Stochastic log-gases, multiple SLEs, and Gaussian free fields

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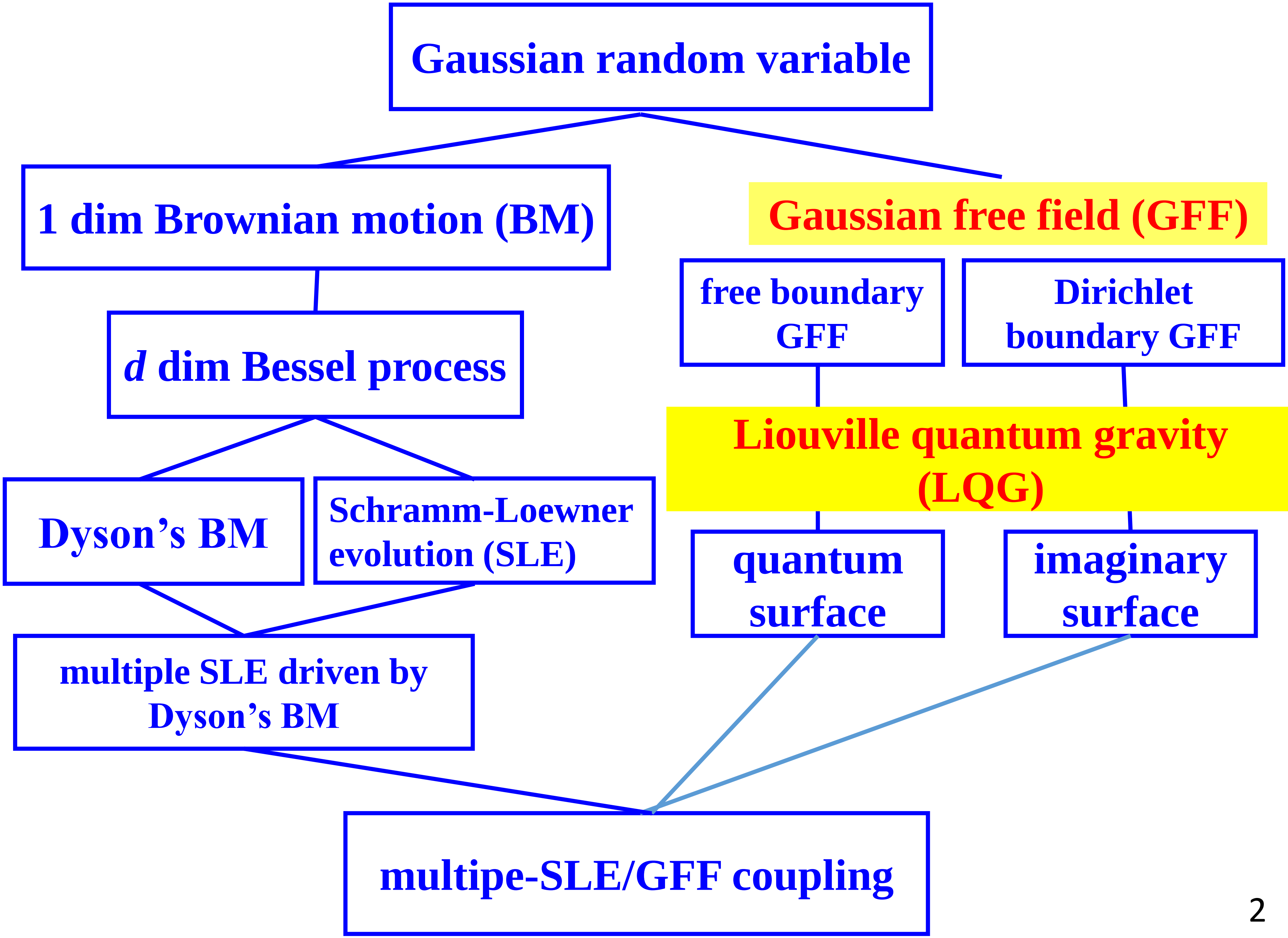
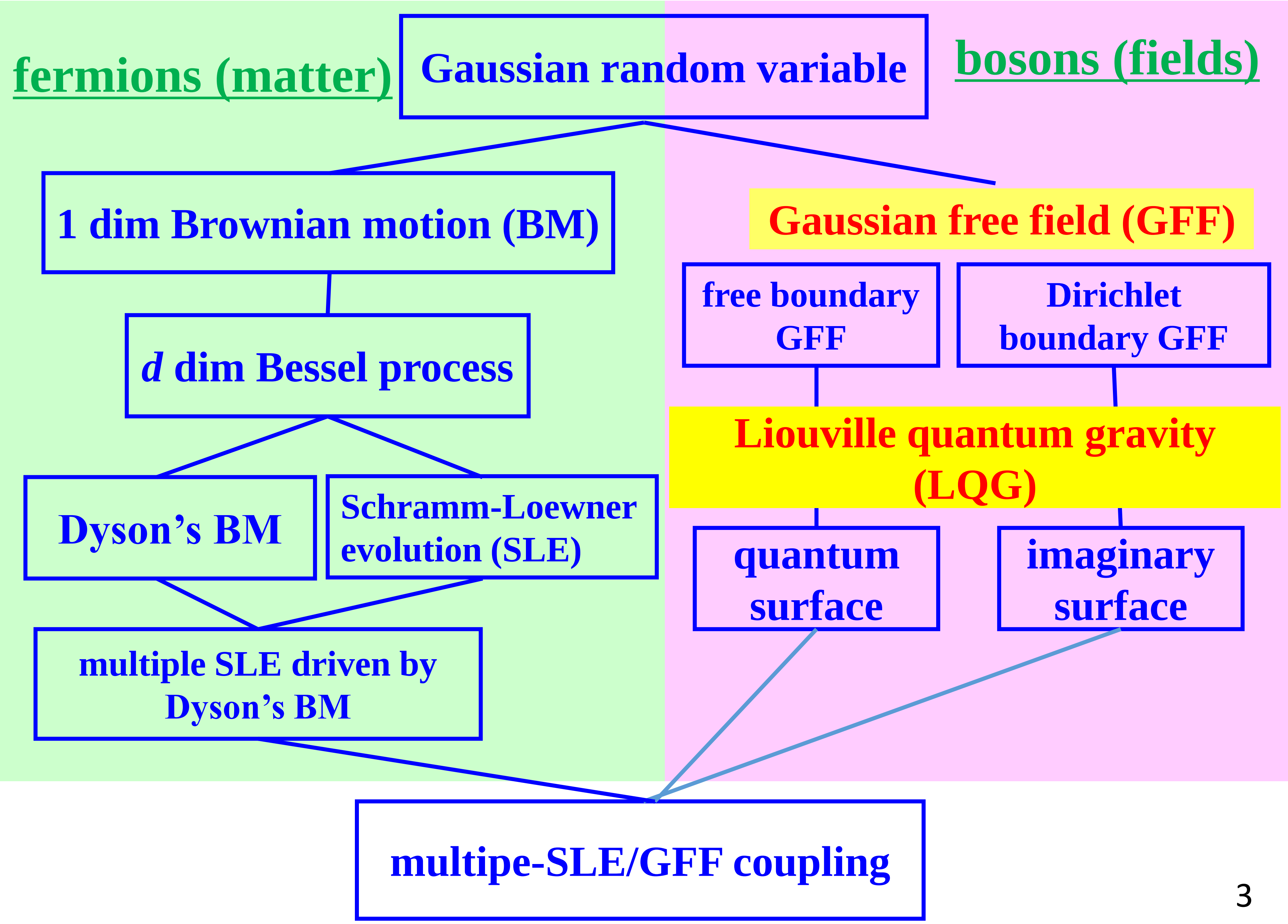
OIST Workshop 2023
New trends of conformal theory from probability to gravity

OIST, Okinawa, Japan
31 July – 4 August 2023


Gaussian random variable
1 dim Brownian motion (BM)
Gaussian free field (GFF)
free boundary GFF
Dirichlet boundary GFF
d dim Bessel process
Liouville quantum gravity (LQG)
Dyson's BM
Schramm-Loewner evolution (SLE)
quantum surface
imaginary surface
multiple SLE driven by Dyson's BM
multipe-SLE/GFF coupling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Plan of my talk

1. **From Brownian motion to Dyson's BM model**
   (random points: stochastic log-gas)

2. **Schramm-Loewner evolution (SLE) and Gaussian free fields**
   (random curves and random surfaces)

3. **Multiple SLE/GFF coupling driven by Dyson's BM model**
   (random points/curves/surfaces)

4. **Concluding Remarks**

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1 dim standard Brownian motion (BM)

Let $(\Omega, \mathcal{F}, \mathrm{P})$ be the probability space of 1-dim. standard Brownian motion, $\{B(t) : t \geq 0\}$ adapted to the filtration $(\mathcal{F}_t)_{t \geq 0}$.

(BM1) $B(0) = 0$ **with probability 1.**

(BM2) $B(t)$ **has a continuous path almost surely.**

(BM3) **Given any $t > 0$. For an arbitrary $n \in \mathbb{N} := \{1, 2, \dots\}$, and for any sequence of times, $t_0 := 0 < t_1 < \cdots < t_n := t$,**

- $B(t_m) - B(t_{m-1}), m = 1, \dots, n$, **are independent from each other.**
- $B(t_m) - B(t_{m-1}) \sim \mathrm{N}(0, t_m - t_{m-1}), \quad m = 1, 2, \dots, n.$

It means that, for any $0 \leq s < t < \infty$ and $a < b$,

$$\mathrm{P}[B(t) - B(s) \in [a, b]] = \int_a^b p(t - s, z|0) dz,$$

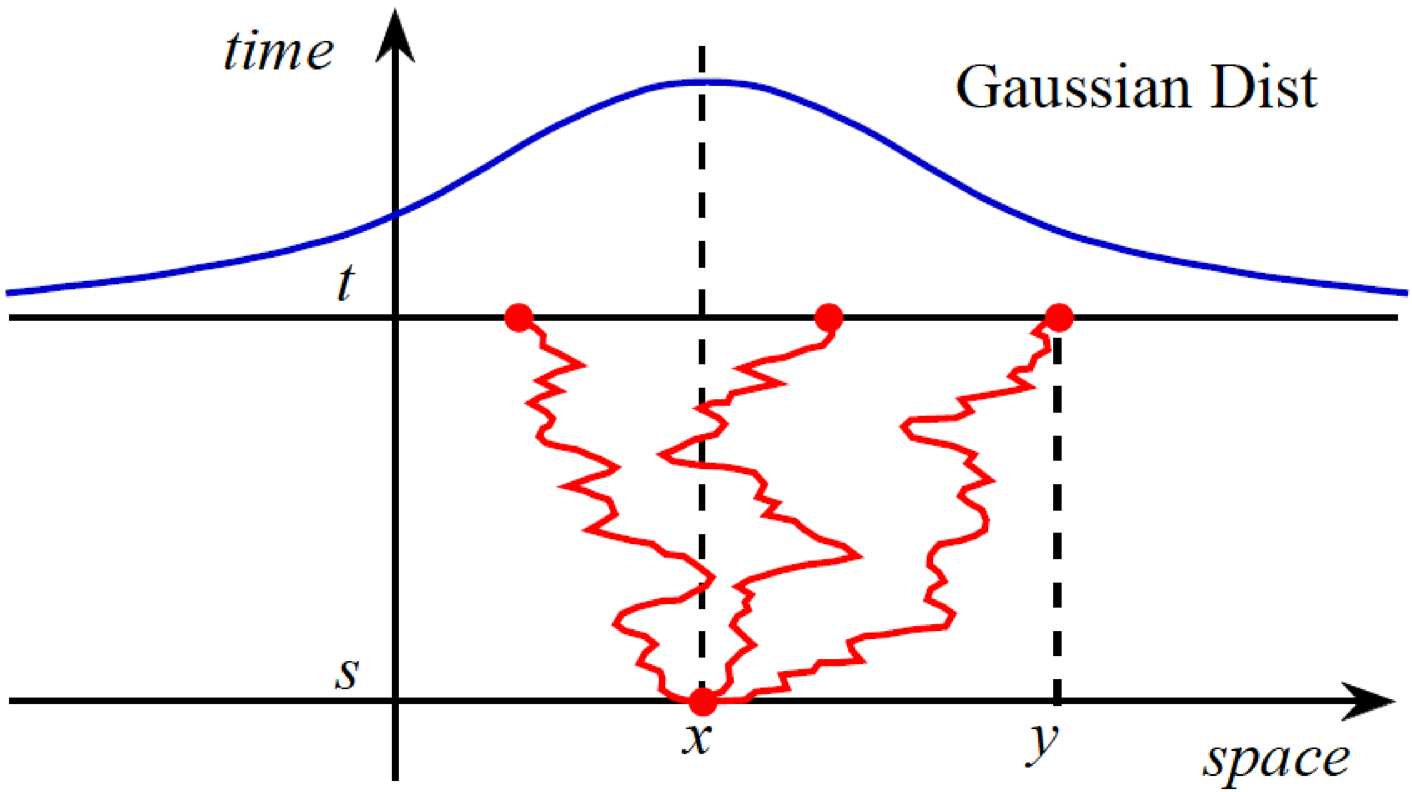
where for $x, y \in \mathbb{R}$

$$p(t, y|x) = \begin{cases} \frac{1}{\sqrt{2\pi t}} e^{-(x-y)^2/2t}, & \text{for } t > 0, \\ \delta(x - y), & \text{for } t = 0, \end{cases}$$

transition prob. density.

The mean is constant in time;

$$\begin{aligned}\mathrm{E}[B(t)|\mathcal{F}_s] &= \mathrm{E}[(B(t)-B(s))+B(s)|\mathcal{F}_s] = \mathrm{E}[B(t)-B(s)|\mathcal{F}_s]+\mathrm{E}[B(s)|\mathcal{F}_s] \\ &= 0 + B(s) = B(s), \quad 0 \le s < t < \infty, \quad \textbf{a.s.}\end{aligned}$$



For each time interval $[0,t], t>0$, **put** $n \in \mathbb{N}$ **and consider a subdivision of** $[0,t]$

$$t_0 := 0 < t_1 < \cdots < t_n := t, \quad \textbf{s.t.} \max_{1\le m\le n} |t_m - t_{m-1}| \to 0 \quad \textbf{as } n \to \infty.$$

The quadratic variation of $B(t)$ **is defined by**

$$\langle B, B\rangle_t := \mathrm{P}\text{–}\lim_{n\to\infty} \sum_{m=1}^{n} (B(t_m) - B(t_{m-1}))^2 \quad \textbf{convergence in probability}$$

$$\Longleftrightarrow \quad \lim_{n\to\infty} \mathrm{P}\left[\left| \sum_{m=1}^{n} (B(t_m) - B(t_{m-1}))^2 - \langle B, B\rangle_t \right| > \varepsilon \right] = 0, \quad \forall \varepsilon > 0.$$

We can prove

$$\langle B, B\rangle_t = t, \quad t \ge 0.$$

We can also prove that the total variation diverges,

$$\mathrm{P}\text{–}\lim_{n\to\infty} \sum_{m=1}^{n} |B(t_m) - B(t_{m-1})| \to \infty \quad \textbf{as } n \to \infty \textbf{ in probability}.$$

In summary,

$$\left.\begin{array}{l} \mathrm{E}[(B(t)|\mathcal{F}_s] = B(s), \quad 0 \le s < t < \infty, \textbf{ a.s.} \\ \displaystyle\sum_{m=1}^{n} |B(t_m) - B(t_{m-1})| \xrightarrow{\mathrm{P}} \infty, \\ \displaystyle\sum_{m=1}^{n} (B(t_m) - B(t_{m-1}))^2 \xrightarrow{\mathrm{P}} \langle B, B \rangle_t = t \uparrow \infty \textbf{ as } t \uparrow \infty \end{array}\right\} \textbf{martingale properties}$$

Consider the process $\left(B(t)^2 - \langle B, B\rangle_t\right)_{t\geq 0} = \left(B(t)^2 - t\right)_{t\geq 0}$**.**
This is martingale;

$$\begin{aligned}
\mathrm{E}[B(t)^2 - t|\mathcal{F}_s] &= \mathrm{E}[(B(t) - B(s))^2 + 2(B(t) - B(s))B(s) + B(s)^2 - t|\mathcal{F}_s] \\
&= \mathrm{E}[(B(t) - B(s))^2|\mathcal{F}_s] + 2\mathrm{E}[(B(t) - B(s))B(s)|\mathcal{F}_s] + \mathrm{E}[B(s)^2|\mathcal{F}_s] - t \\
&= (t - s) + 0 + B(s)^2 - t \\
&= B(s)^2 - s, \quad 0 \leq s < t < \infty, \quad \textbf{a.s.}
\end{aligned}$$

$$\mathrm{E}[B(t)|\mathcal{F}_s] = B(s) \iff \mathrm{E}[B(t) - B(s)|\mathcal{F}_s] = 0$$
$$\Longrightarrow \mathrm{E}[dB(t)] = 0$$

Reasonable: **increment** $dB(t) \sim$ **symmetric dist.** $\mathrm{N}(0,\ dt)$

$$\mathrm{E}[B(t)^2 - t|\mathcal{F}_s] = B(s)^2 - s \iff \mathrm{E}[(B(t)^2 - B(s)^2) - (t - s)|\mathcal{F}_s] = 0$$
$$\iff \mathrm{E}[(B(t)^2 - t) - (B(s)^2 - s)|\mathcal{F}_s] = 0$$
$$\Longrightarrow \mathrm{E}[d(B(t)^2 - t)] = 0$$
$$\iff 2\mathrm{E}[B(t)dB(t)] - dt = 0$$

(**Note again:** $dB(t) \sim \mathrm{N}(0, dt) \Longrightarrow \mathrm{E}[B(t)dB(t)] = 0$)

$$\iff -dt = 0 \quad \textbf{contradiction!}$$

This implies

$$dB(t)^2 \neq 2B(t)dB(t).$$

We need additional term to cancel $-dt$**.**
That new term should be called the Ito term:

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$$
\begin{aligned}
dB(t)^2 &= 2B(t)dB(t) + dt = 2B(t)dB(t) + \frac{1}{2} \times 2 \times dt \\
\Longleftrightarrow \quad dB(t)^2 &= df(B(t)) \qquad \qquad \textbf{(with } f(y) = y^2\textbf{)} \\
&= \left.\frac{df(y)}{dy}\right|_{y=B(t)} dB(t) + \frac{1}{2} \left.\frac{d^2 f(y)}{dy^2}\right|_{y=B(t)} d\langle B, B\rangle_t \\
&= 2y\Big|_{y=B(t)} dB(t) + \frac{1}{2} \times 2 \times d\langle B, B\rangle_t
\end{aligned}
$$

If we follow this Ito calculus (stochastic analysis), then for $(B(t)^2 - t)_{t \geq 0}$**,**

$$
\mathrm{E}[d(B(t)^2 - t)] = \mathrm{E}[2B(t)dB(t) + dt] - dt = 2\mathrm{E}[B(t)dB(t)] + dt - dt = 2\mathrm{E}[B(t)dB(t)] = 0,
$$

and the quadratic variation is $\langle 2BdB, 2BdB\rangle_t = 4B(t)^2 \langle dB, dB\rangle_t = 4B(t)^2 dt$**.**

Let $D \in \mathbb{N}$ and consider identical but independent D BMs, $B_1(t), \ldots, B_D(t)$. By definition, $B_i(t) + B_j(t)$ and $B_i(t) - B_j(t)$, $1 \le i, j \le D$ are martingales. Then their squares are not, but the followings shall be martingales;

$$\begin{aligned}(B_i(t) + B_j(t))^2 - \langle B_i + B_j, B_i + B_j \rangle_t \\ (B_i(t) - B_j(t))^2 - \langle B_i - B_j, B_i - B_j \rangle_t.\end{aligned}$$

and hence, the difference of these two should be also martingale,

$$4B_i(t)B_j(t) - (\langle B_i + B_j, B_i + B_j \rangle_t - \langle B_i - B_j, B_i - B_j \rangle_t).$$

Define the mutual quadratic variation (cross variation) by

$$\langle B_i, B_j \rangle_t := \frac{1}{4}(\langle B_i + B_j, B_i + B_j \rangle_t - \langle B_i - B_j, B_i - B_j \rangle_t).$$

This is calculated as

$$\langle B_i, B_j \rangle_t = \begin{cases} (4\langle B_i, B_i \rangle_t - 0)/4 = t, & (i = j) \\ (2t - 2t)/4 = 0, & (i \ne j) \end{cases} \quad \Longrightarrow \quad d\langle B_i, B_j \rangle_t = \delta_{ij} dt.$$

Itô's formula

- **Let $N \in \mathbb{N}$ and $\boldsymbol{y} := (y_1, \ldots, y_N) \in \mathbb{R}^N$.**
- **Let F be a function of t and $\boldsymbol{y}$; $F = F(t, \boldsymbol{y})$.**
- **Assume that a N-component stochastic process $\boldsymbol{Y}(t) = (Y_1(t), \ldots, Y_N(t)), t \geq 0$ is given.**

