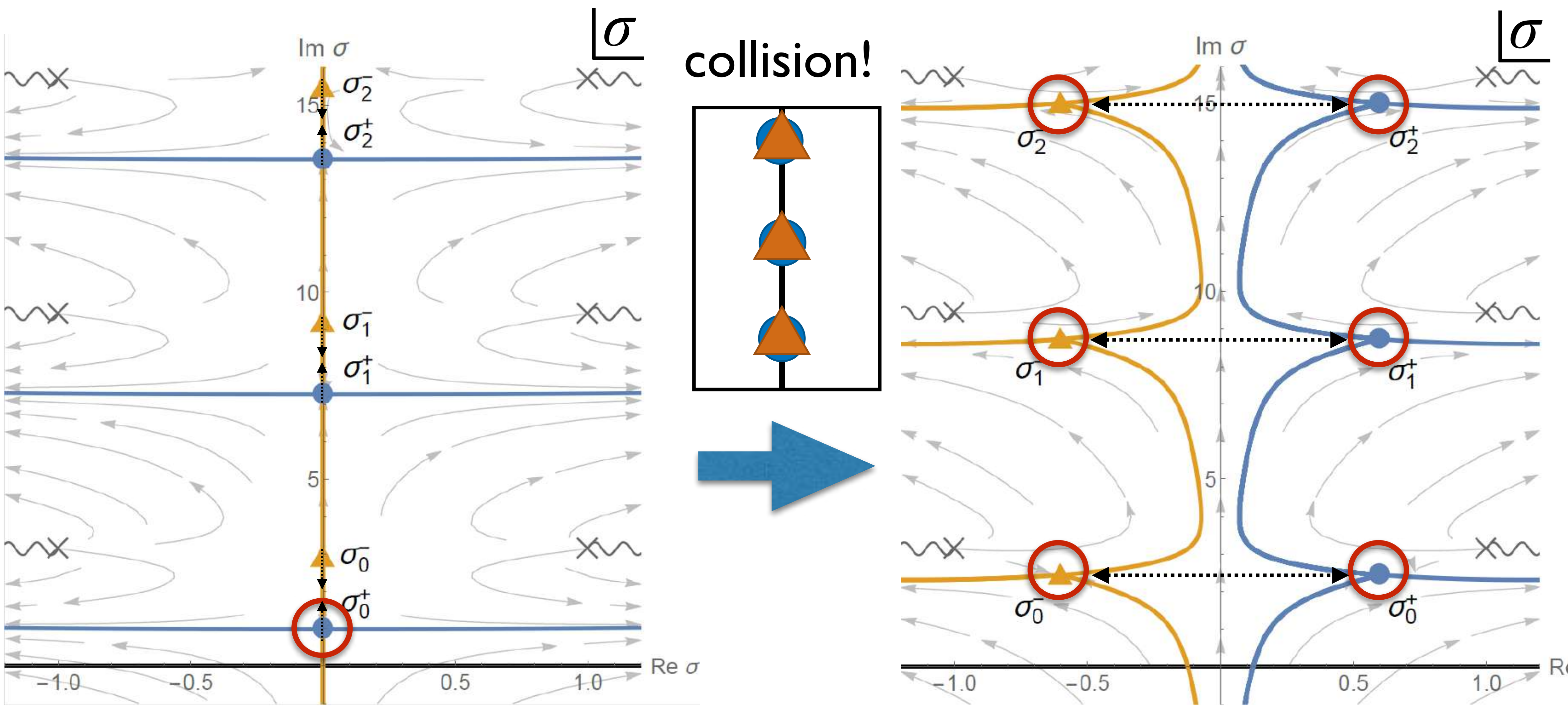
critical point : $\lambda_c \equiv \frac{1}{\sinh m}$

→ 2nd order phase transition

Lefschetz thimble decomposition

Fujimori, Honda, Kamata, TM, Sakai, Yoda (21)

