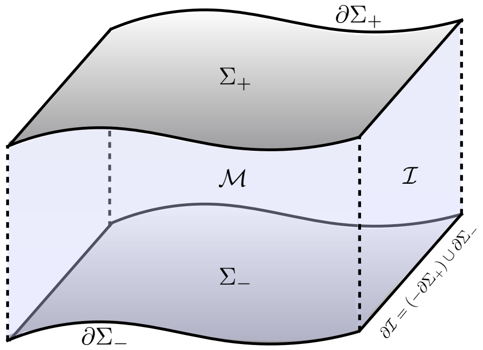
7 Symmetries, Gauge Transformations, and Boundary Charges

In the previous lectures, we have introduced several important notions at different levels of our discussion: physical symmetries, gauge transformations, boundary conditions, and so on. Each of these concepts has a long and rich history which, for obvious time reasons, we will not be able to cover in this course. However, they turn out to be beautifully intertwined and play a major role in our modern understanding of gauge field theories and gravity and its recent developments. The aim of the present lecture is therefore twofold: on the one hand, we will try to recollect these concepts together in a systematic way; on the other hand, we will explore the main aspects of their relations and differences (at least those that will be useful for the second part of the course) within the framework of the covariant phase space formalism introduced in the previous lectures.

7.1 Lagrangian symmetries and boundary conditions

Let us then start by briefly recalling the setup of Lagrangian field theories we are working with and the role of boundary conditions. We consider a generic classical field theory governed by the action

$$S = \int_{\mathcal{M}} L + \int_{\partial\mathcal{M}} \ell \ ,$$



where, $\dim \mathcal{M} = D$, L is the bulk Lagrangian $(D,0)$-form, ℓ is the boundary Lagrangian $(D-1,0)$-form, and the spacetime boundary $\partial\mathcal{M} = \mathcal{I} \cup \Sigma_- \cup \Sigma_+$, with $\Sigma_\pm$ the past (Σ_-) and future (Σ_+) Cauchy surfaces enclosing the points in $\mathcal{M}$ and $\mathcal{I}$ the time-like boundary between them (cfr. side figure). Given an orientation on $\mathcal{M}$, the orientations of the various space-like and time-like hypersurfaces are chosen compatibly with the orientation of $\partial\mathcal{M}$ being the natural one induced by Stokes'theorem, namely $\int_{\mathcal{M}} \mathrm{d}\bullet = \int_{\partial\mathcal{M}} \bullet$. Specifically, the boundary volume form $\epsilon_{\partial\mathcal{M}}$ on $\partial\mathcal{M}$ is related to the volume form ϵ on $\mathcal{M}$ as $\epsilon = n \wedge \epsilon_{\partial\mathcal{M}}$, n being the outward pointing normal form. The orientation on any Cauchy slice Σ is the one corresponding to viewing it as the boundary of its past in $\mathcal{M}$. This means that the past boundary Σ_- of $\mathcal{M}$ in the picture has to be thought of as equipped with the opposite orientation compared to Σ_+. Finally, the boundary $\partial\mathcal{I} = (-\partial\Sigma_+) \cup (\partial\Sigma_-)$ of $\mathcal{I}$ is the union of the boundaries of Σ_+ and Σ_- with opposite orientation due to the opposite orientations of the normals when approaching the boundary from the time-like or space-like directions.

Two kinds of boundary conditions should then be distinguished:

- **boundary conditions at future/past boundaries -** these specify a state within the theory, i.e. the values of the fields at past and future boundaries specify a field history in solution space. Therefore, if we are interested in field variations which do change the state as e.g. when looking at physical symmetries, such boundary conditions are not fixed.

- **boundary conditions at spatial boundaries -** these are instead part of the definition of the theory and are the ones specified for instance also for canonical fields.