Then, for a stochastic process $(F(t, \boldsymbol{Y}(t)))_{t \geq 0}$, the difference is given by the following formula, which is called Itô's formula,

$$dF(t, \boldsymbol{Y}(t)) = \frac{\partial F(t, \boldsymbol{y})}{\partial t} dt + \sum_{i=1}^{N} \left. \frac{\partial F(t, \boldsymbol{y})}{\partial y_i} \right|_{\boldsymbol{y}=\boldsymbol{Y}(t)} dY_i(t)$$

$$+ \frac{1}{2} \sum_{1 \leq i,j \leq N} \left. \frac{\partial^2 F(t, \boldsymbol{y})}{\partial y_i \partial y_j} \right|_{\boldsymbol{y}=\boldsymbol{Y}(t)} d\langle Y_i, Y_j \rangle_t, \quad t \geq 0.$$

D-dim BM and D-dim Bessel process

For the BM starting from $x \in \mathbb{R}$, we write

$$B^x(t) := x + B(t), \quad t \geq 0.$$

The probability law and the expectation are written as P^x and E^x; for $\mathcal{F}_t$-measurable function f,

$$\mathrm{E}^x[f(B(t))] = \mathrm{E}[f(B^x(t))].$$

Consider D independent BMs, $B_i^{x_i}(t)$, $t \geq 0$ starting from $x_i, i = 1, \dots D$. Then define the D-dim. BM started at $\boldsymbol{x}$ as the vector-valued BM,

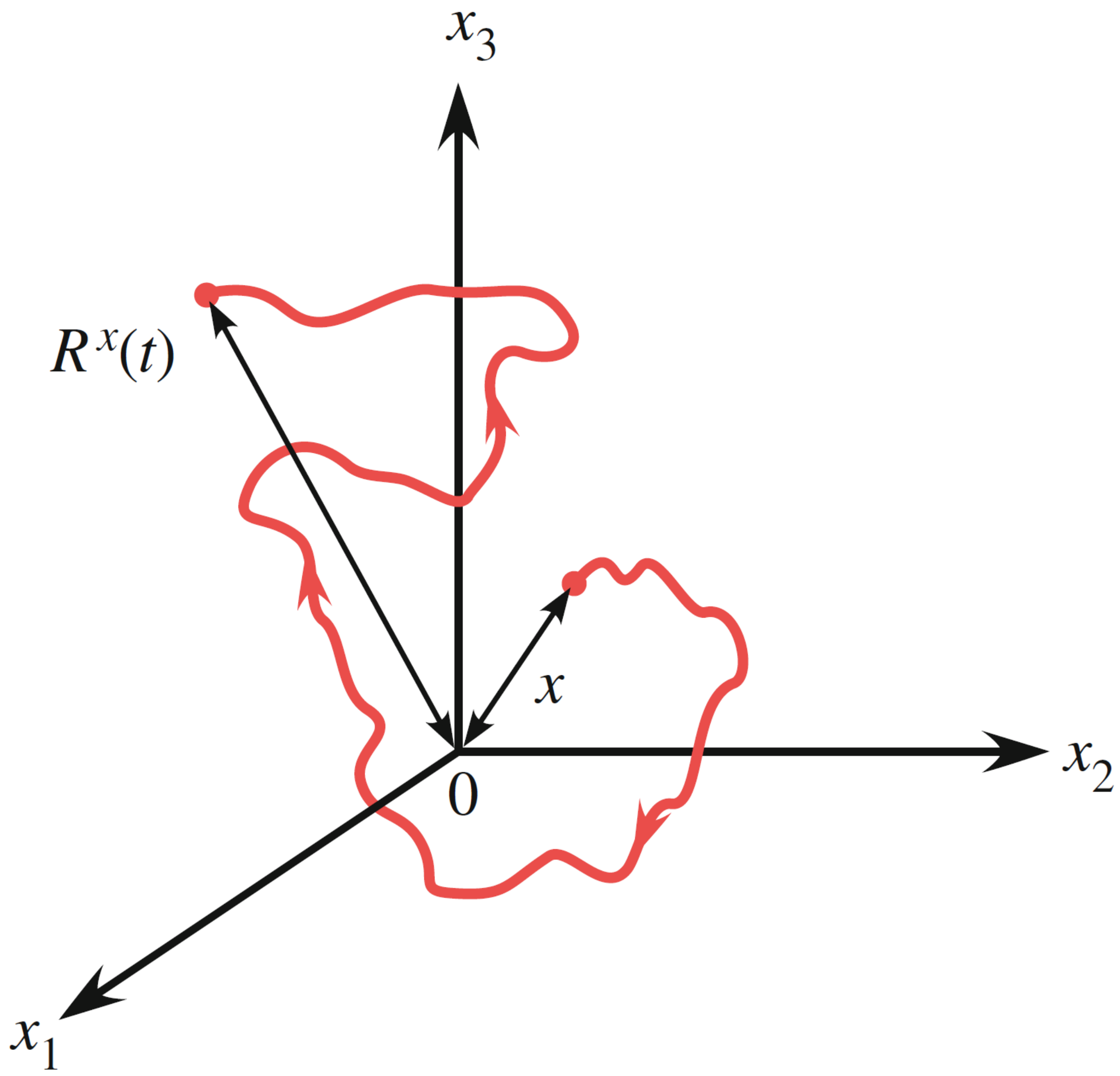
$$\boldsymbol{B}^{\boldsymbol{x}} := (B_1^{x_1}(t), B_2^{x_2}(t), \dots, B_D^{x_D}), \quad t \geq 0.$$

Then the D-dim. Bessel process, BES_D, is defined as the radial coordinate of the D-dim. BM,

$$R^x(t) := |\boldsymbol{B}^{\boldsymbol{x}}(t)| = \sqrt{B_1^{x_1}(t)^2 + \cdots + B_D^{x_D}(t)^2}, \quad t \geq 0,$$

where the initial value is given by $R^x(0) = x := |\boldsymbol{x}| = \sqrt{x_1^2 + \cdots + x_D^2} \geq 0$.

3 dim Bessel process, BES3



BES$_D$, $(R^x(t))_{t\geq 0}$, is defined as $R^x(t) = F(\boldsymbol{B}^{\boldsymbol{x}}(t))$, $t \geq 0$, with $F(\boldsymbol{y}) = \sqrt{y_1^2 + \cdots + y_D^2}$.
We see that

$$\frac{\partial F(\boldsymbol{y})}{\partial t} = 0, \quad \frac{\partial F(\boldsymbol{y})}{\partial y_i} = \frac{y_i}{F(\boldsymbol{y})}, \quad \frac{\partial^2 F(\boldsymbol{y})}{\partial y_i \partial y_j} = \frac{\delta_{ij}}{F(\boldsymbol{y})} - \frac{y_i y_j}{F(\boldsymbol{y})^3}, \quad i, j = 1, \ldots, D,$$

$$d\langle B_i^{x_i}, B_j^{x_j} \rangle_t = \delta_{ij} dt, \quad i, j = 1, \ldots, D.$$

Then the drift terms (finite variation part) become

$$\begin{aligned}
\frac{1}{2} \sum_{1 \leq i,j \leq D} \left. \frac{\partial^2 F(t, \boldsymbol{y})}{\partial y_i \partial y_j} \right|_{\boldsymbol{y}=\boldsymbol{B}^{\boldsymbol{x}}(t)} d\langle B_i^{x_i}, B_j^{x_j} \rangle_t &= \frac{1}{2} \sum_{1 \leq i,j \leq D} \left\{ \frac{\delta_{ij}}{F(\boldsymbol{B}^{\boldsymbol{x}}(t))} - \frac{B_i^{x_i}(t) B_j^{x_j}(t)}{F(\boldsymbol{B}^{\boldsymbol{x}}(t))^3} \right\} \delta_{ij} dt \\
&= \frac{1}{2} \left\{ \frac{\sum_{1 \leq i \leq D} 1}{F(\boldsymbol{B}^{\boldsymbol{x}}(t))} - \frac{\sum_{1 \leq i \leq D} B_i^{x_i}(t)^2}{F(\boldsymbol{B}^{\boldsymbol{x}}(t))^3} \right\} \delta_{ij} dt \\
&= \frac{D-1}{2} \frac{1}{F(\boldsymbol{B}^{\boldsymbol{x}}(t))} dt = \frac{D-1}{2} \frac{dt}{R^x(t)}.
\end{aligned}$$

BES$_D$, $(R^x(t))_{t\geq 0}$, **is defined as** $R^x(t) = F(\boldsymbol{B}^{\boldsymbol{x}}(t))$, $t \geq 0$, **with** $F(\boldsymbol{y}) = \sqrt{\sum_{i=1}^{D} y_i^2}$.

We see that

$$\frac{\partial F(\boldsymbol{y})}{\partial t} = 0, \qquad \frac{\partial F(\boldsymbol{y})}{\partial y_i} = \frac{y_i}{F}, \qquad \frac{\partial^2 F(\boldsymbol{y})}{\partial y_i \partial y_j} = \frac{\delta_{ij}}{F} - \frac{y_i y_j}{F^3}, \qquad i, j = 1, \dots, D,$$

$$d\langle B_i^{x_i}, B_j^{x_j} \rangle_t = \delta_{ij} dt, \qquad i, j = 1, \dots, D.$$

The martingale part is obtained as

$$\sum_{i=1}^{D} \frac{\partial F(t, \boldsymbol{y})}{\partial y_i} \Bigg|_{\boldsymbol{y} = \boldsymbol{B}^{\boldsymbol{x}}(t)} dB_i^{x_i}(t) = \frac{1}{R^x(t)} \sum_{i=1}^{D} B_i^{x_i}(t) dB_i^{x_i}(t).$$

Its quadratic variation is calculated as

$$\left\langle \frac{1}{R^x} \sum_{i=1}^{D} B_i^{x_i} dB_i^{x_i}, \frac{1}{R^x} \sum_{j=1}^{D} B_j^{x_j} dB_j^{x_j} \right\rangle_t = \frac{1}{R^x(t)^2} \sum_{i=1}^{D} \sum_{j=1}^{D} B_i^{x_i}(t) B_j^{x_j}(t) \langle dB_i^{x_i}, dB_j^{x_j} \rangle_t$$

$$= \frac{1}{R^x(t)^2} \sum_{i=1}^{D} \sum_{j=1}^{D} B_i^{x_i}(t) B_j^{x_j}(t) \delta_{ij} dt = \frac{1}{R^x(t)^2} \sum_{i=1}^{D} B_i^{x_i}(t)^2 \delta_{ij} dt = dt.$$

This implies that, with another BM $(B(t))_{t\geq 0}$, **we have**

$$\left(\frac{1}{R^x(t)} \sum_{i=1}^{D} B_i^{x_i}(t) dB_i^{x_i}(t) \right)_{t\geq 0} \overset{\text{(law)}}{=} (B(t))_{t\geq 0}.$$

Now we assume $D > 1$ (not only integers). Consider BES_D starting from $x \geq 0$ as the solution of the following stochastic differential equaiton (SDE),

$$dR^x(t) = dB^x(t) + \frac{D-1}{2} \frac{dt}{R^x(t)}, \qquad 0 \leq t < T^x,$$

where $T^x = \inf\{t > 0 : R^x(t) = 0\}$.

The first term in the RHS is a martingale part
and the second term is a finite variation part (drift term).
The process consisting of these two parts is called a semi-martingale.

Dyson's BM model

- **Fix $N \in \mathbb{N}$.**
- **Prepare N BMs,**

$$B_{ii}^{x_i}(t) = x_i + B_{ii}(t), \quad t \geq 0, \quad x_i \in \mathbb{R}, \quad i = 1, \dots, N,$$

and $N(N-1)/2$ pairs of BMs

$$\{B_{ij}(t), \widetilde{B}_{ij}(t)\}_{1 \leq i < j \leq N}, \quad t \geq 0, \quad \text{starting from the origin.}$$

- **Then, we have a total of $N + 2 \times N(N-1)/2 = N^2$ BMs, each of them independent from the rest.**
- **Then consider an $N \times N$ Hermitian-matrix-valued Brownian motion,**

$$\mathsf{B}^{\boldsymbol{x}}(t) = \begin{pmatrix} B_{11}^{x_1}(t) & \dfrac{B_{12}(t) + \sqrt{-1}\widetilde{B}_{12}(t)}{\sqrt{2}} & \cdots & \dfrac{B_{1N}(t) + \sqrt{-1}\widetilde{B}_{1N}(t)}{\sqrt{2}} \\ \dfrac{B_{12}(t) - \sqrt{-1}\widetilde{B}_{12}(t)}{\sqrt{2}} & B_{22}^{x_2}(t) & \cdots & \dfrac{B_{2N}(t) + \sqrt{-1}\widetilde{B}_{2N}(t)}{\sqrt{2}} \\ \cdots & \cdots & \cdots & \cdots \\ \dfrac{B_{1N}(t) - \sqrt{-1}\widetilde{B}_{1N}(t)}{\sqrt{2}} & \dfrac{B_{2N}(t) - \sqrt{-1}\widetilde{B}_{2N}(t)}{\sqrt{2}} & \cdots & B_{NN}^{x_N}(t). \end{pmatrix}$$

- **By definition, the initial state of this Brownian motion is the diagonal matrix $\mathsf{B}^{\boldsymbol{x}}(0) = \mathrm{diag}(x_1, x_2, \dots, x_N)$. We assume $x_1 \leq x_2 \leq \cdots \leq x_N$.**

- **Corresponding to calculating the absolute value of $\boldsymbol{B}^{\boldsymbol{x}}(t)$, by which BES_D was introduced, here we calculate the eigenvalues of $\mathsf{B}^{\boldsymbol{x}}(t)$.**

- **For any $t \geq 0$, there is a family of $N \times N$ unitary-matrix-valued process, $\{\mathsf{U}(t)\}$ which diagonalizes $\mathsf{B}^{\boldsymbol{x}}(t)$,**

$$\mathsf{U}^*(t)\mathsf{B}^{\boldsymbol{x}}(t)\mathsf{U}(t) = \mathrm{diag}(\lambda_1(t), \ldots, \lambda_N(t)) =: \Lambda(t), \quad t \geq 0.$$

- **Consider a subspace of $\mathbb{R}^N$ defined by**

$$\mathbb{W}_N := \{\boldsymbol{x} = (x_1, x_2, \ldots, x_N) \in \mathbb{R}^N : x_1 < x_2 < \cdots < x_N\},$$

which is called the Weyl chamber. If we impose the condition $(\lambda_i(t))_{i=1}^N \in \mathbb{W}_N$, $\mathsf{U}(t)$ is uniquely determined at each time $t \geq 0$.

Itô's formula

If $\mathsf{X}(t) = (X_{ij}(t))$ and $\mathsf{Y}(t) = (Y_{ij}(t))$ are $N \times N$ matrices with semi-martingale elements, then

$$d(\mathsf{X}^*(t)\mathsf{Y}(t)) = d\mathsf{X}^*(t)\mathsf{Y}(t) + \mathsf{X}^*(t)d\mathsf{Y}(t) + \langle d\mathsf{X}^*, d\mathsf{Y}\rangle_t, \quad t \geq 0,$$

where $\langle d\mathsf{X}^*, d\mathsf{Y}\rangle_t$ denotes an $N \times N$ matrix-valued process, whose (i, j)-th element is given by the finite-variation process $\sum_{k=1}^{N} \langle d\overline{X_{ki}}, dY_{kj}\rangle_t$, $1 \leq i, j \leq N$.

Applying Itô's formula to $\Lambda(t) = \mathsf{U}^*(t)B^{\boldsymbol{x}}(t)\mathsf{U}(t), t \geq 0$, we have the equality

$$d\Lambda(t) = d\mathsf{U}^*(t)\mathsf{B}^{\boldsymbol{x}}(t)\mathsf{U}(t) + \mathsf{U}^*(t)d\mathsf{B}^{\boldsymbol{x}}(t)\mathsf{U}(t) + \mathsf{U}^*(t)\mathsf{B}^{\boldsymbol{x}}(t)d\mathsf{U}(t)$$
$$+ \ \langle d\mathsf{U}^*, d\mathsf{B}^{\boldsymbol{x}}\mathsf{U}\rangle_t + \langle d\mathsf{U}^*, \mathsf{B}^{\boldsymbol{x}}d\mathsf{U}\rangle_t + \langle \mathsf{U}^*d\mathsf{B}^{\boldsymbol{x}}, d\mathsf{U}\rangle_t, \quad t \geq 0.$$

Since $\mathsf{U}^*(t)\mathsf{U}(t) = \mathsf{I}$ for each time t, Itô's formula gives,

$$0 = d(\mathsf{U}^*(t)\mathsf{U}(t)) = d\mathsf{U}^*(t)\mathsf{U}(t) + \mathsf{U}^*(t)d\mathsf{U}(t) + \langle d\mathsf{U}^*, d\mathsf{U}\rangle_t.$$

The following lemma is established.

Lemma 1.1 **The eigenvalue process** $(\lambda_i(t))_{i=1}^N$, $t \geq 0$ **of the Hermitian-matrix-valued Brownian motion** $(\mathsf{B}(t))_{t\geq 0}$ **starting from** $\mathrm{diag}(x_1, \dots, x_N)$**, satisfies the SDEs,**

$$d\lambda_i(t) = dB_i^{x_i}(t) + \sum_{\substack{1\leq j\leq N, \\ j\neq i}} \frac{dt}{\lambda_i(t) - \lambda_j(t)}, \quad t \geq 0, \quad i = 1, \dots, N,$$

where $(B_i^{x_i}(t))_{i=1}^N$, $t \geq 0$ **are independent BMs different from the** N^2 **BMs used to define** $\mathsf{B}^{\boldsymbol{x}}(t)$ **above.**

We consider the following one-parameter ($\beta > 0$) extension of the above eigenvalue-process.
Define an interacting Brownian motions, $\boldsymbol{X}(t) = (X_1(t), X_2(t), \dots, X_N(t))$, $t \geq 0$ as a solution of the following system of SDEs, with the condition $x_1 < x_2 < \cdots < x_N$ for initial positions $x_i = X_i(0), 1 \leq i \leq N$,

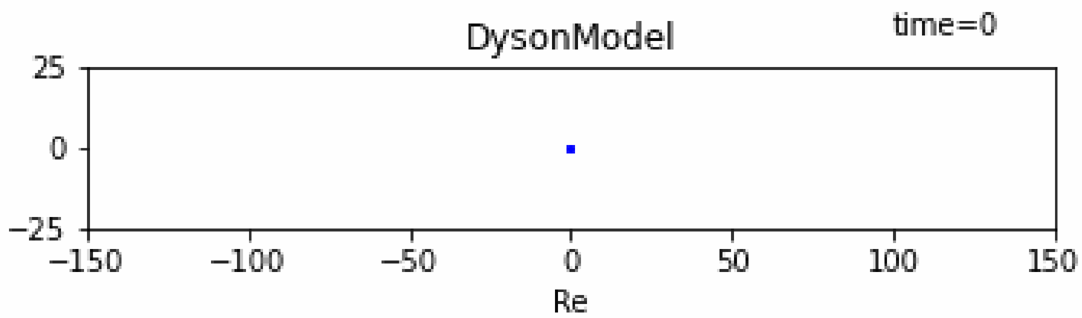
$$dX_i(t) = dB_i(t) + \frac{\beta}{2} \sum_{\substack{1 \leq j \leq N, \\ j \neq i}} \frac{dt}{X_i(t) - X_j(t)}, \quad t \in [0, T^{\boldsymbol{x}}), \quad 1 \leq i \leq N,$$

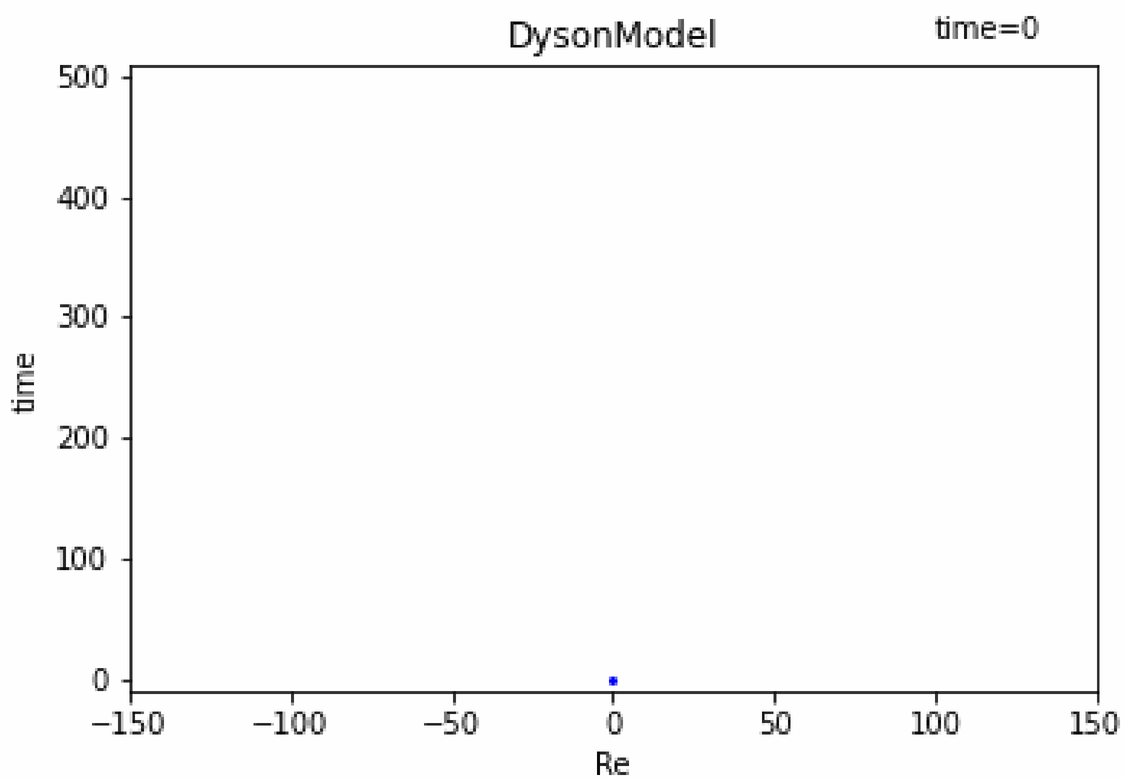
where $\{B_i(t)\}_{i=1}^N$, $t \geq 0$ are independent BMs and

$$\begin{aligned} T_{ij}^{\boldsymbol{x}} &= \inf\{t > 0 : X_i(t) = X_j(t)\}, \quad 1 \leq i < j \leq N, \\ T^{\boldsymbol{x}} &= \min_{1 \leq i < j \leq N} T_{ij}^{\boldsymbol{x}}. \end{aligned}$$

It is called Dyson's Brownian motion model with parameter β with N particles, abbreviated as DYS_β^N here.