$\boxed{\lambda < \lambda_c}$ $\boxed{\lambda \geq \lambda_c}$



Only a trivial saddle contributes

An infinite number of saddles contribute

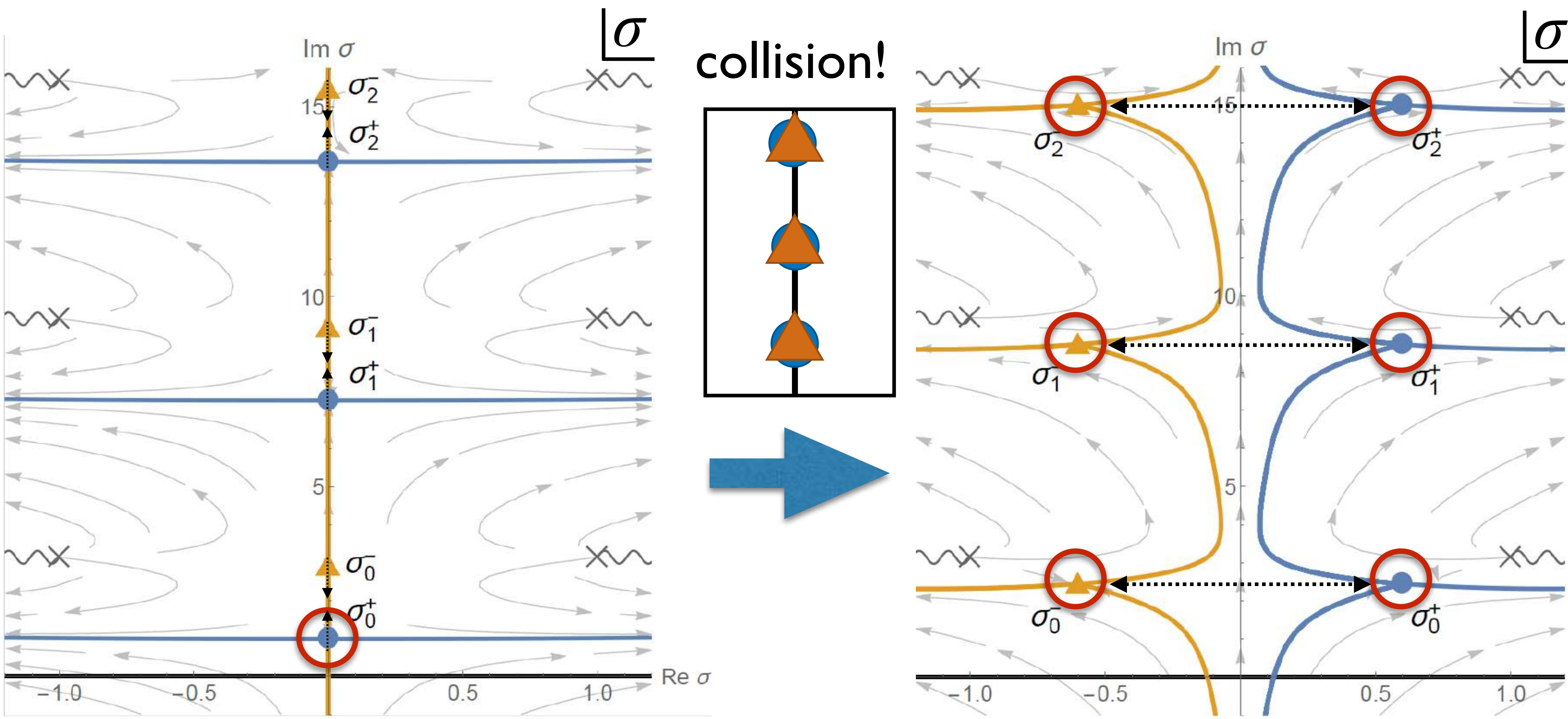
$$\mathrm{Im}[S[\varphi_i]] = \mathrm{Im}[S[\varphi_j]]$$
$$\mathrm{Re}[S[\varphi_i]] = \mathrm{Re}[S[\varphi_j]]$$

for pair saddles

Lefschetz thimble decomposition

Fujimori, Honda, Kamata, TM, Sakai, Yoda (21)

$\boxed{\lambda < \lambda_c}$ $\boxed{\lambda \geq \lambda_c}$



Only a trivial saddle contributes

An infinite number of saddles contribute

- At $\lambda=\lambda_c$, two of pair saddles collide and scatter with $\pi/2$.
- At $\lambda=\lambda_c$, both Stokes and anti-Stokes phenomena simultaneously occur!