At the level of the Lagrangian variational principle, this amounts to the requirement of the action to be stationary up to past/future boundary terms. Therefore, boundary conditions on $\mathcal{I}$ must be specified accordingly to guarantee it. Specifically, as we have discussed in Sec. 5.7, the following boundary conditions are imposed

$$(\theta + \delta\ell)\Big|_{\mathcal{I}} = \mathrm{d}C \ , \tag{7.1}$$

for some local $(D-2,1)$-form C constructed from the dynamical and background fields, their derivatives, and the variations of the dynamical fields[9]. Indeed, we have

$$\begin{aligned} \delta S &= \int_{\mathcal{M}} E \cdot \delta\phi + \int_{\Sigma_+} (\theta + \delta\ell) - \int_{\Sigma_-} (\theta + \delta\ell) + \int_{\mathcal{I}} \underbrace{(\theta + \delta\ell)}_{\mathrm{d}C} \\ &= \int_{\mathcal{M}} E \cdot \delta\phi + \int_{\Sigma_+ - \Sigma_-} (\theta + \delta\ell) + \int_{\partial\mathcal{I}} C \\ &= \int_{\mathcal{M}} E \cdot \delta\phi + \int_{\Sigma_+ - \Sigma_-} (\theta + \delta\ell - \mathrm{d}C) \ , \end{aligned}$$

where in the second line we used Stokes' theorem for the last integral, while in the third line we used again Stokes'theorem to rewrite the integral on $\partial\mathcal{I} = (-\partial\Sigma_+) \cup (\partial\Sigma_-)$ as an integral on $\Sigma_\pm$ and the minus sign for the last term of the second integral is due to the orientations discussed above when approaching the boundary of the Cauchy slice from the time-like or space-like normal directions[10].

Now, what do we mean by symmetry and in particular by physical symmetry? There are various notions of "symmetries" and already in the previous sections we have anticipated that not all symmetries of the action which leave the equations of motion unchanged are then physical symmetries on the space of solutions, as there might be redundancies due to our descriptions that need to be removed. In order to systematically distinguish among the various notions of symmetries, let us start by defining a so-called *Lagrangian symmetry*.

Lagrangian symmetry - A Lagrangian symmetry is a transformation of the fields that leaves the Lagrangian invariant up to past/future boundary terms, i.e. such that the (bulk) equations of motion are not affected by such a transformation. More precisely, continuous symmetries of a Lagrangian field theory are associated with vector fields $X \in \mathfrak{X}(\mathcal{C})$ on the configuration space $\mathcal{C}$[11] such that

$$\mathcal{L}_X L = I_X \delta L = \mathrm{d}\ell_X \ , \tag{7.2}$$

where I_X denotes the contraction of a field-space vertical form with the vector field X, and in the first equality we used Cartan's magic formula $\mathcal{L}_X = I_X\delta + \delta I_X$ for field space differential calculus $(\delta, I_X, \mathcal{L}_X)$ (to be distinguished from spacetime differential calculus $(\mathrm{d}, \iota_\xi, \mathcal{L}_\xi)$, with the distinction between Lie derivative along $X \in \mathfrak{X}(\mathcal{C})$ and $\xi \in \mathfrak{X}(\mathcal{M})$ being clear from the

[9] We recall from Sec. 4.4, that e.g. for generally covariant theories background fields, if any, can only leave on the boundary.

[10] Note that, since C is defined only on $\mathcal{I}$, such a rewriting requires an arbitrary extension of C to $\Sigma_\pm$. This is not a problem as in any case only its value on $\partial\Sigma_\pm$ contributes as indicated by the fact that only $\mathrm{d}C$ and not C appears when inside the integral on $\Sigma_\pm$.

[11] Recall that $\mathcal{C}$ is the space of field histories obeying certain given spatial boundary conditions but not necessarily the equations of motion.

context) and the fact that L is a 0-form on field space ($I_X L = 0$). ℓ_X on the r.h.s. of Eq. (7.2) is a $(D-1,0)$-form locally constructed out of dynamical (and background, if any) fields and it satisfies the boundary conditions

$$(\ell_X + I_X\delta\ell)\Big|_{\mathcal{I}} = \mathrm{d}\alpha_X \ , \tag{7.3}$$

for some $(D-2,0)$-form α_X locally constructed out of dynamical and background fields on the time-like boundary $\mathcal{I}$ and the variation of the former. The boundary conditions (7.3) in fact ensure the action to be invariant up to past and future boundary terms, namely

$$\mathcal{L}_X S = \int_{\mathcal{M}} I_X\delta L + \int_{\partial\mathcal{M}} I_X\delta\ell = \int_{\partial\mathcal{M}=\mathcal{I}\cup\Sigma_+\cup\Sigma_-}(\ell_X + I_X\delta\ell) = \int_{\Sigma_+-\Sigma_-}(\ell_X + I_X\delta\ell - \mathrm{d}\alpha_X) \tag{7.4}$$