$\mathbf{DYS}_2^N \stackrel{\text{(law)}}{=}$ **eigenvalue process of the Hermitian-matrix-valued BM**





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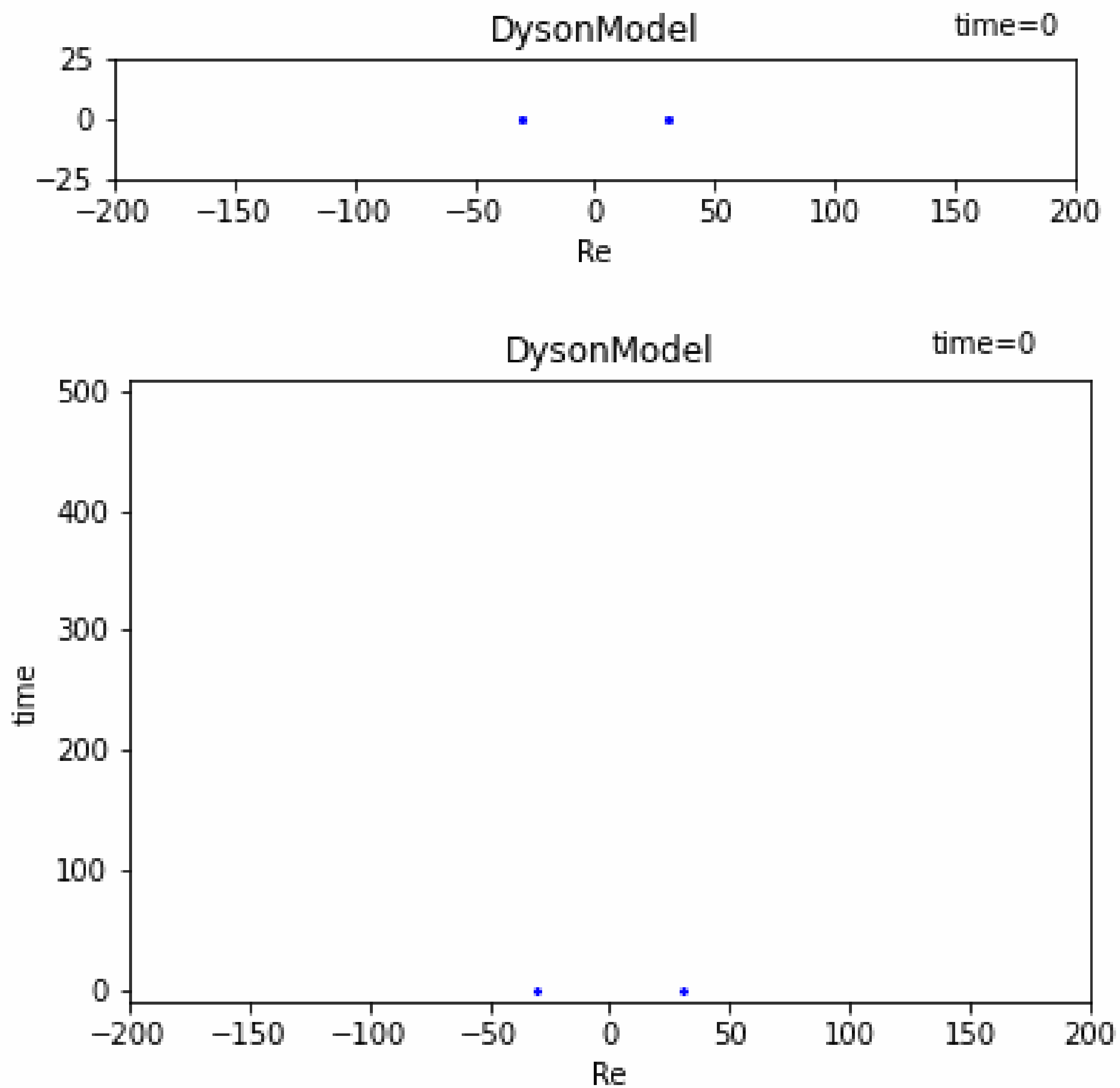
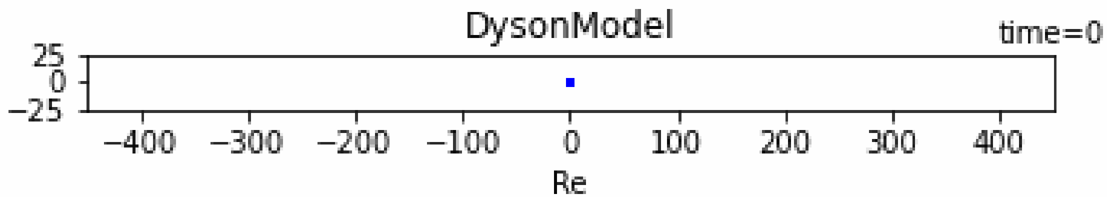
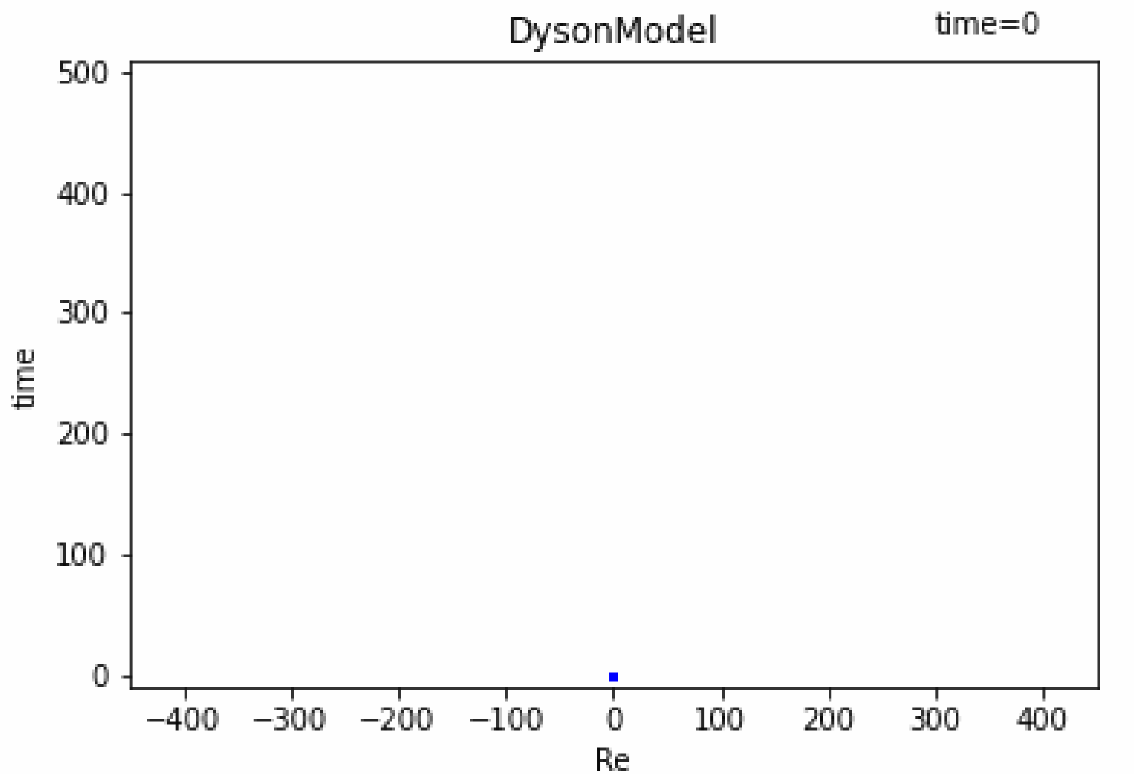
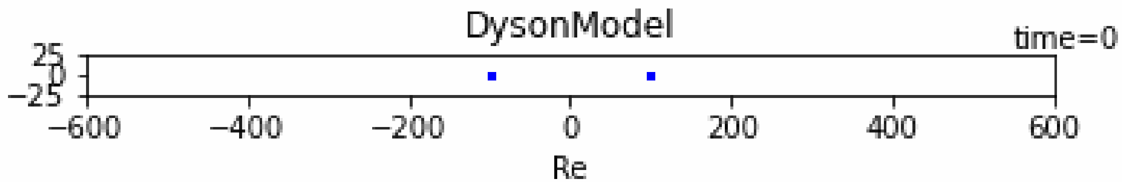


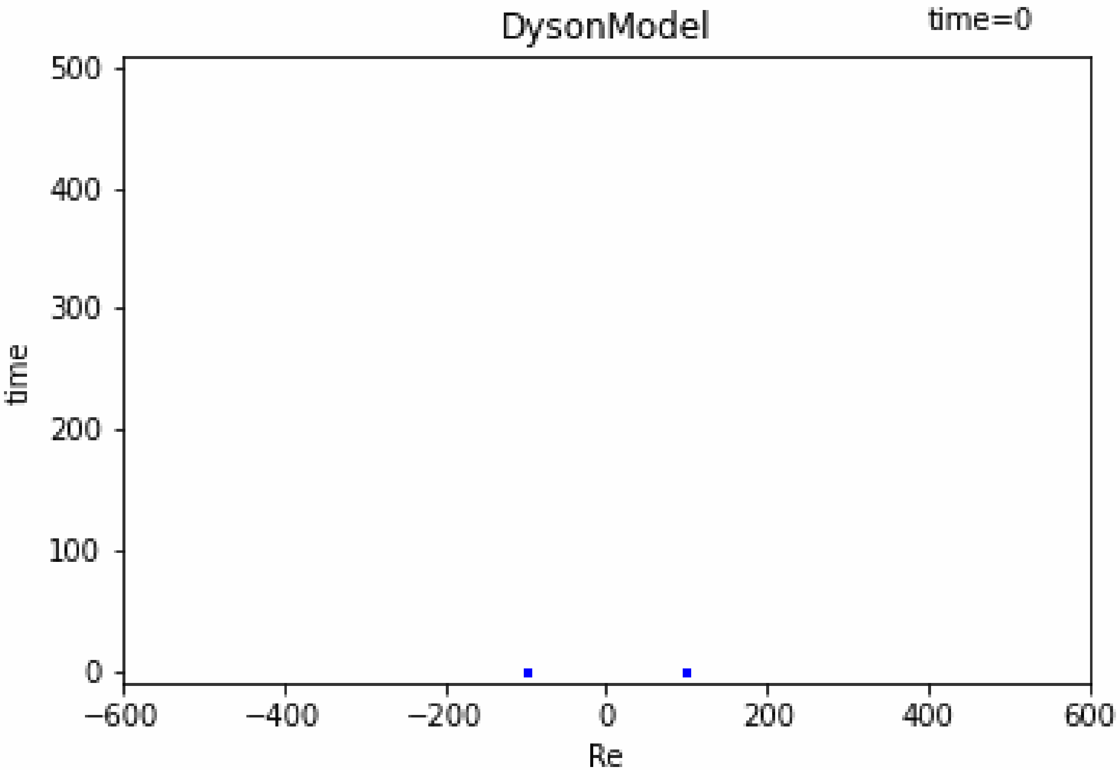
$\mathbf{DYS}_2^N \overset{\text{(law)}}{=}$ eigenvalue process of the Hermitian-matrix-valued BM





$\mathbf{DYS}_2^N \overset{\text{(law)}}{=}$ **eigenvalue process of the Hermitian-matrix-valued BM**





1. From Brownian motion to Dyson's BM model
(random points: stochastic log-gas)

2. Schramm-Loewner evolution (SLE) and Gaussian free fields
(random curves and random surfaces)

3. Multiple SLE/GFF coupling driven by Dyson's BM model
(random points/curves/surfaces)

4. Concluding Remarks

Complexification of Bessel process flow

We start from BES$_D$, $(R^x(t))_{t\geq 0}$, introduced in **Section 1** :

$$dR^x(t) = dB^x(t) + \frac{D-1}{2}\frac{dt}{R^x(t)}, \quad x > 0, \quad 0 \leq t < T^x.$$

- This is a diffusion process on $\mathbb{R}_{\geq 0} := \mathbb{R}_+ \cup \{0\}$.
- x in the supper script indicates the initial position on $\mathbb{R}_{\geq 0}$.

Denote the upper half complex plane as $\mathbb{H} := \{z \in \mathbb{C} : \mathrm{Im}\, z > 0\}$,
and let $\overline{\mathbb{H}} := \mathbb{H} \cup \mathbb{R}$.
We set $Z^z(t) = X^z(t) + \sqrt{-1}Y^z(t) \in \overline{\mathbb{H}} \setminus \{0\},\ t \geq 0$ and complexificate BES$_D$ as

$$dZ^z(t) = dB(t) + \frac{D-1}{2}\frac{dt}{Z^z(t)}$$

with the initial condition $Z^z(0) = z = x + \sqrt{-1}y \in \overline{\mathbb{H}} \setminus \{0\}$.

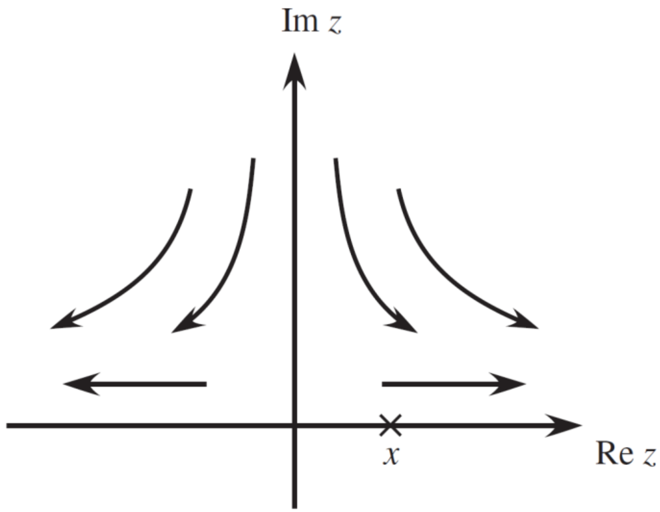
- The crucial point of this complexification of BES$_D$ is that the BM remains real, $B(t) \in \mathbb{R},\ t \geq 0$.

Then, there is an asymmetry between the real part and the imaginary part of the flow in $\mathbb{H}$,

$$dX^z(t) = dB(t) + \frac{D-1}{2}\frac{X^z(t)}{(X^z(t))^2 + (Y^z(t))^2}dt,$$
$$dY^z(t) = -\frac{D-1}{2}\frac{Y^z(t)}{(X^z(t))^2 + (Y^z(t))^2}dt.$$

Assume $D > 1$.

- **As indicated by the minus sign in RHS of the second equation, the flow is downward in $\overline{\mathbb{H}}$.**
- **If the flow goes down and arrives at the real axis, the imaginary part vanishes, $Y^z(t) = 0$, then equation is reduced to be the same equation as equation for the BES$_D$, which is now considered for $\mathbb{R} \setminus \{0\} = \mathbb{R}_+ \cup \mathbb{R}_-$.**



For $z \in \overline{\mathbb{H}} \setminus \{0\}$, $t \geq 0$, **put**

$$\widehat{g}_t(z) := Z^z(t) + B(t).$$

Then,

$$dZ^z(t) = dB(t) + \frac{D-1}{2}\frac{dt}{Z^z(t)} \quad \Longrightarrow \quad \frac{\partial \widehat{g}_t(z)}{\partial t} = \frac{D-1}{2}\frac{1}{\widehat{g}_t(z) - B(t)}, \quad t \geq 0.$$

Put

$$\kappa = \frac{4}{D-1} \quad \Longleftrightarrow \quad D = 1 + \frac{4}{\kappa},$$

and set $g_t(z) = \sqrt{\kappa}\widehat{g}_t(z)$. **Then we have the equation in the form**

$$\boxed{\frac{\partial g_t(z)}{\partial t} = \frac{2}{g_t(z) - \sqrt{\kappa}B(t)}, \quad t \geq 0.}$$

This equation is called the Schramm–Loewner evolution with parameter κ; **SLE**$_\kappa$.

Loewer equation

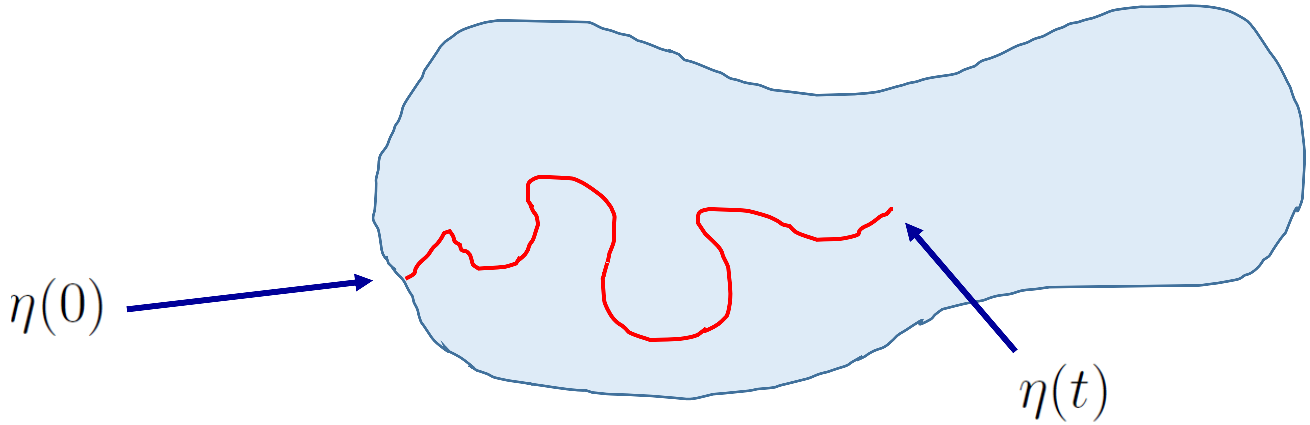
- Let D be a simply connected domain in $\mathbb{C}$ which does not complete the plane; $D \subsetneq \mathbb{C}$.
- Its boundary is denoted by ∂D.
- We consider a slit in D, which is defined as a trace

$$\eta = \{\eta(t) : t \in (0, \infty)\}$$

of a simple curve $\eta(t) \in D, 0 < t < \infty$; $\eta(s) \neq \eta(t)$ for $s \neq t$.