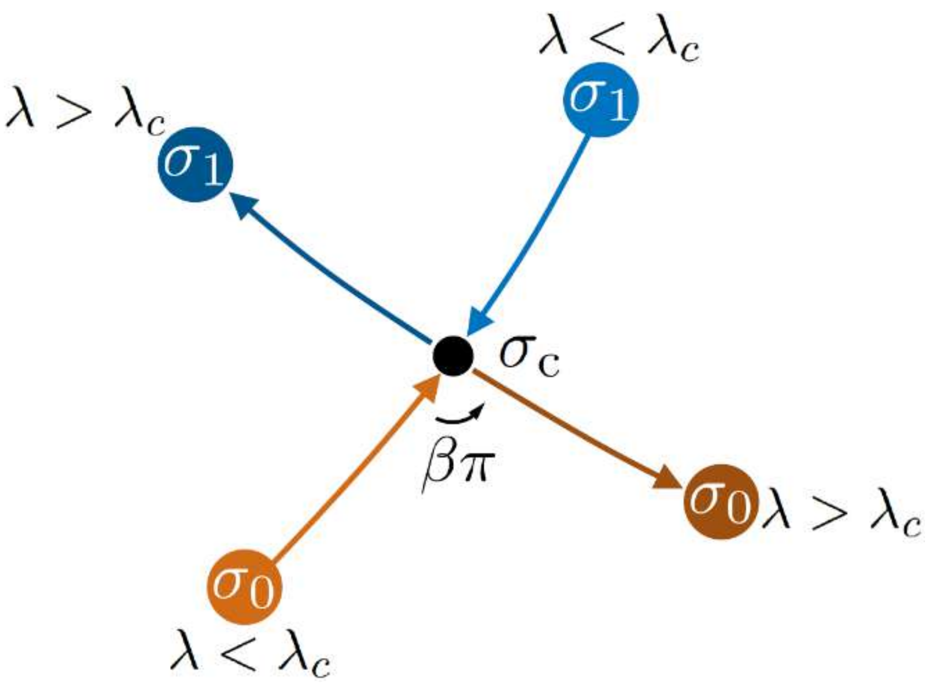
Generic argument on phase transition

Fujimori, Honda, Kamata, TM, Sakai, Yoda (21)

$\boxed{\text{Theorem}}$

Assume action in expression as

$$e^{-NF(\lambda)} = \int \mathrm{d}\sigma \; e^{-N\tilde{S}(\lambda;\sigma)}$$



When n saddles collide with angle $\beta\pi$,
phase transition of order $\lceil (n+1)\beta \rceil$ occurs, where Stokes and anti-Stokes phenomena simultaneously occur.

ceiling function, cf.) $\lceil (2+1)(1/2) \rceil = 2$

- Simple-model phase transitions are understood in terms of thimbles.
- It means the phase transitions can be detected from perturbative series!

Generic argument on phase transition

Fujimori, Honda, Kamata, TM, Sakai, Yoda (21)

ex.) Airy integral

$$\tilde{S}(\lambda;\sigma) = \frac{i\sigma^3}{3} - i\lambda\sigma.$$

$$\tilde{S}_\pm = \mp\frac{2i}{3}(\delta\lambda)^{3/2}.$$

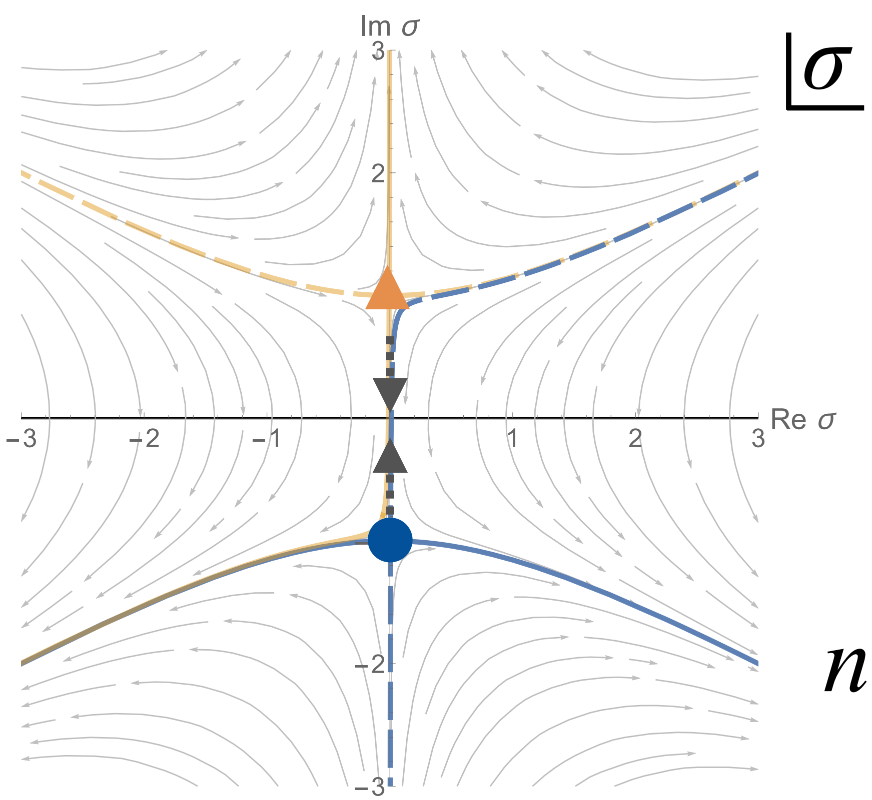
$$\sigma_{\rm c} = 0, \quad \lambda_{\rm c} = 0, \quad \delta\sigma_\pm = \pm\delta\lambda^{1/2} \quad \sigma_\pm = \pm\lambda^{1/2}$$

