GENERAL RELATIVITY – MIDTERM EXAM

Exercise 1. *In full GR, the action for a massive particle in a gravitational field reads:*

$$S = -m \int_\gamma \sqrt{-g_{\mu\nu}(x) dx^\mu dx^\nu} \ , \tag{1}$$

where $g_{\mu\nu}(x)$ is the curved spacetime metric.

1. *Denoting $g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x)$, expand the action (1) to first order in small $h_{\mu\nu}$. Identify the linearized gravitational interaction term from the lectures.*

2. *Write this interaction term for vanishing particle velocity $\mathbf{v} = 0$. Compare with the Lagrangian of Newtonian gravity, and deduce the proportionality coefficient between h_{tt} and the Newtonian potential ϕ.*

3. *Now, consider a purely spatial and $\mathbf{x}$-independent $h_{\mu\nu}$ (as one would encounter in a large-wavelength gravitational wave):*

$$h_{tt} = h_{ti} = 0 \ ; \quad h_{ij} = h_{ij}(t) \ . \tag{2}$$

   From the linearized action of Part 1, derive a conserved momentum $\mathbf{p}$ as a function of m, velocity $\mathbf{v}$, and h_{ij}. In the small-$\mathbf{v}$ limit, use this to find the acceleration $\mathbf{a}$ as a function of $\partial_t h_{ij}$ and $\mathbf{v}$.

Exercise 2. *In this exercise, we will solve the linearized Einstein equation with sources.*

1. *Consider the scalar wave equation with source:*

$$\Box\phi(t, \mathbf{x}) = -4\pi\rho(t, \mathbf{x}) \ . \tag{3}$$

   Show that this equation is solved by:

$$\phi(t, \mathbf{x}) = \int d^3\mathbf{y} \, \frac{\rho(t - |\mathbf{y} - \mathbf{x}|, \mathbf{y})}{|\mathbf{y} - \mathbf{x}|} \ . \tag{4}$$

2. *Now, consider a conserved 4-current $j^\mu(t, \mathbf{x})$ with $\partial_\mu j^\mu = 0$, inducing the following electromagnetic potential:*

$$A^\mu(t, \mathbf{x}) = \int d^3\mathbf{y} \, \frac{j^\mu(t - |\mathbf{y} - \mathbf{x}|, \mathbf{y})}{|\mathbf{y} - \mathbf{x}|} \ . \tag{5}$$

   Show that this A^μ satisfies the Lorentz gauge condition $\partial_\mu A^\mu = 0$ (Hint: it may be useful to switch the integration variable to $\mathbf{r} \equiv \mathbf{y} - \mathbf{x}$).

3. Show that the potential (5) *solves the Maxwell equations (in units where* $k = 1$*):*

$$\Box A^\mu - \partial^\mu \partial_\nu A^\nu = -4\pi j^\mu \ . \tag{6}$$

4. Now, consider a conserved stress-energy tensor $T^{\mu\nu}(t, \mathbf{x})$ *with* $\partial_\mu T^{\mu\nu} = 0$*. Using the results above, find a solution* $h_{\mu\nu}(t, \mathbf{x})$ *to the linearized Einstein equations:*

$$\Box \tilde{h}^{\mu\nu} - 2\partial^{(\mu}\partial_\rho \tilde{h}^{\nu)\rho} + \eta^{\mu\nu}\partial_\rho\partial_\sigma \tilde{h}^{\rho\sigma} = -16\pi G T^{\mu\nu} \ ; \quad \tilde{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h^\rho_\rho \ . \tag{7}$$