- We assume that the initial point of the slit is located in ∂D,

$$^{\exists}\eta(0) := \lim_{t \downarrow 0} \eta(t) \in \partial D.$$

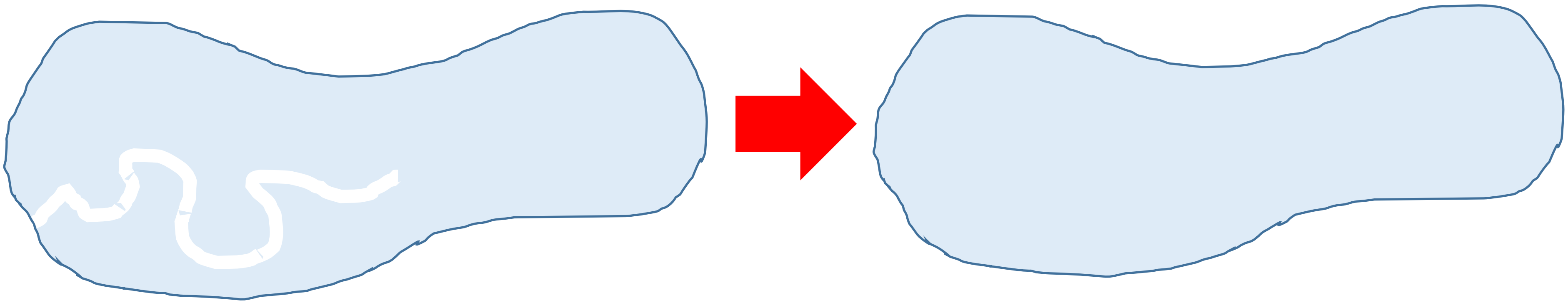


- **Let**

$$\eta(0,t] := \{\eta(s) : s \in (0,t]\} \quad \textbf{and} \quad D_t^\eta := D \setminus \eta(0,t], \quad t \in (0,\infty).$$

- **The Loewner theory (1923) describes a slit** η **by encoding the curve into a time-dependent analytic function** $g_{D_t^\eta} : t \in (0,\infty)$ **such that**

$$g_{D_t^\eta} : \textbf{conformal map } D_t^\eta \to D, \quad t \in (0,\infty).$$



Here we consider the Loewner theory for

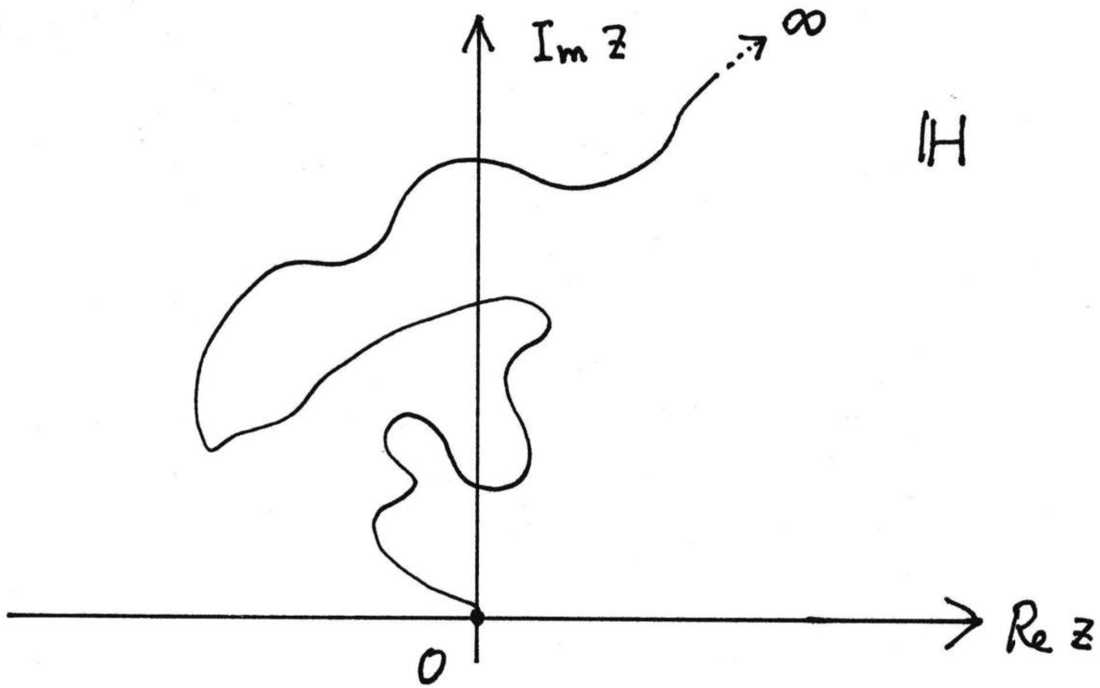
$$D = \mathbb{H} := \{z \in \mathbb{C} : \operatorname{Im} z > 0\} \qquad \textbf{(the upper half plane)}.$$

$\eta := \{\eta(t) : t \in (0, \infty)\}$ **is a simple curve**,

$\eta(0) := \lim_{t \to 0} \eta(t) = 0 \in \mathbb{R}$,

$\eta(0, t] \subset \mathbb{H}, \quad \forall t \in (0, \infty)$,

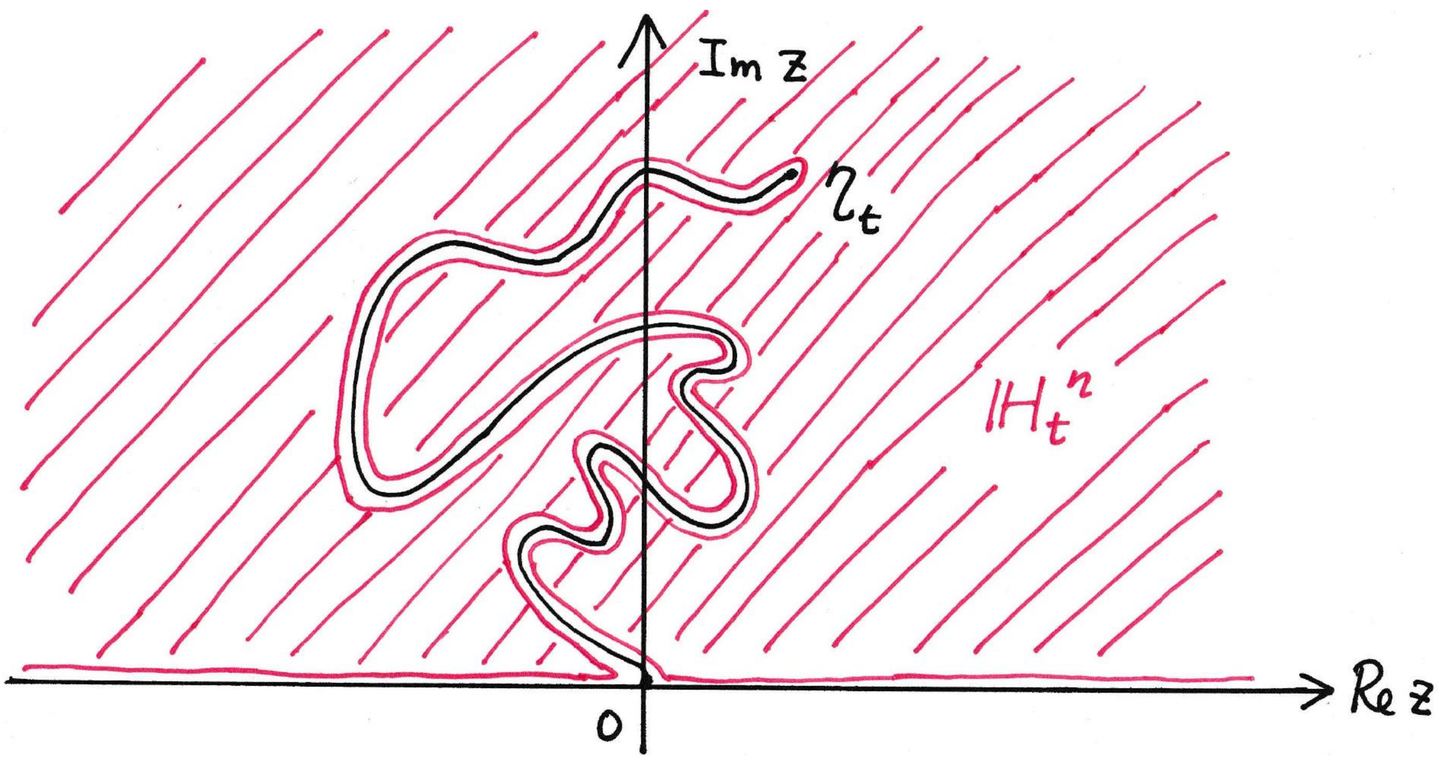
$\lim_{t \to \infty} \eta(t) = \infty$.



- **For each time $t \in (0, \infty)$,**

$$\mathbb{H}_t^\eta := \mathbb{H} \setminus \eta(0, t]$$

is a simply connected domain in $\mathbb{C}$.



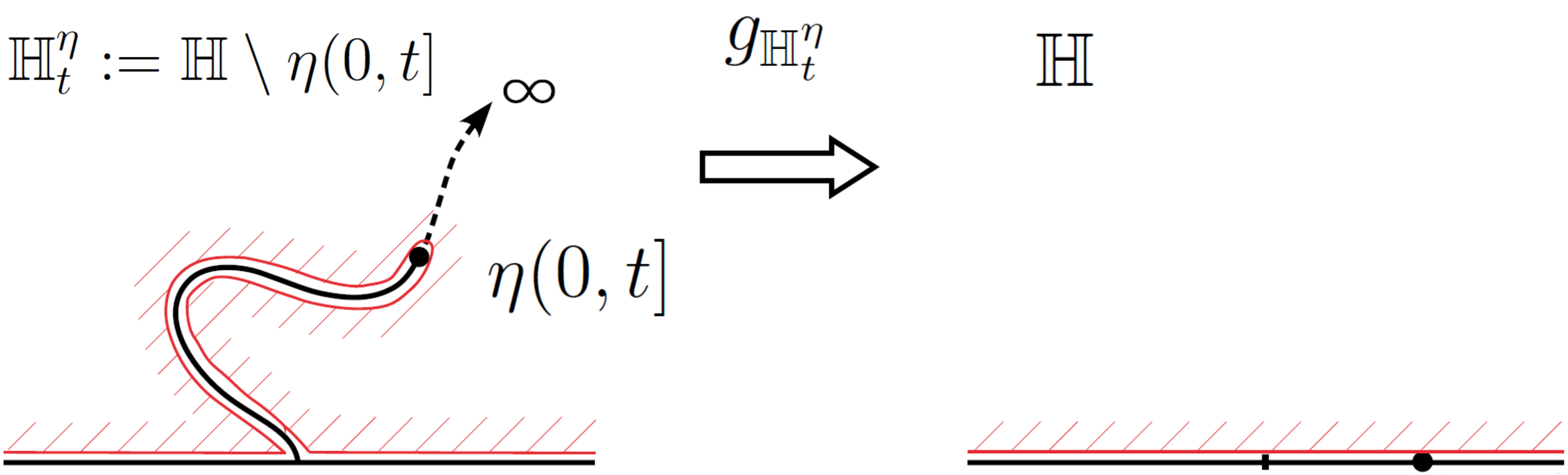
- **For each time** $t \in (0, \infty)$**,**

$$\mathbb{H}_t^\eta := \mathbb{H} \setminus \eta(0,t]$$

  is a simply connected domain in $\mathbb{C}$**.**

- **And there exist analytic functions** $g_{\mathbb{H}_t^\eta}$ **such that**

$$g_{\mathbb{H}_t^\eta} : \textbf{conformal map } \mathbb{H}_t^\eta \to \mathbb{H}.$$



- **We specify the conformal map by putting the condition**

$$g_{\mathbb{H}_t^\eta}(z) = z + \frac{c_t}{z} + \mathrm{O}(|z|^{-2}) \quad \textbf{as } z \to \infty$$

for some $c_t > 0$, in which the coefficient of z is unity and no constant term appears. This is called the hydrodynamic normalization.

- **The coefficient c_t gives the half-plane capacity of $\eta(0,t]$ and denoted by $\mathrm{hcap}(\eta(0,t])$.**

Theorem 2.1 **(Löwner (1923), Kufarev–Sobolev–Sporyševa (1968))** **Let** η **be a slit in** $\mathbb{H}$ **for which the parameterization by** t **is arranged so that**

$$c_t = \mathrm{hcap}(\eta(0,t]) = 2t, \quad t \in (0,\infty).$$

Then there exists a unique continuous driving function $V(t) \in \mathbb{R}, t \in (0,\infty)$ **such that the solution** g_t **of the differential equation**

$$\frac{dg_t(z)}{dt} = \frac{2}{g_t(z) - V(t)}, \quad t \geq 0,$$

under the initial condition

$$g_0(z) = z \in \mathbb{H}$$

gives $g_t = g_{\mathbb{H}_t^\eta}, t \in (0,\infty)$**.**

The above equation is called the chordal Loewner equation.

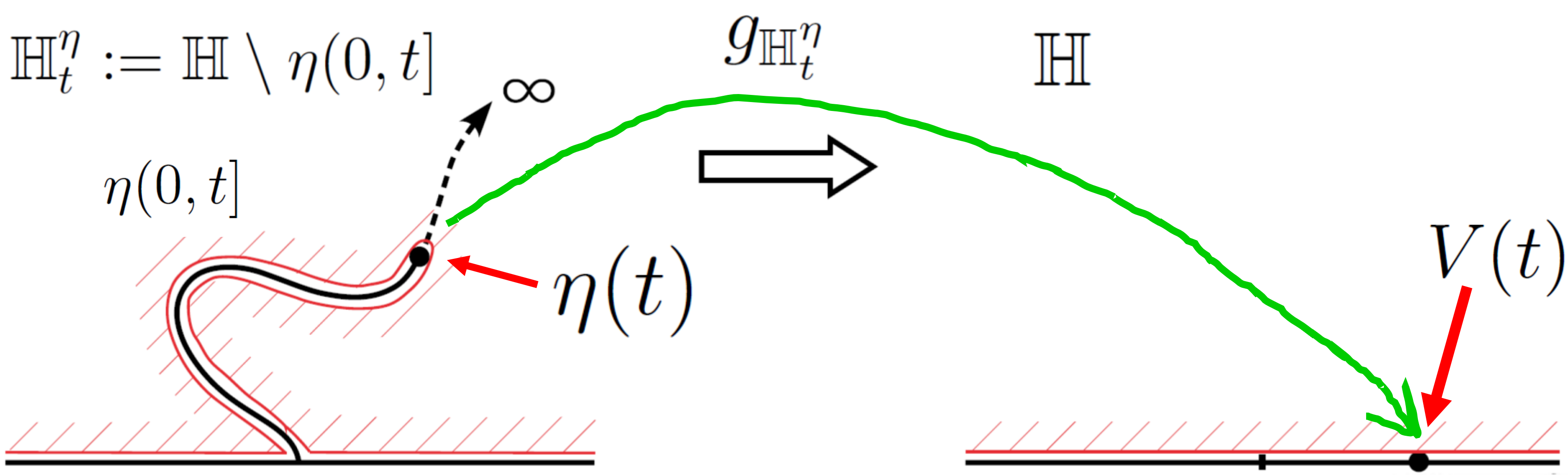
- **Note that at each time $t \in (0, \infty)$, the tip of slit $\eta(t)$ and the value of $V(t)$ satisfy the following relations,**

$$V(t) = \lim_{\substack{z \to 0, \\ \eta(t)+z \in \mathbb{H}_t^\eta}} g_{\mathbb{H}_t^\eta}(\eta(t) + z) \quad \Longleftrightarrow \quad \eta(t) = \lim_{\substack{z \to 0, \\ z \in \mathbb{H}}} g_{\mathbb{H}_t^\eta}^{-1}(V(t) + z), \quad t \geq 0.$$

- **Moreover, $V(t) = \lim_{s<t, s \to t} g_{\mathbb{H}_s^\eta}(\eta(t))$ and $t \mapsto V(t)$ is continuous.**

- **We write**

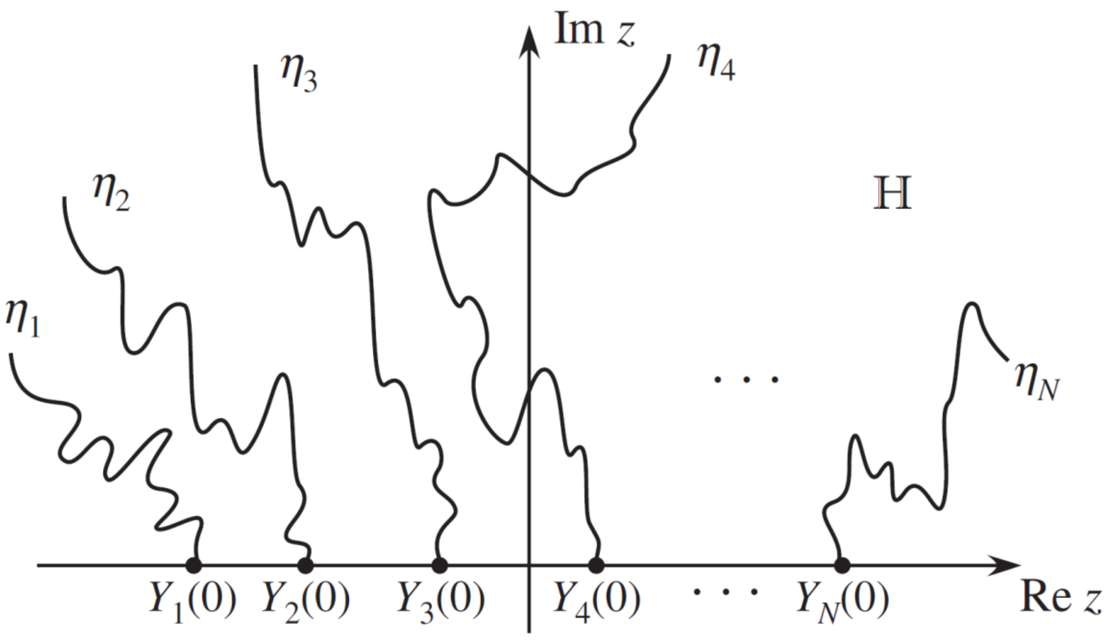
$$g_{\mathbb{H}_t^\eta}(\eta(t)) = V(t) \in \mathbb{R}, \quad \Longleftrightarrow \quad \eta(t) = g_{\mathbb{H}_t^\eta}^{-1}(V(t)) \in \partial \mathbb{H}_t^\eta, \quad t \geq 0.$$



Loewer equation with multi-slit

Theorem 2.1 can be extended to the situation such that η in $\mathbb{H}$ is given by a multi-slit (Roth–Schleissinger (2017), K–Koshida (2021)).

- Let $N \in \mathbb{N} := \{1, 2, \dots\}$ and assume that we have N slits $\eta_i = \{\eta_i(t) : t \in (0, \infty)\} \subset \mathbb{H}$, $i = 1, \dots, N$, such that
  - they are simple curves, disjoint with each other, $\eta_i \cap \eta_j = \emptyset, i \neq j$,
  - starting from N distinct points $\lim_{t \to 0} \eta_i(t) =: \eta_i(0)$ on $\mathbb{R}$;
    $$\eta_1(0) < \cdots < \eta_N(0),$$
  - and all going to infinity; $\lim_{t \to \infty} \eta_i(t) = \infty$, $i = 1, \dots, N$.



- A multi-slit is defined as a union of them, $\bigcup_{i=1}^{N} \eta_i$, and

$$\mathbb{H}_t^\eta := \mathbb{H} \setminus \bigcup_{i=1}^{N} \eta_i(0,t]$$

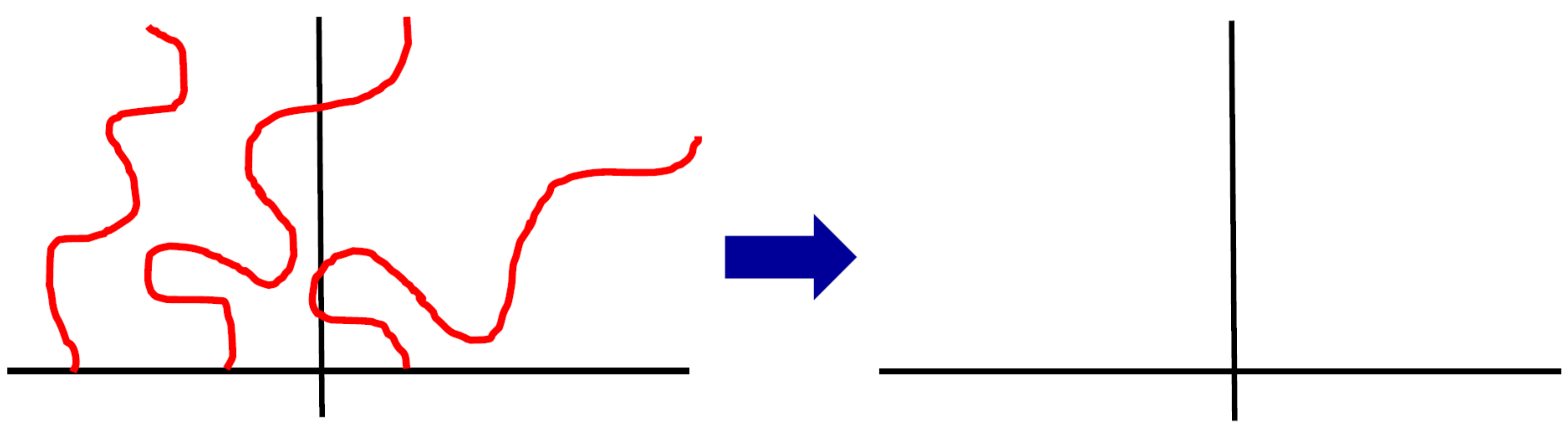
for each $t > 0$ with $\mathbb{H}_0^\eta := \mathbb{H}$.