where in the second equality we used the definition (7.2) of Lagrangian symmetry and Stokes' theorem, while in the third equality we first split the integral on $\partial\mathcal{M} = \mathcal{I}\cup\Sigma_+\cup\Sigma_-$ into its various pieces and then used the boundary conditions (7.3) followed again by Stokes's theorem to rewrite the integral over $\mathcal{I}$ as $\int_{\mathcal{I}}\mathrm{d}\alpha_X = \int_{\partial\mathcal{I}=-\partial\Sigma_+ +\partial\Sigma_-}\alpha_X = -\int_{\Sigma_+-\Sigma_-}\mathrm{d}\alpha_X$, with the minus sign originating from the above discussed orientation conventions.

Lagrangian symmetries can be further differentiated into:

- **Global symmetries -** Continuous symmetries parameterised by a finite number of real parameters (actually these could also be discrete symmetries but we will only deal with continuous transformations in this course).
- **Local symmetries -** Continuous symmetries whose parameters are local functions or functionals. The latter case corresponds to field-dependent transformations and we will come back to it in Sec. 7.3.
- **Physical symmetries -** True symmetries of the theory, not of our description of the system. Such symmetries are characterised by the existence of conserved charges and they change the physical state of the system (i.e., they do not act as the identity transformation on solution space). We will come back on this point in the next section.
- **Unphysical symmetries/Gauge redundancies -** These are fictitious symmetries associated with redundancies of our description of the system. In particular, as discussed in Sec. 6.2, they are often due to the local character of our description, the non-local physical quantities being not always easily accessible. As such, these transformations are not symmetries of the system itself and have no associated conserved charges on the physical phase space. Such transformations in fact do not change the physical state of the system and need to be modded out when constructing the physical phase space, thus resulting into a trivial action on the space of solutions.

The traditional lore in undergraduate courses is to identify global symmetries with physical symmetries and local symmetries with unphysical gauge redundancies. However, such an identification is too strict and turns out to be not always true. In particular, this is the case in presence of boundaries which make the distinction between physical symmetries and gauge redundancy more subtle. As we will try to explain in the following, there are local symmetries supported on the boundary[12] which are indeed physical and satisfy all the relevant properties (including existence of a non-trivial conserved charges).

[12] These are sometimes referred to as *boundary symmetries* or *large gauge symmetries* depending on the

7.2 Noether's theorem and conserved charges

In order to explicitly see the above distinctions and the related subtleties in presence of boundaries, let us have a closer look at the consequences of the definition (7.2) of Lagrangian symmetries from the phase space point of view.

Using the relation $\delta L = E \cdot \delta\phi + \mathrm{d}\theta$, the Lie derivative of the action along a vector field $X \in \mathfrak{X}(\mathcal{C})$ can be written as

$$
\begin{aligned}
\mathcal{L}_X S &= \int_{\mathcal{M}} I_X \delta L + \int_{\partial\mathcal{M}} I_X \delta\ell \\
&= \int_{\mathcal{M}} (I_X(E \cdot \delta\phi) + \mathrm{d}I_X\theta) + \int_{\partial\mathcal{M}} I_X \delta\ell \\
&= \int_{\mathcal{M}} I_X(E \cdot \delta\phi) + \int_{\partial\mathcal{M}} I_X(\theta + \delta\ell) \,,
\end{aligned}
\tag{7.5}
$$

where in the second line we used the property that I_X commutes with the spacetime exterior derivative d, and in the last line we used Stokes' theorem and the linearity of I_X. But, if X is a Lagrangian symmetry, then comparing Eq. (7.5) with the second equality of Eq. (7.4) yields

$$
\int_{\partial\mathcal{M}} I_X\theta - \ell_X = -\int_{\mathcal{M}} I_X(E \cdot \delta\phi) \,. \tag{7.6}
$$