Free energy

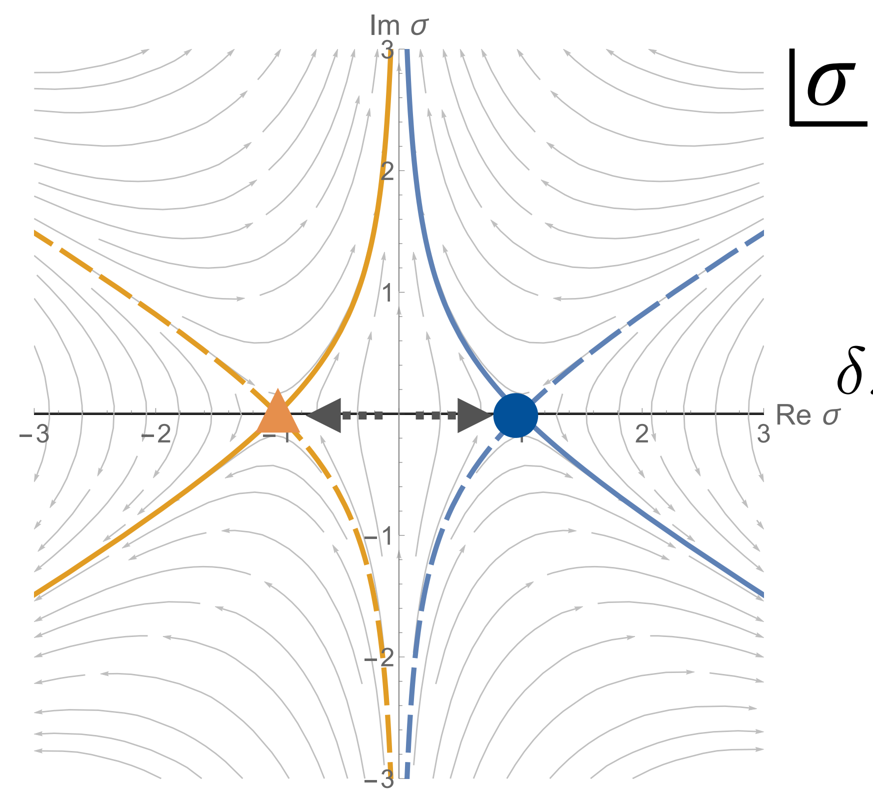
$$F \simeq \begin{cases} \tilde{S}_+ & = \frac{2}{3}(-\delta\lambda)^{3/2} \quad \text{for } \delta\lambda < 0 \\ \tilde{S}_+ + \tilde{S}_- & = 0 \quad \text{for } \delta\lambda > 0 \end{cases}$$

2nd order phase transition

$\delta\lambda < 0$



n=2, β=1/2



$\delta\lambda > 0$

Generic argument on phase transition

Fujimori, Honda, Kamata, TM, Sakai, Yoda (21)

ex.) Airy integral

$$\tilde{S}(\lambda;\sigma) = \frac{i\sigma^3}{3} - i\lambda\sigma.$$

$$\tilde{S}_\pm = \mp\frac{2i}{3}(\delta\lambda)^{3/2}.$$

$$\sigma_c = 0, \quad \lambda_c = 0, \quad \delta\sigma_\pm = \pm\delta\lambda^{1/2} \quad \sigma_\pm = \pm\lambda^{1/2}$$