- For each time $t \in (0, \infty)$, $\mathbb{H}_t^\eta$ is a simply connected domain in $\mathbb{C}$ and then there exists a unique analytic function $g_{\mathbb{H}_t^\eta}$ such that

$$g_{\mathbb{H}_t^\eta} : \text{conformal map } \mathbb{H}_t^\eta \to \mathbb{H},$$

satisfying the hydrodynamic normalization condition

$$g_{\mathbb{H}_t^\eta}(z) = z + \frac{\mathrm{hcap}(\bigcup_{i=1}^{N} \eta_i(0,t])}{z} + \mathrm{O}(|z|^{-2}) \quad \text{as } z \to \infty.$$



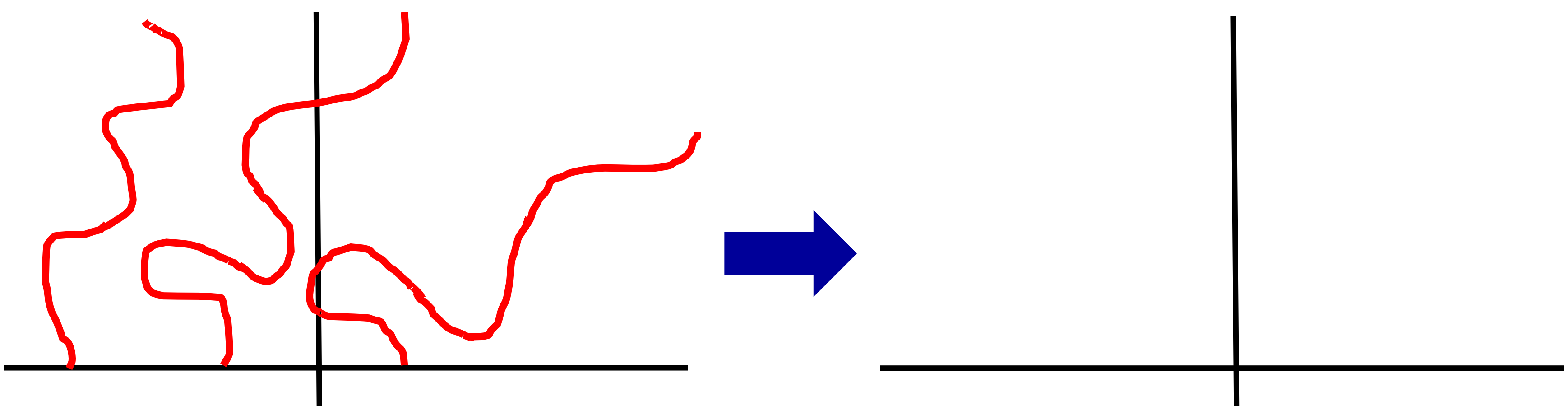
Theorem 2.2 For $N \in \mathbb{N}$, let $\bigcup_{i=1}^{N} \eta_i$ be a multi-slit in $\mathbb{H}$ such that

$$\operatorname{hcap}\Big(\bigcup_{i=1}^{N} \eta(0,t]\Big) = 2Nt, \quad t \in (0,\infty).$$

Then there exists a set of weight functions $w_i(t) \geq 0, t \geq 0, i = 1, \ldots, N$ satisfying $\sum_{i=1}^{N} w_i(t) = 1, t \geq 0$ and an N-variate continuous driving function $\boldsymbol{V}(t) = (V_1(t), \ldots, V_N(t)) \in \mathbb{R}^N, t \in (0,\infty)$ such that the solution g_t of the differential equation

$$\frac{dg_t(z)}{dt} = \sum_{i=1}^{N} \frac{2Nw_i(t)}{g_t(z) - V_i(t)}, \quad t \geq 0, \quad g_0(z) = z,$$

gives $g_t = g_{\mathbb{H}_t^\eta}, t \in (0,\infty)$.



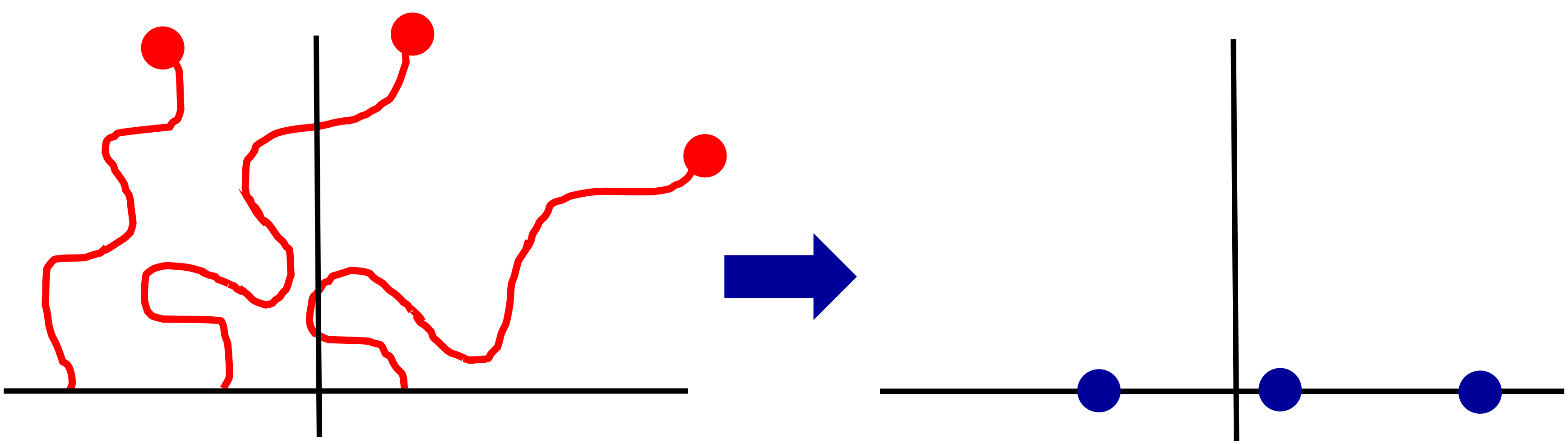
Similarly to the previous single-slit case, the following relations hold,

$$V_i(t) = \lim_{\substack{z\to 0,\\ \eta_i(t)+z\in\mathbb{H}_t^\eta}} g_{\mathbb{H}_t^\eta}(\eta_i(t)+z) \quad \Longleftrightarrow \quad \eta_i(t) = \lim_{\substack{z\to 0,\\ z\in\mathbb{H}}} g_{\mathbb{H}_t^\eta}^{-1}(V_i(t)+z), \quad i=1,\dots,N, \quad t\geq 0,$$

and we write for the multiple tips $\eta_i(t)$**,** $i=1,\dots,N$**,** $t\geq 0$**,**

$$g_{\mathbb{H}_t^\eta}(\eta_i(t)) = V_i(t) \in \mathbb{R}, \quad i=1,\dots,N, \quad t\geq 0$$

in the above sense.



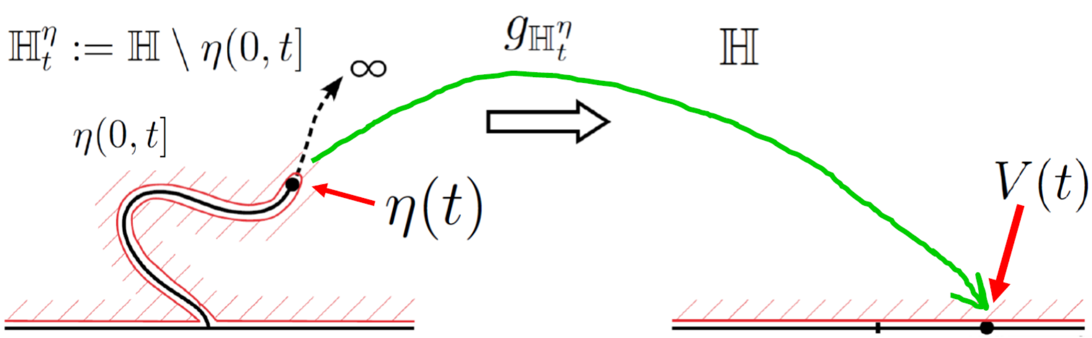
Schramm– Loewner evolution (SLE)

- So far we have considered the problem such that, given time-evolution of a single slit $\eta(0,t], t \geq 0$ or a multi-slit $\bigcup_{i=1}^{N} \eta(0,t], t \geq 0$ in $\mathbb{H}$, time-evolution of the conformal map from $\mathbb{H}_t^{\eta}$ to $\mathbb{H}$, $t \geq 0$ is asked.

- The answers are given by the solution of the Loewner equation in Theorem 2.1 for a single slit and by the solution of the multiple Loewner equation in Theorem 2.2 for a multi-slit, which are driven by a function $(V(t))_{t\geq 0}$ and by a multi-variate function $\boldsymbol{V}(t) = (V_1(t), \dots, V_N(t)) \in \mathbb{R}^N, t \geq 0$, respectively.

- The both processes are defined in $\mathbb{R}$ and deterministic:

single slit $\eta(0,t] \in \mathbb{H}, t \geq 0$	driving function $(V(t))_{t\geq 0}$ on $\mathbb{R}$
multi-slit $\bigcup_{i=1}^{N} \eta(0,t] \in \mathbb{H}, t \geq 0$	multi-variate $(\boldsymbol{V}(t))_{t\geq 0}$ on $\mathbb{R}^N$



- For $\mathbb{H}$ with a single slit, Schramm (2000) considered an inverse problem in a probabilistic setting.
  - He first asked a suitable family of driving stochastic processes $(Y(t))_{t\geq 0}$ on $\mathbb{R}$.
  - Then he asked the probability law of a random slit in $\mathbb{H}$, which will be determined by the above mentioned relations from $(Y(t))_{t\geq 0}$ and the solution $g_t = g_{\mathbb{H}_t^\eta}, t \geq 0$ of the Loewner equation):

random curve $\eta(0,t] \in \mathbb{H}, t \geq 0$ ⬅ driving stochastic process $(Y(t))_{t\geq 0}$ on $\mathbb{R}$

- **Schramm argued that conformal invariance implies that the driving process $(Y(t))_{t\geq 0}$ should be a continuous Markov process which has in a particular parameterization independent increments.**

- **Hence $Y(t)$ can be a constant time change of a one-dimensional standard Brownian motion $(B(t))_{t\geq 0}$, and it is expressed as**

$$(\sqrt{\kappa}B(t))_{t\geq 0} \overset{\text{(law)}}{=} (B(\kappa t))_{t\geq 0} \quad \textbf{with a parameter } \kappa > 0.$$

The solution of the Loewner equation driven by $Y(t) = \sqrt{\kappa}B(t), t \geq 0$,

$$\frac{dg_{\mathbb{H}_t^\eta}(z)}{dt} = \frac{2}{g_{\mathbb{H}_t^\eta}(z) - \sqrt{\kappa}B(t)}, \quad t \geq 0, \quad g_{\mathbb{H}_0^\eta}(z) = z \in \mathbb{H},$$

is called the chordal Schramm–Loewner evolution (chordal SLE) with parameter $\kappa > 0$ and is written as SLE_κ for short.

The following was proved by Lawler, Schramm, and Werner (2004) for $\kappa = 8$ and by Rohde and Schramm (2005).

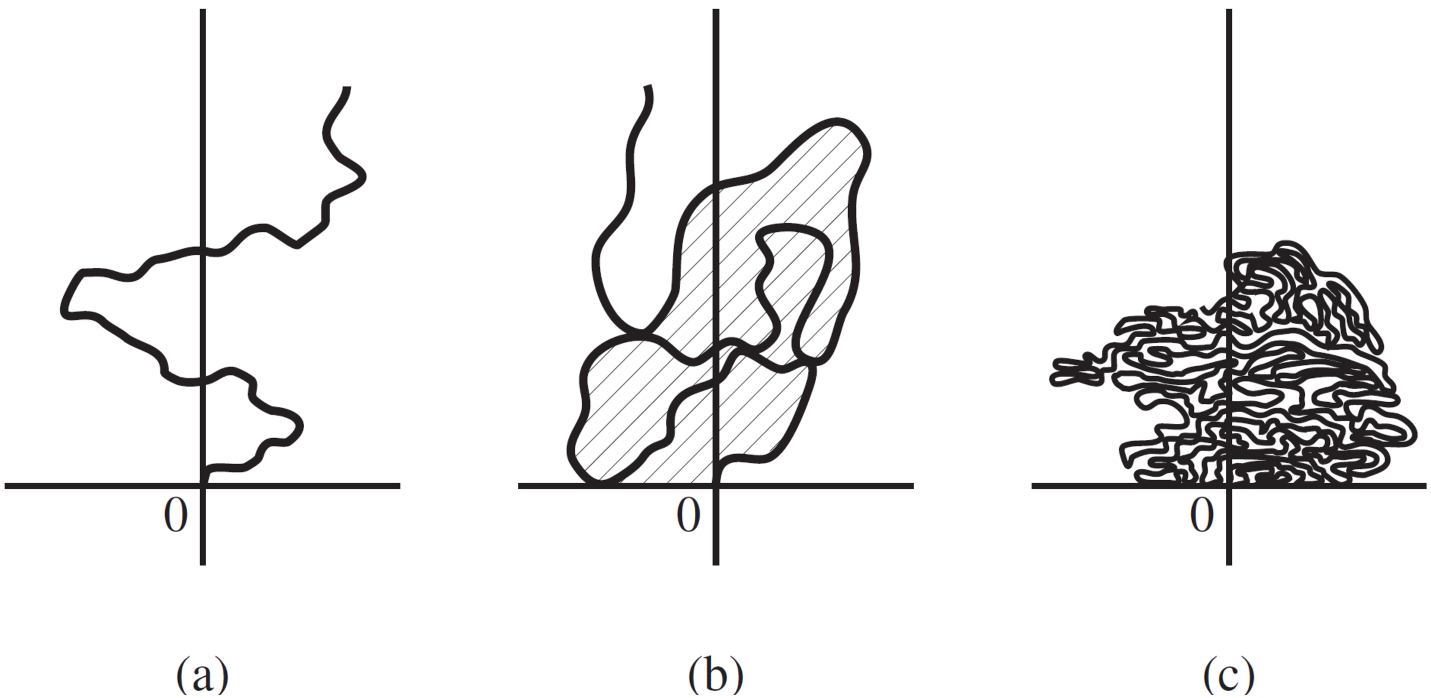
Proposition 2.3 A chordal SLE$_\kappa$ $g_{\mathbb{H}_t^\eta}, t \in (0, \infty)$ determines a continuous curve such that

$$\eta = \{\eta(t) : t \in (0, \infty)\} \subset \mathbb{H} \cup \mathbb{R}$$
$$\eta(0) := \lim_{t \downarrow 0} \eta(t) = 0,$$
$$\lim_{t \to \infty} |\eta(t)| = \infty$$

with probability one.

- The continuous curve η determined by an SLE$_\kappa$ is called an SLE$_\kappa$ curve (or SLE$_\kappa$ trace)

- **The probability law of an SLE_κ curve depends on κ.**
  - **As a matter of fact, SLE_κ curve becomes self-intersecting and can touch the real axis $\mathbb{R}$ when $\kappa > 4$, so it is no more a slit, since a slit has been defined as a trace of a continuous simple curve.**
  - **When $\kappa > 4$, the domain $\mathbb{H} \setminus \eta(0,t]$ is divided into many components, only one of which is unbounded.**



(a) $0 < \kappa \leq 4$ **(b)** $4 < \kappa < 8$ **(c)** $\kappa = 8$

So here we change the definition of $\mathbb{H}_t^\eta, t \geq 0$ as follows,

$$\mathbb{H}_t^\eta := \text{the unbounded component of } \mathbb{H} \setminus \eta(0,t], \quad t \geq 0.$$

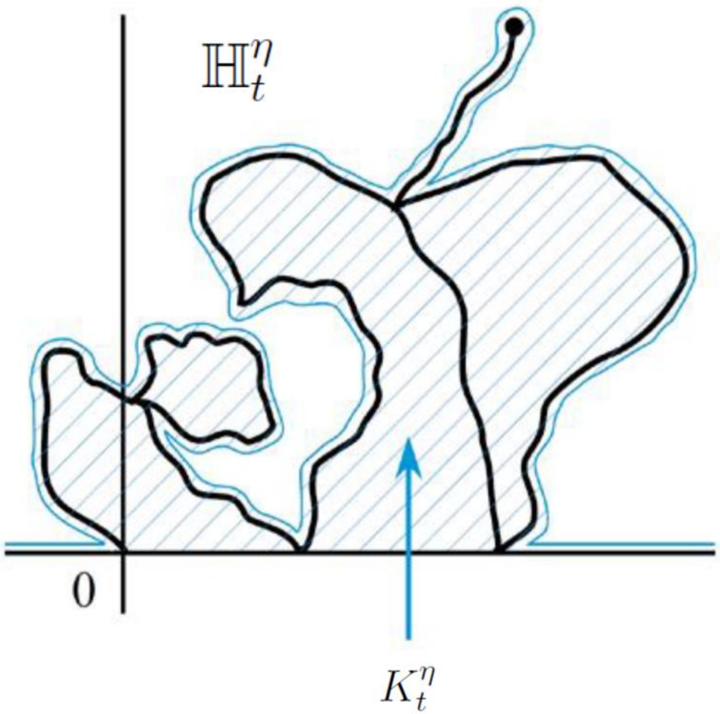
Then

$$g_{\mathbb{H}_t^\eta} : \text{conformal map } \mathbb{H}_t^\eta \to \mathbb{H}, \quad t \geq 0.$$

We also define

$$K_t^\eta := \overline{\mathbb{H} \setminus \mathbb{H}_t^\eta}, \quad t \geq 0,$$

and call it the SLE hull.



Selection problem of driving process for multiple SLE

For simplicity, we assume the equal weight $w_i(t) \equiv 1/N, t \geq 0, i = 1, \dots, N$ in **Theorem 2.2.**

Then the Loewner equation for the multi-slit in $\mathbb{H}$ is written as

$$\frac{dg_{\mathbb{H}_t^\eta}(z)}{dt} = \sum_{i=1}^{N} \frac{2}{g_{\mathbb{H}_t^\eta}(z) - Y_i(t)}, \quad t \geq 0, \quad g_{\mathbb{H}_0^\eta}(z) = z \in \mathbb{H}.$$

Then we ask what is the suitable family of driving stochastic processes of N particles on $\mathbb{R}$, $\boldsymbol{Y}(t) = (Y_1(t), \dots, Y_N(t)), t \geq 0$;

multiple **SLE**$_\kappa$ curves $\bigcup_{i=1}^{N} \eta_i(0, t] \in \mathbb{H}, t \geq 0$

$\Longleftarrow$ driving many-particle stochastic process $(\boldsymbol{Y}(t))_{t \geq 0}$ on $\mathbb{R}^N$?

- **The same argument with Schramm (2000) will give that $\boldsymbol{Y}(t)$ should be a continuous Markov process.**

- **Moreover, Bauer, Bernard, and Kytölä (2005), Graham (2007), and Dubédat (2007) argued that $(Y_i^{\mathbb{R}}(t))_{t\geq 0}, i = 1, \dots, N$ are semi-martingales and the quadratic variations should be given by $\langle dY_i, dY_j \rangle_t = \kappa \delta_{ij} dt, t \geq 0$, $1 \leq i, j \leq N$ with $\kappa > 0$.**

- **Then we will be able to assume that the system of SDEs for $(\boldsymbol{Y}(t))_{t\geq 0}$ is in the form,**

$$dY_i(t) = \sqrt{\kappa} dB_i(t) + F_i(\boldsymbol{Y}(t))dt, \quad t \geq 0, \quad i = 1, \dots, N,$$

where $(B_i(t))_{t\geq 0}, i = 1, \dots, N$ are independent one-dimensional standard Brownian motions, $\kappa > 0$, and $\{F_i(\boldsymbol{x})\}_{i=1}^N$ are suitable functions of $\boldsymbol{x} = (x_1, \dots, x_N)$ which do not explicitly depend on t.

In the following, we will give a theory so that the driving process $(\boldsymbol{Y}(t))_{t\geq 0}$ should be a time change of DYS_β with $\beta = 8/\kappa$ to construct a proper multiple SLE.