Looking at the integral on the l.h.s. of (7.6), we have

$$
\begin{aligned}
\int_{\partial\mathcal{M}} I_X\theta - \ell_X &= \int_{\Sigma_+ - \Sigma_-} I_X\theta - \ell_X + \int_{\mathcal{I}} I_X\theta - \underbrace{\ell_X}_{\underset{(7.3)}{=} \mathrm{d}\alpha_X - I_X\delta\ell} \\
&= \int_{\Sigma_+ - \Sigma_-} I_X\theta - \ell_X + \mathrm{d}\alpha_X + \int_{\mathcal{I}} I_X \underbrace{(\theta + \delta\ell)}_{\underset{(7.1)}{=} \mathrm{d}C} \\
&= \int_{\Sigma_+ - \Sigma_-} I_X\theta - \ell_X + \mathrm{d}(\alpha_X - I_X C) \\
&= \int_{\Sigma_+ - \Sigma_-} I_X\theta - \ell_X + \int_{\partial\Sigma_+ - \partial\Sigma_-} \alpha_X - I_X C \,,
\end{aligned}
\tag{7.7}
$$

where we recall that in converting the integrals over $\mathcal{I}$ involving $\mathrm{d}\alpha_X$ and $\mathrm{d}C$ into integrals over $\partial\Sigma_\pm$ we used Stokes' theorem and the fact that $\partial\mathcal{I} = (-\partial\Sigma_+) \cup \partial\Sigma_-$. Plugging this result back into Eq. (7.6), we get

$$
\int_{\Sigma_+ - \Sigma_-} I_X\theta - \ell_X + \int_{\partial\Sigma_+ - \partial\Sigma_-} \alpha_X - I_X C = -\int_{\mathcal{M}} I_X(E \cdot \delta\phi) \,. \tag{7.8}
$$

This means that the quantity

$$
H_X := \int_{\Sigma} I_X\theta - \ell_X + \int_{\partial\Sigma} \alpha_X - I_X C \,, \tag{7.9}
$$

context and the type of boundary involved, the latter being finite or asymptotic, space-like, time-like, or null. The name large gauge transformations is historically rooted in the analysis of asymptotic (null) boundaries of asymptotically flat spacetimes to emphasise the distinction with gauge redundancies which are instead sometimes referred to as *small gauge transformations*. These large gauge symmetries played a very important role in the recent developments about the infrared structure of gauge theories and gravity with interesting connections to QFT soft theorems and memory effects. We will not be able to cover such topics in this class and refer the interested students to Strominger's lectures reported in the further readings at the end of this lecture.

is conserved on-shell ($E = 0$) and is independent of the choice of the Cauchy slice Σ, namely $H_X\big|_{\Sigma_+} - H_X\big|_{\Sigma_-} = 0$ on-shell. This is the statement of Noether's (first) theorem which tells us that for any Lagrangian symmetry X there exist an associated conserved quantity H_X. The $(D-1,0)$-form

$$J_X := I_X\theta - \ell_X \ , \tag{7.10}$$

is called the (bulk) *Noether current*, while the functional H_X is called the *Noether charge* associated to the Lagrangian symmetry X and generalises the familiar result in absence of boundaries. In general, however, it should be kept in mind that the conserved quantity is H_X and not J_X whose name *Noether current* is somewhat misleading and should be better called *Noether potential*. Note that, using Stoke's theorem, Eq. (7.9) can be also written as an integral over Σ only

$$H_X = \int_\Sigma I_X(\theta - \mathrm{d}C) - \ell_X + \mathrm{d}\alpha_X \ , \tag{7.11}$$

whose integrand $(D-1,0)$-form satisfies the properties of

- being closed w.r.t. the spacetime exrerior differential d on-shell:

$$\begin{aligned} \mathrm{d}\left(I_X(\theta - \mathrm{d}C) - \ell_X + \mathrm{d}\alpha_X\right) &= I_X \mathrm{d}\theta - \mathrm{d}\ell_X \\ &= -I_X(E\cdot\delta\phi) + \underbrace{I_X\delta L - \mathrm{d}\ell_X}_{\overset{=}{\scriptstyle (7.2)} 0} \\ &= 0 \qquad (\text{on-shell} \ \ E = 0) \ . \end{aligned} \tag{7.12}$$

- having vanishing pull-back on $\mathcal{I}$:

$$\left(I_X(\theta - \mathrm{d}C) - \ell_X + \mathrm{d}\alpha_X\right)\Big|_{\mathcal{I}} \overset{(7.1),(7.3)}{=} -I_X\delta\ell + I_X\delta\ell = 0 \ . \tag{7.13}$$

compatibly with H_X being independent of Σ.