Free energy

$$F \simeq \begin{cases} \tilde{S}_+ & = \frac{2}{3}(-\delta\lambda)^{3/2} \quad \text{for } \delta\lambda < 0 \\ \tilde{S}_+ + \tilde{S}_- & = 0 \quad \text{for } \delta\lambda > 0 \end{cases}$$

2nd order phase transition

(i) Contributing saddles jump as $\sigma_+ \longrightarrow \sigma_+, \sigma_-$
(ii) The two saddles collide and scatter with a scattering angle $\pi/2$
(iii) Stokes and anti-Stokes phenomena occur simultaneously

$$[(2+1)(1/2)] = 2$$

Generic argument on phase transition

Fujimori, Honda, Kamata, TM, Sakai, Yoda (21)

ex.) Airy integral

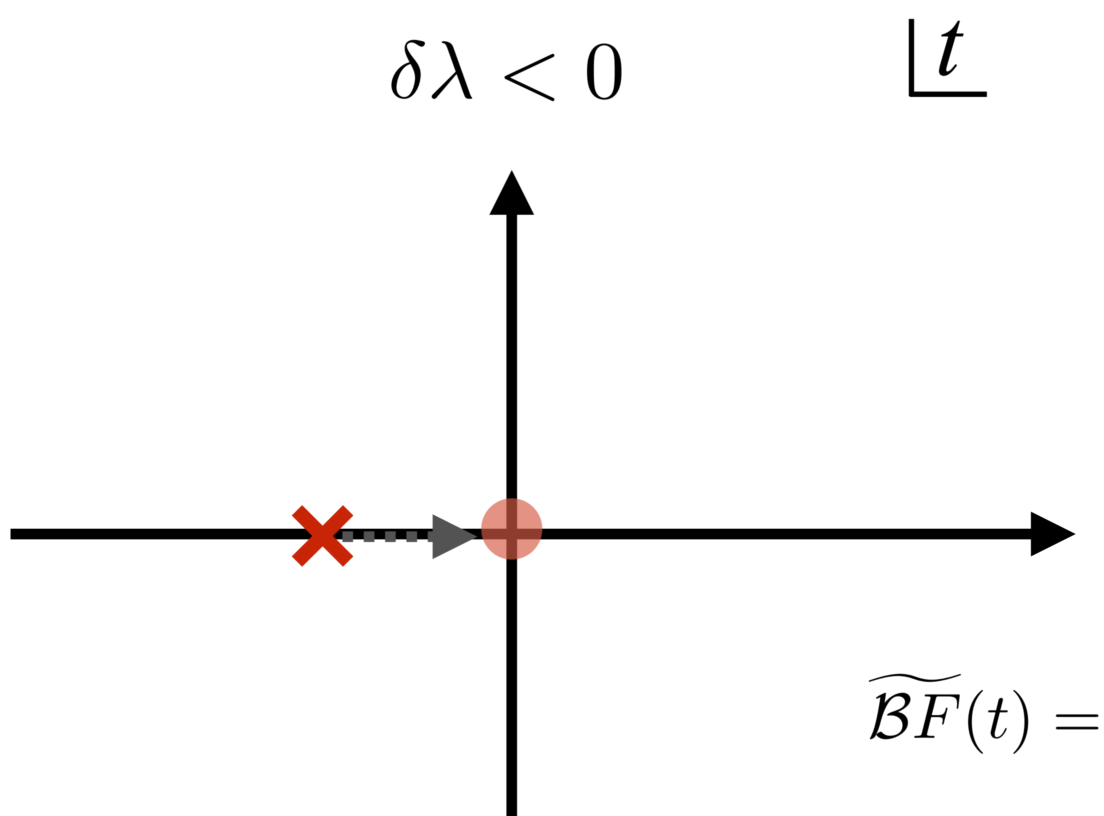
$$\tilde{S}(\lambda;\sigma) = \frac{i\sigma^3}{3} - i\lambda\sigma. \quad \Rightarrow \quad \tilde{S}_{\pm} = \mp\frac{2i}{3}(\delta\lambda)^{3/2}.$$