- We consider the Gaussian free field (GFF) and its generalization called the imaginary surface with parameter χ, which are considered as the distribution-valued random fields on $\mathbb{H}$.

- Under the relation
$$\chi = \frac{2}{\sqrt{\kappa}} - \frac{\kappa}{\sqrt{2}},$$
we regard the SLE/GFF coupling studied by Dubédat, Sheffield, and Miller as a temporally stationary field.

- We extend it to multiple cases. We prove that the multiple SLE/GFF coupling is established, if and only if the driving N-particle process on $\mathbb{R}$ is identified with $\mathrm{DYS}_{8/\kappa}$.

Our answer to this selection problem of driving process for the multiple SLE will be given by

$$dY_i(t) = \sqrt{\kappa} dB_i(t) + 4 \sum_{\substack{1 \le j \le N, \\ j \ne i}} \frac{1}{Y_i(t) - Y_j(t)} dt, \quad t \ge 0, \quad i = 1, \dots, N,$$

- Since $(\sqrt{\kappa} B(t))_{t \ge 0} \stackrel{\text{(law)}}{=} (B(\kappa t))_{t \ge 0}$, we perform a time change $\kappa t \to t$ and define $\boldsymbol{X}(t) := \boldsymbol{Y}(\kappa t), t \ge 0$. Then we have the following set of SDEs for $(\boldsymbol{X}(t))_{t \ge 0}$,

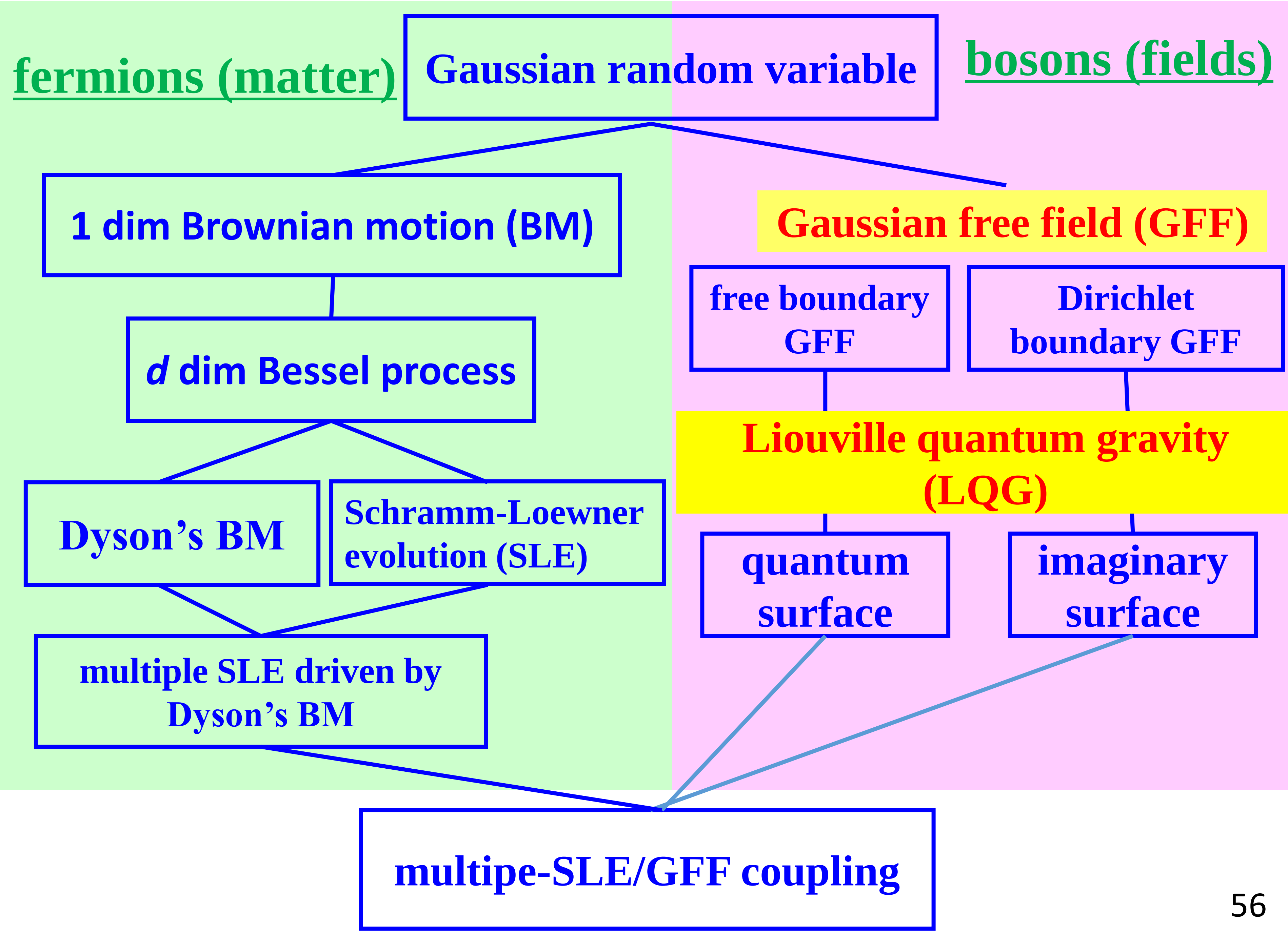
$$dX_i(t) = dB_i(t) + \frac{4}{\kappa} \sum_{\substack{1 \le j \le N, \\ j \ne i}} \frac{1}{X_i(t) - X_j(t)} dt$$

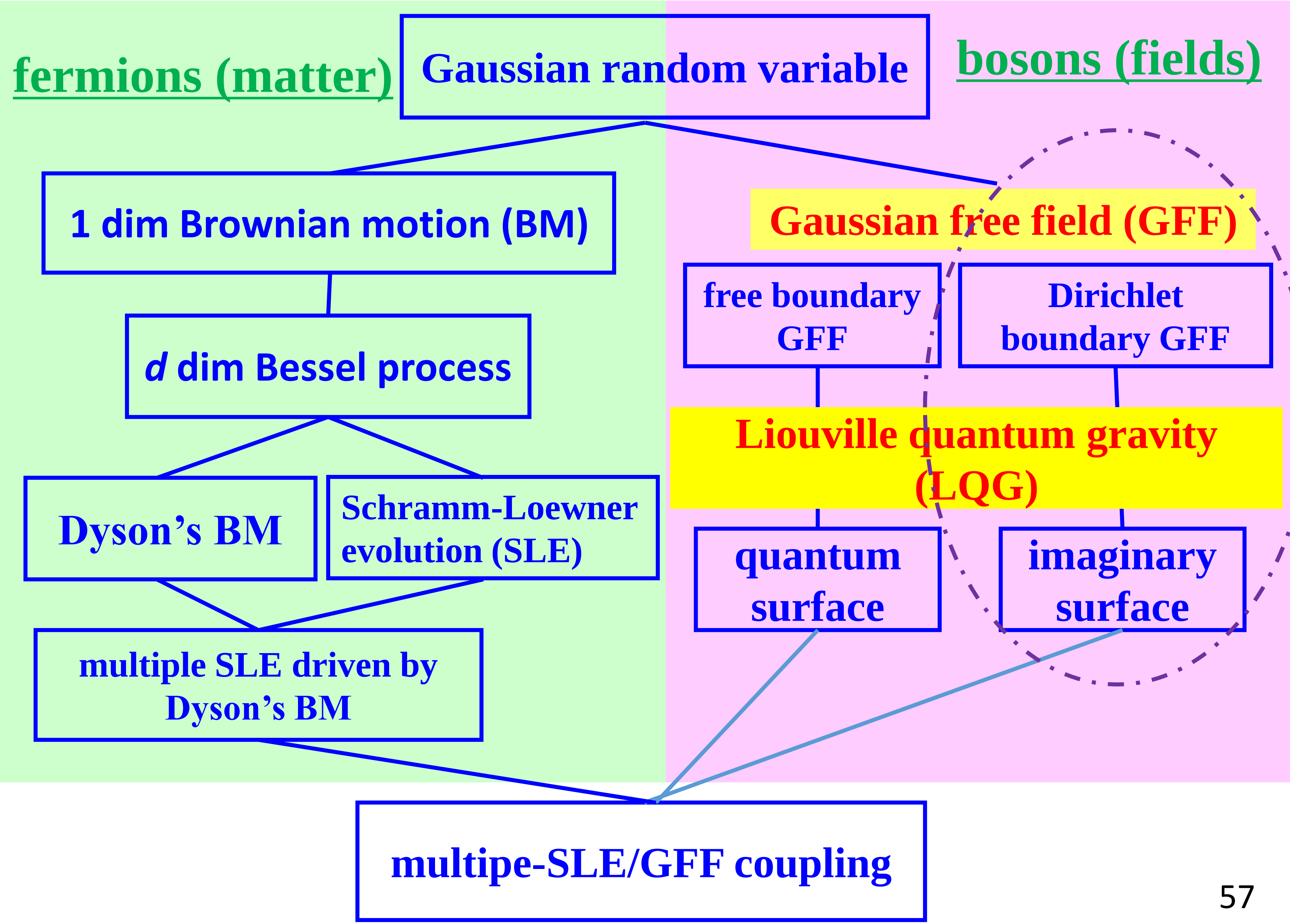
$$\iff \quad dX_i(t) = dB_i(t) + \frac{\beta}{2} \sum_{\substack{1 \le j \le N, \\ j \ne i}} \frac{1}{X_i(t) - X_j(t)} dt, \quad t \ge 0, \quad i = 1, \dots, N,$$

with

$$\beta = \frac{8}{\kappa}.$$

- Hence, we can say that the N-particle system $(\boldsymbol{Y}(t))_{t \ge 0}$ is a time change of the Dyson model with $\beta = 8/\kappa$; $\mathrm{DYS}_{8/\kappa}$.


fermions (matter)
Gaussian random variable
bosons (fields)
1 dim Brownian motion (BM)
Gaussian free field (GFF)
free boundary GFF
Dirichlet boundary GFF
d dim Bessel process
Liouville quantum gravity (LQG)
Dyson's BM
Schramm-Loewner evolution (SLE)
quantum surface
imaginary surface
multiple SLE driven by Dyson's BM
multipe-SLE/GFF coupling


fermions (matter)
Gaussian random variable
bosons (fields)
1 dim Brownian motion (BM)
Gaussian free field (GFF)
free boundary GFF
Dirichlet boundary GFF
d dim Bessel process
Liouville quantum gravity (LQG)
Dyson's BM
Schramm-Loewner evolution (SLE)
quantum surface
imaginary surface
multiple SLE driven by Dyson's BM
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Dirichlet boundary Gaussian free field (GFF)

- For two functions $f, g \in \mathcal{C}_c^\infty(D)$, their Dirichlet inner product is defined as

$$\langle f, g \rangle_\nabla := \frac{1}{2\pi} \int_D (\nabla f)(z) \cdot (\nabla g)(z) \mu(dz).$$

- The Hilbert space completion of $\mathcal{C}_c^\infty(D)$ with respect to this Dirichlet inner product will be denoted by $W(D)$. We write $\|f\|_\nabla = \sqrt{\langle f, f \rangle_\nabla}, f \in W(D)$.

- If we set

$$u_n = \sqrt{\frac{2\pi}{\lambda_n}} \quad e_n, n \in \mathbb{N},$$

then by integration by parts, we have

$$\langle u_n, u_n \rangle_\nabla = \frac{1}{2\pi} \langle u_n, (-\Delta) u_m \rangle = \delta_{nm}, \qquad n, m \in \mathbb{N}.$$

Therefore $\{u_n\}_{n \in \mathbb{N}}$ forms a CONS of $W(D)$.

Assume that $D, D' \subsetneq \mathbb{C}$ are simply connected domains and let

$$\varphi : D' \to D \quad \text{conformal map}.$$

Lemma 2.5 The Dirichlet inner product is conformally invariant, that is,

$$\int_D (\nabla f)(z) \cdot (\nabla g)(z) \mu(dz) = \int_{D'} (\nabla (f \circ \varphi))(z) \cdot (\nabla (g \circ \varphi))(z) \mu(dz)$$

for $f, g \in \mathcal{C}_c^\infty(D)$.

Definition 2.6 (Dirichlet boundary GFF) A Gaussian free field (GFF) with Dirichlet boundary condition is defined as a pair $((\Omega^{\mathrm{GFF}}, \mathcal{F}^{\mathrm{GFF}}, \mathbb{P}^{\mathrm{GFF}}), H)$ of a probability space $(\Omega^{\mathrm{GFF}}, \mathcal{F}^{\mathrm{GFF}}, \mathbb{P}^{\mathrm{GFF}})$ and an **isometry**

$$H : W(D) \to L^2(\Omega^{\mathrm{GFF}}, \mathcal{F}^{\mathrm{GFF}}, \mathbb{P}^{\mathrm{GFF}})$$

such that each

$$H(f) := \langle H, f \rangle_{\nabla}, \quad f \in W(D)$$

is a centered (mean zero) Gaussian random variable, whose characteristic function is given by

$$\mathbb{E}^{\mathrm{GFF}}[e^{\sqrt{-1}\theta \langle H, f \rangle_{\nabla}}] = \exp\left(-\frac{\theta^2}{2}\mathrm{Var}\left[\langle H, f \rangle_{\nabla}\right]\right), \quad \theta \in \mathbb{R}, \quad f \in W(D).$$

Here

$$\mathrm{Var}[\langle H, f \rangle_{\nabla}] := \mathbb{E}^{\mathrm{GFF}}\left[\langle H, f \rangle_{\nabla}^2\right] = \|f\|_{\nabla}^2, \quad f \in W(D).$$

- By **Definition 2.6** we have the following simple formula for covariances,

$$\mathrm{Cov}[\langle H, f\rangle_\nabla, \langle H, g\rangle_\nabla] := \mathbb{E}^{\mathrm{GFF}}\Big[\langle H, f\rangle_\nabla \langle H, g\rangle_\nabla\Big] = \langle f, g\rangle_\nabla, \quad f, g \in W(D).$$

- Fron **Lemma 2.5,** we see that the pull-back

$$\varphi^* : W(D) \ni f \mapsto f \circ \varphi \in W(D')$$

is an isomorphism.

- This allows one to consider a GFF on an unbounded domain:
  - Assume that D' is bounded on which a Dirichlet GFF H is defined, but D is unbounded.
  - We can define a family $\{\langle \varphi_* H, f\rangle_\nabla : f \in W(D)\}$ by

$$\langle \varphi_* H, f\rangle_\nabla := \langle H, \varphi^* f\rangle_\nabla, \quad f \in W(D).$$

- By **Lemma 2.5,** we can prove the equality,

$$\boxed{\mathrm{Cov}[\langle \varphi_* H, f\rangle_\nabla, \langle \varphi_* H, g\rangle_\nabla] = \mathrm{Cov}[\langle H, f\rangle_\nabla, \langle H, g\rangle_\nabla].}$$

That is, the covariance structure does not change under a conformal map φ. We say that the GFF is conformally invariant.

Green's function

By a formal integration by parts, we see that

$$\langle H, f\rangle_\nabla = \frac{1}{2\pi}\int_D (\nabla H)(z)\cdot(\nabla f)(z)\mu(dz) = \frac{1}{2\pi}\int_D H(z)(-\Delta f)(z)\mu(dz)$$
$$= \frac{1}{2\pi}\langle H, (-\Delta)f\rangle.$$

Motivated by this observation, we define

$$\langle H, f\rangle := 2\pi\langle H, (-\Delta)^{-1}f\rangle_\nabla \quad \text{for } \in \mathsf{D}((-\Delta)^{-1}),$$

where $\mathsf{D}((-\Delta)^{-1})$ denotes the domain of $(-\Delta)^{-1}$ in $W(D)$.
The action of $(-\Delta)^{-1}$ is expressed as an integral operator and the integral kernel is known as the Green's function. Namely,

$$((-\Delta)^{-1}f)(z) = \frac{1}{2\pi}\int_D G_D(z,w)f(w)\mu(dw), \quad \text{a.e. } z\in D, \quad f\in\mathsf{D}((-\Delta)^{-1}),$$

where $G_D(z,w)$ denotes the Green's function of D under the Dirichlet boundary condition.

- **The covariance of $\langle H, f \rangle$ and $\langle H, g \rangle$ with $f, g \in \mathsf{D}((-\Delta)^{-1})$ is written as**

$$\mathbb{E}^{\mathrm{GFF}}[\langle H, f \rangle \langle H, g \rangle] = \int_{D \times D} f(z) G_D(z, w) g(w) \mu(dz) \mu(dw).$$

- **When we symbolically write**

$$\langle H, f \rangle = \int_D H(z) f(z) \mu(dz), \quad f \in \mathsf{D}((-\Delta)^{-1}),$$

the covariance structure can be expressed as

$$\mathbb{E}[H(z) H(w)] = G_D(z, w), \quad z, w \in D, \quad n \neq w.$$

When D is the upper half plane $\mathbb{H}$,

$$G_{\mathbb{H}}(z,w) = \log\left|\frac{z-\overline{w}}{z-w}\right| = \log|z-\overline{w}| - \log|z-w|$$
$$= \text{Re}\,\log(z-\overline{w}) - \text{Re}\,\log(z-w),\, z,w\in\mathbb{H}, z\neq w.$$

- **The conformal invariance of GFF implies that for a conformal map $\varphi : D' \to D$, we have the equality,**

$$G_{D'}(z,w) = G_D(\varphi(z),\varphi(w)), \qquad z,w\in D'.$$

- **In the following, we will regard the upper half plane $\mathbb{H}$ as the representative of the simply connected domain $D \subsetneq \mathbb{C}$.**

- **Since each $D' \subsetneq \mathbb{C}$ is specified by the conformal transformation $\varphi : D' \to \mathbb{H}$, we put this in the superscript and write**

$$G^{\varphi}(z,w) := G_{\varphi^{-1}(\mathbb{H})}(z,w) = G_{\mathbb{H}}(\varphi(z),\varphi(w)).$$

- **By the explicit form of $G_{\mathbb{H}}(z,w)$ given above, we have**

$$G^{\varphi}(z,w) = \log|\varphi(z)-\overline{\varphi(w)}| - \log|\varphi(z)-\varphi(w)|$$
$$= \text{Re}\,\log(\varphi(z)-\overline{\varphi(w)}) - \text{Re}\,\log(\varphi(z)-\varphi(w)).$$

1. From Brownian motion to Dyson's BM model
(random points: stochastic log-gas)

2. Schramm-Loewner evolution (SLE) and Gaussian free fields
(random curves and random surfaces)

3. Multiple SLE/GFF coupling driven by Dyson's BM model
(random points/curves/surfaces)

4. Concluding Remarks

We write the probability space of the multiple SLE as $(\Omega^{\text{SLE}}, \mathcal{F}^{\text{SLE}}, (\mathcal{F}^{\text{SLE}})_{\geq 0}, \mathbb{P}^{\text{SLE}})$. Now we define the direct product of this space and that for the Dirichlet boundary GFF,

$$(\Omega, \mathcal{F}, \mathbb{P}) = (\Omega^{\text{SLE}} \times \Omega^{\text{GFF}}, \mathcal{F}^{\text{SLE}} \times \mathcal{F}^{\text{GFF}}, \mathbb{P}^{\text{SLE}} \times \mathbb{P}^{\text{GFF}}).$$

We consider the multiple SLE as well as GFF in this extended probability space, and the multiple SLE is assumed to be adapted to the filtration,

$$\mathcal{F}_t = \mathcal{F}_t^{\text{SLE}} \times \{\emptyset, \Omega^{\text{GFF}}\}, \quad t \geq 0.$$

In summary, we assume the following;

$$\frac{dg_{\mathbb{H}_t^\eta}(z)}{dt} = \sum_{i=1}^{N} \frac{2}{g_{\mathbb{H}_t^\eta}(z) - Y_i(t)}, \quad t \geq 0, \quad g_{\mathbb{H}_0^\eta}(z) = g_{\mathbb{H}}(z) = z \in \mathbb{H},$$

$$Y_i(t) = g_{\mathbb{H}_t^\eta}(\eta_i(t)) := \lim_{z \to \eta_i(t), z \in \mathbb{H}_t^\eta} g_{\mathbb{H}_t^\eta}(z), \quad i = 1, \dots, N, \quad t \geq 0,$$

$$g_{\mathbb{H}_t^\eta} : \ \textbf{conformal} : \mathbb{H}_t^\eta \to \mathbb{H}.$$

GFF transformed by multiple SLE

- **Put**

$$G^{g_{\mathbb{H}_t^\eta}}(z,w) := G_{\mathbb{H}}(g_{\mathbb{H}_t^\eta}(z), g_{\mathbb{H}_t^\eta}(w)), \quad t \geq 0.$$

- **Then define the Dirichlet boundary GFF, $g_{\mathbb{H}_t^\eta *} H$, on $\mathbb{H}_t^\eta, t \geq 0$, such that its Green's function is given by the following equation.**

Proposition 3.1 For $t \geq 0$,

$$\frac{dG^{g_{\mathbb{H}_t^\eta}}(z,w)}{dt} = -4 \sum_{i=1}^{N} \mathrm{Im}\left(\frac{1}{g_{\mathbb{H}_t^\eta}(z) - Y_i(t)}\right) \mathrm{Im}\left(\frac{1}{g_{\mathbb{H}_t^\eta}(w) - Y_i(t)}\right), \quad z, w \in \mathbb{H}_t^\eta.$$

- **For a given domain $A \subset \mathbb{H}$, we assume that $\mathrm{supp}\,(f)$ of $f \in \mathcal{C}_c^\infty(\mathbb{H})$ satisfies $\mathrm{supp}\,(f) \subset A$. Then we define the Dirichlet energy of GFF by**

$$E_A^{g_{\mathbb{H}_t^\eta}}(f) := \int_{A \times A} f(z) G^{g_{\mathbb{H}_t^\eta}}(z,w) f(w) m(dz) m(dw).$$

By **Proposition 3.1,** we have

$$\frac{dE_A^{g_{\mathbb{H}_t^\eta}}(f)}{dt} = -\sum_{i=1}^{N} \left(\int_A \mathrm{Im}\,\frac{2}{g_{\mathbb{H}_t^\eta}(z) - Y_i(t)} f(z) m(dz)\right)^2,$$

It implies that $E_A^{g_{\mathbb{H}_t^\eta}}(f)$ is **non-increasing function of t.**

Multiple SLE/GFF coupling as temporally stationary field

Consider the following time-evolution of extended GFF,

$$H_t := g_{\mathbb{H}_t^\eta\,*} H + \mathfrak{h}_t, \quad t \geq 0,$$

where the process $(\mathfrak{h}_t)_{t\geq 0}$ is not yet determined. The process $(\mathfrak{h}_t)_{t\geq 0}$ shall be a functional of the mutiple SLE $(g_{\mathbb{H}_t^\eta})_{t\geq 0}$.

Definition 3.2 (multiple SLE/GFF coupling) For any domain $A \subset \mathbb{H}$ such that

$$v_A := \inf\{\mathrm{Im}\, z : z \in A\} \geq {}^\exists\delta > 0,$$

define the $(\mathcal{F}_t)_{t\geq 0}$–stopping time by $\tau_A := \sup\{t \geq 0 : A \subset \mathbb{H}_t^\eta\}$ and assume $0 < T < \tau_A$. For the multiple SLE, assume the equality

$$\langle H_0, f\rangle \overset{\text{(law)}}{=} \langle H_t, f\rangle, \quad t \in [0, T] \quad \text{(temporal stationarity)}$$

for any $f \in \mathcal{C}_c^\infty(\mathbb{H})$ with $\mathrm{supp}\,(f) \subset A$. Then we say that the multiple SLE/GFF coupling is established.

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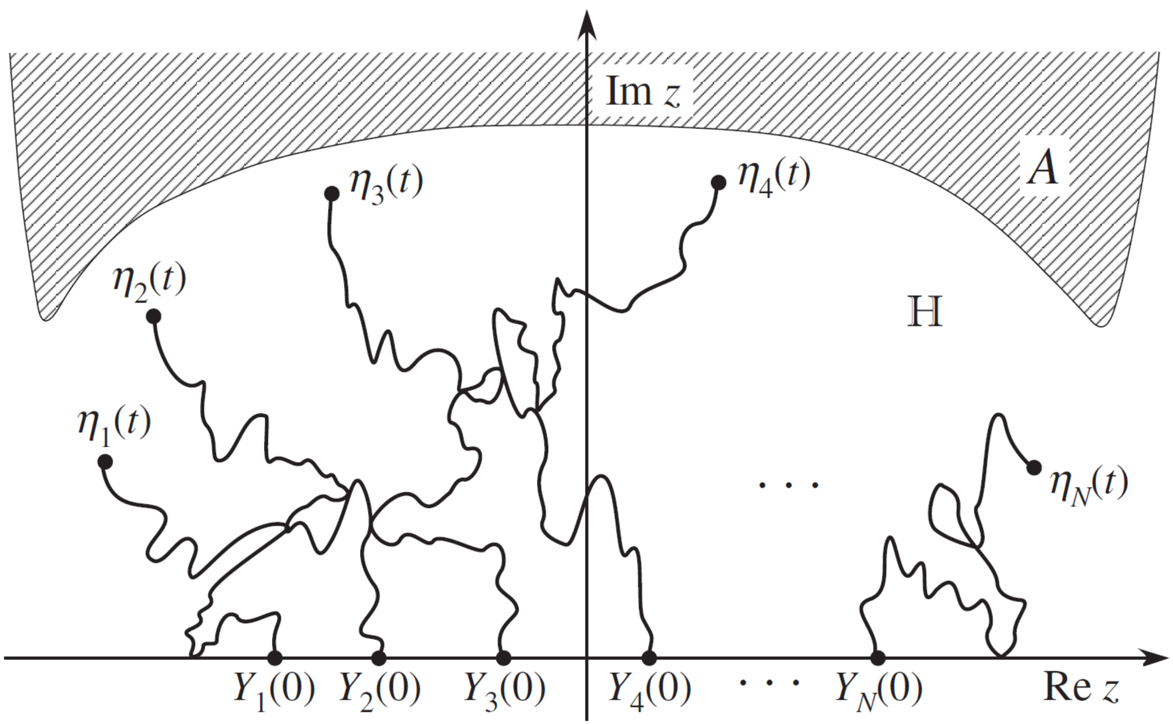

Proposition 3.3 Assume the following.

(A) $(\mathfrak{h}_t)_{t\geq 0}$ is a continuous local martingale and its quadratic covariation satisfies the equalities,

$$d\langle \mathfrak{h}.(z), \mathfrak{h}.(w)\rangle_t = -dG^{g_{\mathbb{H}^\eta_t}}(z,w), \quad z,w \in \mathbb{H}^\eta_t, \quad t \in [0,\infty),$$
$$d\langle\langle \mathfrak{h}., f\rangle, \langle \mathfrak{h}., f\rangle\rangle_t = -dE_A^{g_{\mathbb{H}^\eta_t}}(f) \quad \text{for } f \in \mathcal{C}_c^\infty(\mathbb{H}) \text{ with } \mathrm{supp}\,(f) \subset A \subset \mathbb{H}.$$

Then the multiple SLE/GFF coupling is established for $H_t := g_{\mathbb{H}^\eta_t *} H + \mathfrak{h}_t, t \geq 0$.

Proof. Introduce a real parameter $\theta \in \mathbb{R}$ and consider the characteristic function for $\langle H_t, f\rangle$, $\mathbb{E}[e^{\sqrt{-1}\theta\langle H_t, f\rangle}]$. Here $\mathbb{E}$ denotes the expectation with respect to the joint probability law $\mathbb{P}$ of the multiple SLE and GFF. $\mathfrak{h}_t(\cdot)$ is $\mathcal{F}_t$–measurable,

$$\mathbb{E}\left[e^{\sqrt{-1}\theta\langle H_t, f\rangle}\right] = \mathbb{E}\left[\mathbb{E}\left[\exp(\sqrt{-1}\theta\langle g_{\mathbb{H}^\eta_t *} H, f\rangle)\middle|\mathcal{F}_t\right] e^{\sqrt{-1}\theta\langle \mathfrak{h}_t, f\rangle}\right].$$

Since $\mathrm{supp}\,(f) \subset A$, $\mathrm{Var}[\langle g_{\mathbb{H}^\eta_t *} H, f\rangle] = E_A^{g_{\mathbb{H}^\eta_t}}(f)$. Hence by the definition of centered GFF,

$$\mathbb{E}\left[\exp(\sqrt{-1}\theta\langle g_{\mathbb{H}^\eta_t *} H, f\rangle)\middle|\mathcal{F}_t\right] = \exp\left(-\frac{\theta^2}{2}E_A^{g_{\mathbb{H}^\eta_t}}(f)\right).$$

Then we obtain

$$\mathbb{E}\left[e^{\sqrt{-1}\theta\langle H_t, f\rangle}\right] = \mathbb{E}\left[\exp\left(-\frac{\theta^2}{2}E_A^{g_{\mathbb{H}^\eta_t}}(f) + \sqrt{-1}\theta\langle \mathfrak{h}_t, f\rangle\right)\right].$$

By Itô's formula,

$$
\begin{aligned}
&d\exp\left(-\frac{\theta^2}{2}E_A^{g_{\mathbb{H}_t^\eta}}(f)+\sqrt{-1}\theta\langle\mathfrak{h}_t,f\rangle\right)\\
&=\left\{\sqrt{-1}\theta d\langle\mathfrak{h}_t,f\rangle-\frac{\theta^2}{2}\left(dE_A^{g_{\mathbb{H}_t^\eta}}(f)+d\langle\langle\mathfrak{h}_\cdot,f\rangle,\langle\mathfrak{h}_\cdot,f\rangle\rangle_t\right)\right\}\exp\left(-\frac{\theta^2}{2}E_A^{g_{\mathbb{H}_t^\eta}}(f)+\sqrt{-1}\theta\langle\mathfrak{h}_t,f\rangle\right)\\
&=\sqrt{-1}\theta d\langle\mathfrak{h}_t,f\rangle\exp\left(-\frac{\theta^2}{2}E_A^{g_{\mathbb{H}_t^\eta}}(f)+\sqrt{-1}\theta\langle\mathfrak{h}_t,f\rangle\right),
\end{aligned}
$$

where the assumption (A) was used. Hence $\exp\left(-\frac{\theta^2}{2}E_A^{g_{\mathbb{H}_t^\eta}}(f)+\sqrt{-1}\theta\langle\mathfrak{h}_t,f\rangle\right)$ **is an** $\mathcal{F}_t$**-local martingale. Therefore, the characteristic function, given by the expectation of martingale, is time independent and equals to its initial value,**

$$
\mathbb{E}\left[\exp\left(-\frac{\theta^2}{2}E_A^{g_{\mathbb{H}}}(f)+\sqrt{-1}\theta\langle\mathfrak{h}_0,f\rangle\right)\right]=\mathbb{E}\left[e^{\sqrt{-1}\theta\langle H_0,f\rangle}\right].
$$

The proof is complete. ∎

Complex-valued logarithmic potentials

- **Assume that the driving process $(\boldsymbol{Y}(t))_{t \geq 0}$ of the multiple SLE is given by**

$$dY_i(t) = \sqrt{\kappa} dB_i(t) + F_i(\boldsymbol{Y}(t))dt, \quad t \geq 0, \quad i = 1, \dots, N,$$

- **For $z \in \mathbb{C}, x^{(i)} \in \mathbb{R}, i = 1, \dots, N$, we consider a *sum* of complex-valued logarithmic potentials,**

$$\Phi(z, \boldsymbol{y}) := \sum_{i=1}^{N} \log(z - y^{(i)}).$$

- **And consider a stochastic process** $(\Phi(g_{\mathbb{H}_t^\eta}(z), \boldsymbol{Y}(t)))_{t \geq 0}$.

Following Ito's formula and some calculation based on the multiple SLE, we have ($f'(z) := df(z)/dz$)

$$-\frac{2}{\sqrt{\kappa}} d\Phi(g_{\mathbb{H}_t^\eta}(z), \boldsymbol{Y}(t)) = 2\sum_{i=1}^{N} \frac{1}{g_{\mathbb{H}_t^\eta}(z) - Y_i(t)} dB_i(t)$$
$$- \frac{2}{\sqrt{\kappa}} \sum_{i=1}^{N} \frac{1}{g_{\mathbb{H}_t^\eta}(z) - Y_i(t)} \left[4 \sum_{\substack{1 \le j \le N, \\ j \neq i}} \frac{1}{Y_i(t) - Y_j(t)} - F_i(\boldsymbol{Y}(t)) \right] dt$$
$$+ \frac{1}{\sqrt{\kappa}} \left(2 - \frac{\kappa}{2}\right) d \log g'_{\mathbb{H}_t^\eta}(z).$$

We find the following:

(i) if $\chi = \frac{2}{\sqrt{\kappa}} - \frac{\sqrt{\kappa}}{2}$,

then the last term in RHS $= d\left(\chi \log g'_{\mathbb{H}_t^\eta}(z)\right)$,

(ii) if $F_i(\boldsymbol{Y}) = 4 \sum_{\substack{1 \le j \le N, \\ j \neq i}} \frac{1}{Y_i(t) - Y_j(t)}, \quad i = 1, \dots, N,$ (the drift term of $\mathrm{DYS}_{8/\kappa}$)

then the second term in RHS $= 0$.

Hence we put

$$\mathfrak{h}_t(\cdot) := \mathrm{Im}\left[-\frac{2}{\sqrt{\kappa}}\Phi(g_{\mathbb{H}_t^\eta}(\cdot), \boldsymbol{Y}(t)) - \chi \log g_{\mathbb{H}_t^\eta}{}'(\cdot)\right]$$

$$= -\frac{2}{\sqrt{\kappa}}\sum_{i=1}^{N} \arg\left(g_{\mathbb{H}_t^\eta}(\cdot) - Y_i(t)\right) - \chi \arg g_{\mathbb{H}_t^\eta}{}'(\cdot).$$

Then the above observation implies the following.

Proposition 3.4 **Assume that the driving process of the multiple SLE is given by DYS$_{8/\kappa}$. If $\chi = \frac{2}{\sqrt{\kappa}} - \frac{\sqrt{\kappa}}{2}$, then $(\mathfrak{h}_t(\cdot))_{t\geq 0}$ is a local martingale and satisfies**

$$d\mathfrak{h}_t(z) = 2\sum_{i=1}^{N} \mathrm{Im}\frac{1}{g_{\mathbb{H}_t^\eta}(z) - Y_i(t)} dB_i(t), \quad z \in \mathbb{H}_t^\eta, \quad t \in [0, \infty).$$

Then the assumption (A) holds.

Imaginary surface

Consider the collection

$$\mathsf{S} := \left\{ (D,h) \,\middle|\, D \subsetneq \mathbb{C} : \text{simply connected}, h : \text{extended GFF on } D \right\}.$$

Fixing a parameter $\chi \in \mathbb{R}$, the following equivalence relation in S was introduced.

Definition 3.5 Two pairs (D,h) and $(\widetilde{D},\widetilde{h}) \in \mathsf{S}$ are χ-equivalent if there exists a conformal map $\varphi : \widetilde{D} \to D$ and

$$\widetilde{h} \overset{\text{(law)}}{=} \varphi_* h - \chi \arg \varphi'.$$

In this case, we write $(D,h) \sim (\widetilde{D},\widetilde{h})$.

- Each element belonging to the equivalence class $\mathsf{S}/\sim$ is called an imaginary surface (Miller–Sheffield 2016) (or an AC surface (Sheffield 2016); (AC means a combination of an altimeter and a compass.)

Definition 3.5 Two pairs (D, h) and $(\widetilde{D}, \widetilde{h}) \in \mathsf{S}$ are χ-equivalent if there exists a conformal map $\varphi : \widetilde{D} \to D$ and

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- Each element belonging to the equivalence class $\mathsf{S}/\sim$ is called an imaginary surface (Miller–Sheffield 2016) (or an AC surface (Sheffield 2016); (AC means a combination of an altimeter and a compass.)

- That is, in this equivalence class, a conformal map φ causes not only a coordinate change of a GFF as $h \mapsto \varphi_* h$ associated with changing the domain of definition of the field as $D \mapsto \varphi^{-1}(D)$, but also an addition of a deterministic harmonic function $-\chi \arg \varphi'$ to the field. Notice that this definition depends on one parameter $\chi \in \mathbb{R}$.

The time-dependent field obtained by multiple SLE/GFF coupling is a imaginary field with $\chi = \frac{2}{\sqrt{\kappa}} - \frac{\sqrt{\kappa}}{2}$ associated with the time evolution of the boundary points $(\boldsymbol{Y}(t))_{t \geq 0} \sim \mathrm{DYS}_{8/\kappa}$.

The following is the main result in this talk.

Theorem 3.6 Assume

$$\chi = \frac{2}{\sqrt{\kappa}} - \frac{\sqrt{\kappa}}{2}.$$

Then, if and only if the driving process of multiple SLE $(\boldsymbol{Y}(t))_{t \geq 0}$ is given by the solution of the SDEs

$$dY_i(t) = \sqrt{\kappa} dB_i(t) + 4 \sum_{\substack{1 \leq j \leq N, \\ j \neq i}} \frac{1}{Y_i(t) - Y_j(t)} dt, \quad t \geq 0, \quad i = 1, \dots, N,$$

the multiple SLE/GFF coupling is established. In other words the driving process of multiple SLE is uniquely determined to be the Dyson model with

$$\beta = \frac{8}{\kappa},$$

$\mathrm{DYS}_{8/\kappa}$, if the multiple SLE/GFF coupling holds.

The multiple SLE driven by DYS_β with the relation $\beta = 8/\kappa$, inherits many properties from the original SLE_κ with a single SLE curve. Actually, we have proved that our multiple SLE_κ also shows phase transitions at $\kappa_c = 4$ and $\overline{\kappa}_c = 8$.

Theorem 3.7 For each $i = 1, \ldots, N$, the limit $\eta_i(t) = \lim_{\varepsilon \downarrow 0} g_{\mathbb{H}_t^\eta}(Y_i(t) + \sqrt{-1}\varepsilon)$ exists for all $t \geq 0$ and $\lim_{t \to \infty} \eta_i(t) = \infty$ with probability one. Moreover, the following three phases are observed.

(a) If $0 < \kappa \leq \kappa_c = 4$, $\eta_i(0, \infty), i = 1, \ldots, N$ are simple disjoint curves such that $\eta_i \subset \mathbb{H}, i = 1, \ldots, N$ with probability one.

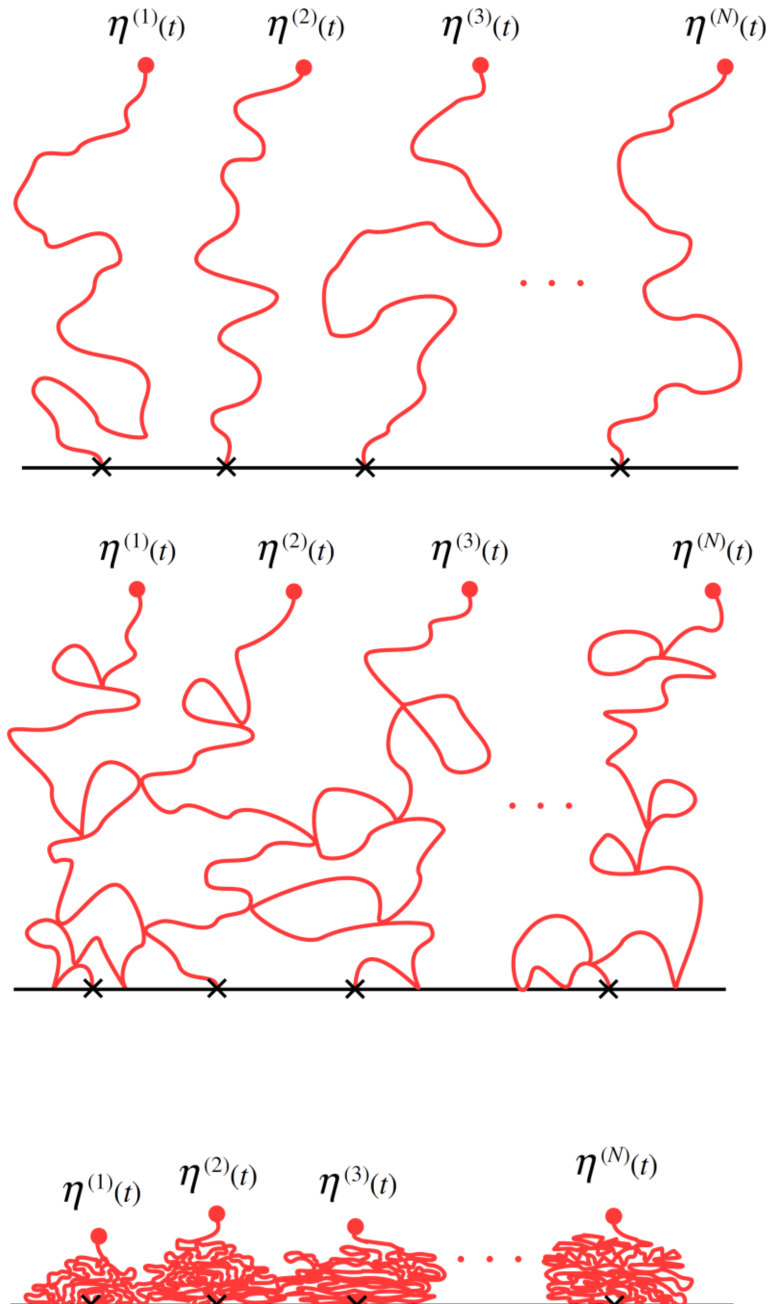
(b) If $\kappa_c = 4 < \kappa < \overline{\kappa}_c = 8$, $\eta_i, i = 1, \ldots, N$ are continuous curves with probability one, and they intersect themselves and $\mathbb{R}$ with positive probability.