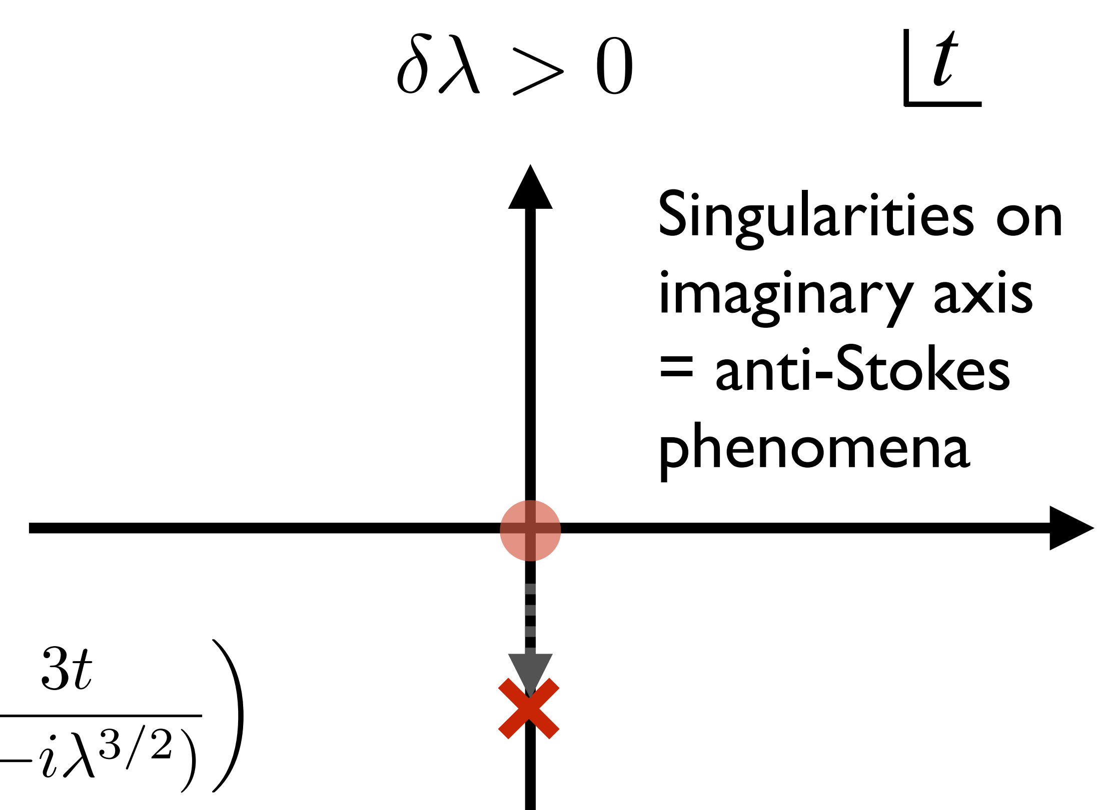
$$\sigma_{\rm c} = 0, \quad \lambda_{\rm c} = 0, \quad \delta\sigma_{\pm} = \pm\delta\lambda^{1/2} \quad \sigma_{\pm} = \pm\lambda^{1/2}$$

Free energy

$$F \simeq \begin{cases} \tilde{S}_+ & = \frac{2}{3}(-\delta\lambda)^{3/2} \quad \text{for } \delta\lambda < 0 \\ \tilde{S}_+ + \tilde{S}_- & = 0 \quad \text{for } \delta\lambda > 0 \end{cases}$$

2nd order phase transition





$$\widetilde{\mathcal{B}F}(t) = {}_2F_1\left(\frac{1}{6}, \frac{5}{6}, 1; \frac{3t}{4(-i\lambda^{3/2})}\right)$$

Summary

1. Resurgence structure in 2D sigma models:

- Analytic continuation is essential for cancellation of imaginary ambiguities,
- Combination of ambiguities at non-pert. orders cancels renormalon,
- Binomial-expansion-type resurgent structure,
- Compactif. leads to infinite-times Stokes pheno. & change of renormalon.

2. Phase transition and resurgence:

- Higher-order phase transitions are understood as collisions of saddles,
- Stokes and anti-Stokes phenomena simultaneously occur there,
- encoded in collision of Borel singularities of perturbative series,
- Theorem: n-saddle collision with angle $\beta\pi$ → transition order $\lceil (n+1)\beta \rceil$

3. Exact resurgence and quantization conditions from EWKB:

- Exact quantization conditions obtained for multi-well and periodic QM,
- Exact resurgent structures in these models are shown,
- Dunne-Unsal (P-NP) relation in some models is derived by exact-WKB.