(c) If $\kappa = 8$, $\eta_i, i = 1, \ldots, N$ are space filling continuous curves with probability one.

(a) $0 < \kappa \leq 4$

(b) $4 < \kappa < 8$

(c) $\kappa = 8$



1. **From Brownian motion to Dyson's BM model**
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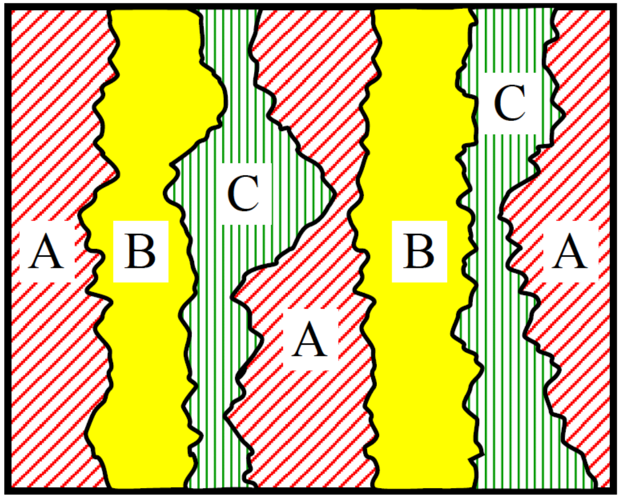
4. **Concluding Remarks**

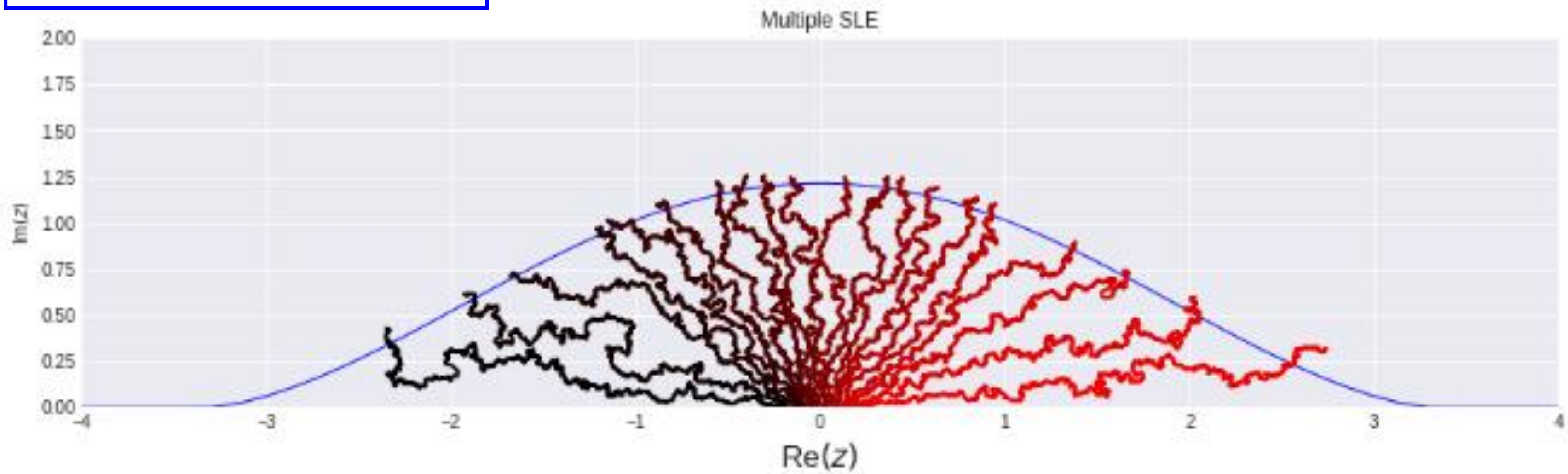
Scaling limits of statistical mechanical models?

Model for coil-globule transitions in 2-dim polymers?

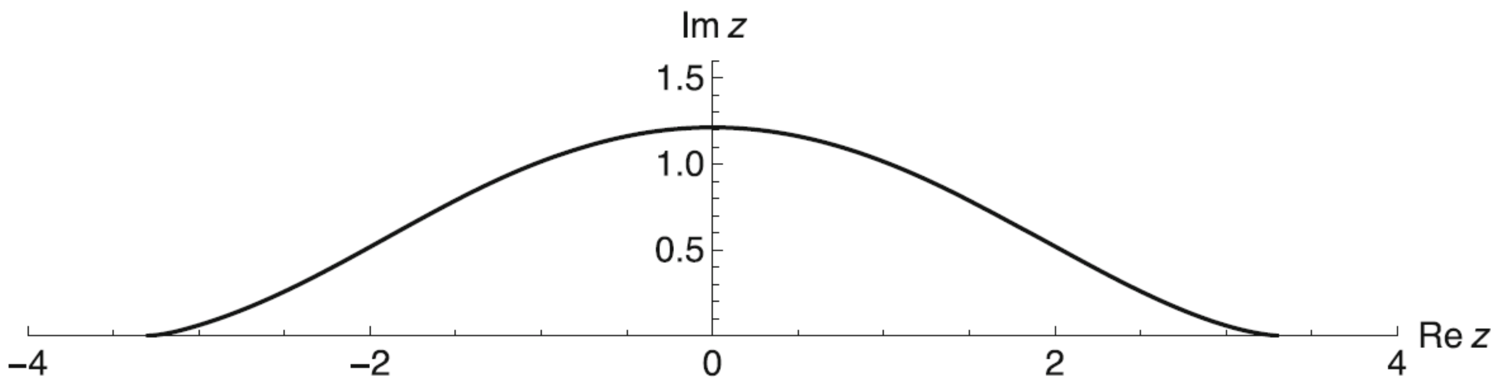


Describing the commensurate-incommensurate transitions studied by Huse and Fisher (Phys. Rev. **B29** (1984)) ?

Infinite N limits ?

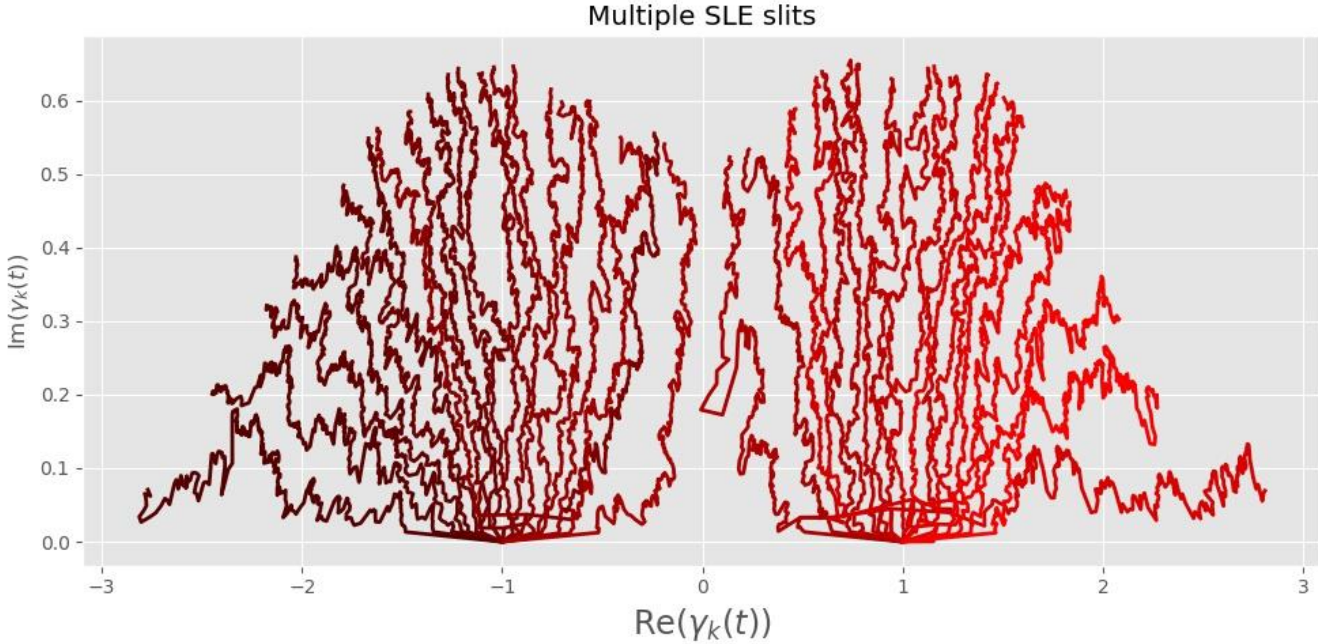


Numerical simulation by S. Schleißinger

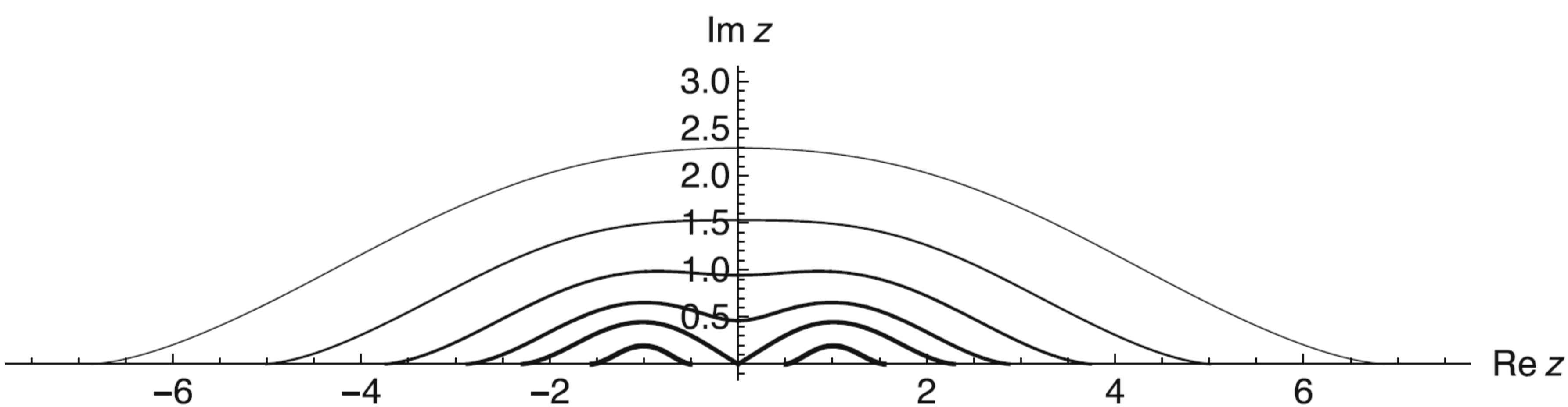


$$\Gamma_t(\varphi) = 2\sqrt{-t}\exp\left(-\sqrt{-1}\varphi - \frac{e^{2\sqrt{-1}\varphi}}{2}\right), \quad -\frac{\pi}{2} \le \varphi \le \frac{\pi}{2}, \quad t \ge 0.$$

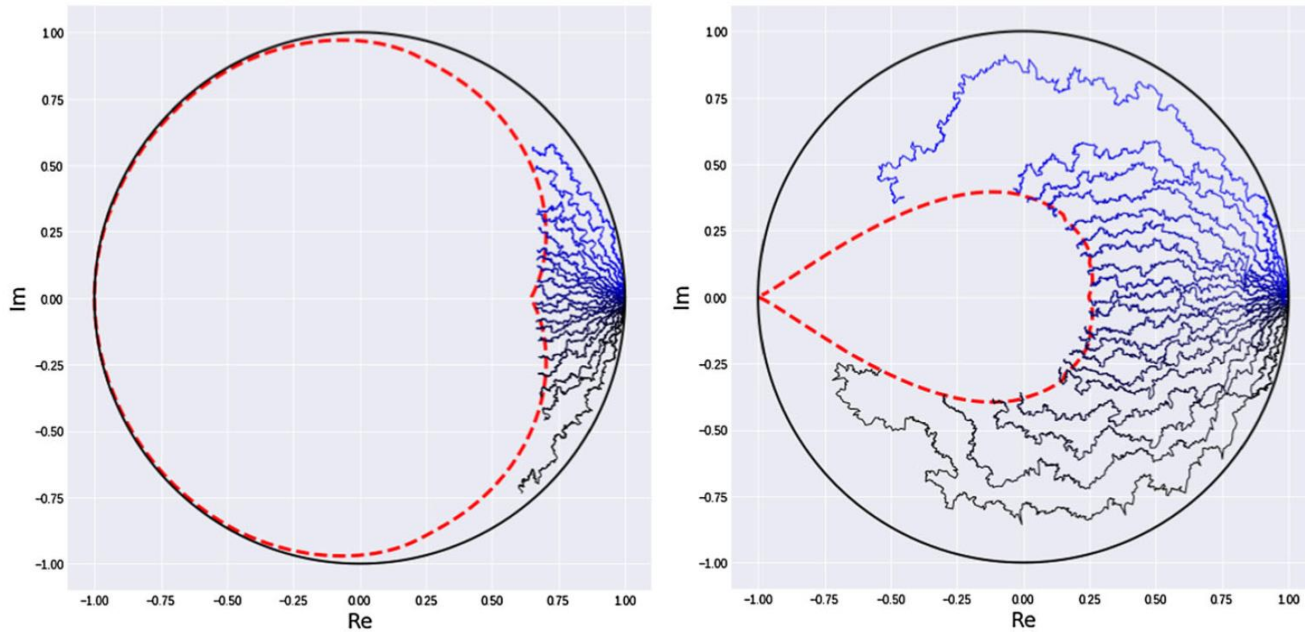
"High-end model" of Wigner's semicircle
(Hotta–Katori: J. Stat. Phys. 171 (2018) 166–188)



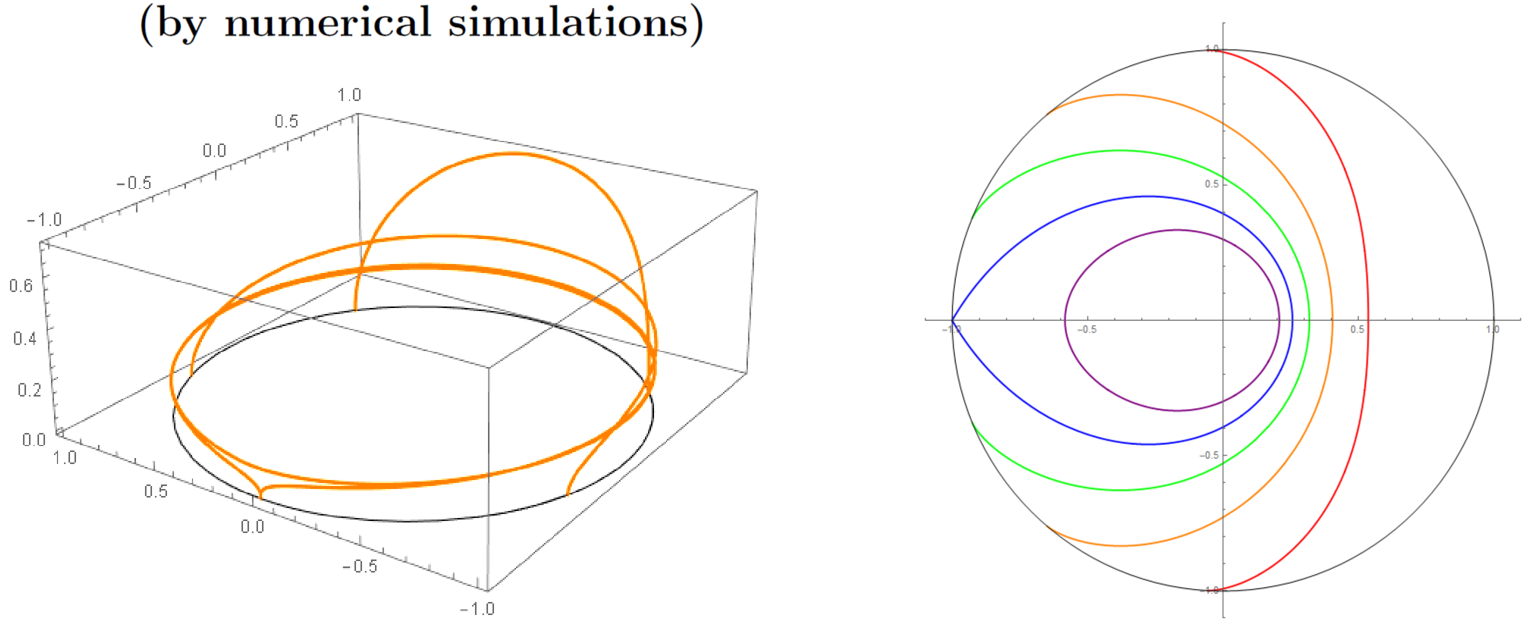
Numerical simulation by S. Schleißinger



"infinite SLE version" of Brézin–Hikami (–Pearcey)
(Hotta–Katori: J. Stat. Phys. 171 (2018) 166–188)



From Hotta–Schleißinger: J. Theor. Probab. 34 (2021) 755–783
(by numerical simulations)



(Endo–Katori–Koshida: in preparation)

Variations like RMT ?

[GUE] **Dyson's BM model on $\mathbb{R}$:** $dY_i(t) = \sqrt{\kappa} dB_i(t) + 4 \sum_{1 \le j \le N, j \ne i} \frac{1}{Y_i(t) - Y_j(t)} dt$

multiple SLE on $\mathbb{H} := \{z \in \mathbb{C} : \mathrm{Im}\, z > 0\}$: $\frac{dg_t(z)}{dt} = \sum_{i=1}^{N} \frac{2}{g_t(z) - Y_i(t)}$

harmonic term (local martingale): $\chi := \frac{2}{\sqrt{\kappa}} - \frac{\sqrt{\kappa}}{2}$

$$\mathfrak{h}_t(z) = -\frac{2}{\sqrt{\kappa}} \sum_{i=1}^{N} \arg\left(g_t(z) - Y_i(t)\right) - \chi \arg g_t'(z)$$

[chiral GUE] **Bru–Wishart(–Laguerre) process on** $\mathbb{R}_+ := \{x \in \mathbb{R} : x > 0\}$**:** $\nu > 0$

$$dY_i(t) = \sqrt{\kappa} dB_i(t) + \frac{8(\nu+1)-\kappa}{2Y_i(t)} dt + 4 \sum_{1 \leq j \leq N, j \neq i} \left(\frac{1}{Y_i(t) - Y_j(t)} + \frac{1}{Y_i(t) + Y_j(t)} \right) dt$$

multiple SLE on the first orthant $\mathbb{O} := \{z \in \mathbb{C} : \mathrm{Re}\, z > 0, \mathrm{Im}\, z > 0\}$**:**

$$\frac{dg_t(z)}{dt} = \sum_{i=1}^{N} \left(\frac{2}{g_t(z) - Y_i(t)} + \frac{2}{g_t(z) + Y_i(t)} \right) + \frac{4\nu}{g_t(z)}$$

harmonic term (local martingale):

$$\mathfrak{h}_t(z) = -\frac{2}{\sqrt{\kappa}} \sum_{i=1}^{N} \Big\{ \arg(g_t(z) - Y_i(t)) + \arg(g_t(z) + Y_i(t)) \Big\} - \chi \arg g_t(z) - \chi \arg {g_t}'(z)$$

[circular unitary ensemble (CUE)] circular Dyson model on $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$:

$$Y_i(t) = e^{\sqrt{-1}\Theta_i(t)}, \quad d\Theta_i(t) = \sqrt{\kappa} dB_i(t) + 2 \sum_{1 \leq j \leq N, j \neq i} \cot \left(\frac{\Theta_i(t) - \Theta_j(t)}{2} \right) dt$$

radial multiple SLE on $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$:

$$\frac{dg_t(z)}{dt} = -\sum_{i=1}^{N} \frac{g_t(z) + Y_i(t)}{g_t(z) - Y_i(t)}$$

harmonic term (local martingale): $\xi_N := \frac{N+2}{\sqrt{\kappa}} - \frac{\sqrt{\kappa}}{2}$

$$\mathfrak{h}_t(z) = -\frac{2}{\sqrt{\kappa}} \sum_{i=1}^{N} \arg\left(g_t(z) - Y_i(t)\right) + \xi_N \arg g_t(z) + \frac{1}{\sqrt{\kappa}} \sum_{i=1}^{N} \arg Y_i(t) - \chi \arg {g_t}'(z)$$

(Endo–Katori–Koshida: in preparation)

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Thank you very much for your attention.