Finally, let us recall from Sec. 5.5, that

$$\Omega_{\Sigma_+} - \Omega_{\Sigma_-} = \int_\Sigma \mathrm{d}\omega - \int_{\mathcal{I}} \omega = -\int_\Sigma \delta \mathrm{d}\theta - \int_{\mathcal{I}} \omega = \int_{\mathcal{M}} \delta(E\cdot\delta\phi) - \int_{\mathcal{I}} \omega \, ,$$

from which, comparing with Eqs. (7.8) and (7.9), it follows that the requirement $\int_{\mathcal{I}} \omega = 0$ of vanishing presymplectic flux through $\mathcal{I}$ boils down to the following requirement

$$\begin{aligned} (I_X\omega_\Sigma - \delta H_X)\big|_{\Sigma_+} - (I_X\omega_\Sigma - \delta H_X)\big|_{\Sigma_-} &= \int_{\mathcal{M}} I_X\delta(E\cdot\delta\phi) + \int_{\mathcal{M}} \delta I_X(E\cdot\delta\phi) \\ &= \int_{\mathcal{M}} \mathcal{L}_X(E\cdot\delta\phi) \\ &= \int_{\mathcal{M}} (I_X\delta E)\cdot\delta\phi + E\cdot(\mathcal{L}_X\delta\phi) \ , \end{aligned} \tag{7.14}$$

with the r.h.s. vanishing for arbitrary field variations in configuration space about any solution of the field equations of motion. This, together with the fact that the quantities on the l.h.s. of Eq. (7.15) are independent of the choice of Σ, means that the vector field X is tangent to pre-phase space and results into a non-trivial transformation on the space of solutions where

$$I_X\Omega_\Sigma = \delta H_X \ , \tag{7.15}$$

i.e., the vector field X (or more precisely its pushforward to the solution space) is a Hamiltonian vector field with Hamiltonian functional given by the Noether charge H_X. Thus, the Hamiltonian flow generated by the Lagrangian symmetry X sends solutions of the equations of motion to other such solutions (physical symmetry).

7.3 Gauge symmetries and boundary charges

Let us now specialise the above discussion to the case in which the Lagrangian symmetry of interest is a gauge symmetry according to the following definition:

Gauge transformation - A *gauge transformation* is a Lagrangian symmetry

$$X_\lambda = \delta_\lambda \phi \cdot \frac{\delta}{\delta \phi} \, , \tag{7.16}$$

whose parameters λ are local function(al)s.

In the case of Maxwell electrodynamics ($\phi = A$), for instance, there is a gauge symmetry $A \mapsto A + \mathrm{d}\lambda$ whose infinitesimal version $\delta_\lambda A = \mathcal{L}_{X_\lambda} A = \mathrm{d}\lambda$ is generated by the vector field

$$X_\lambda = \delta_\lambda A \cdot \frac{\delta}{\delta A} = (\mathrm{d}\lambda) \cdot \frac{\delta}{\delta A} = (\partial_\mu \lambda) \cdot \frac{\delta}{\delta A_\mu} \, . \tag{7.17}$$

For the time being, we will assume for simplicity that the theory we are considering has no boundary Lagrangian ($\ell = 0$) so that imposing Dirichlet boundary conditions on $\mathcal{I}$, which fix the pull-back of the fields to $\mathcal{I}$, is enough for stationarity of the action. This is the case for instance with standard gauge theories like Maxwell electrodynamics or Yang-Mills theory for which the stationarity requirement (up to past/future boundary terms) is satisfied when Dirichlet boundary conditions are imposed on A with no need for ℓ or C. Of course, other choices of boundary conditions are possible depending on the situation we want to describe. Here, we restrict ourselves to such a specific choice to illustrate the peculiarities of gauge theories with boundaries in the simplest possible setting. We will come back to boundary Lagrangians in the next lecture when moving the discussion to (generally) covariant theories.

With these premises, let us consider then Maxwell electrodynamics minimally coupled with matter current whose action is given by

$$S = - \int_{\mathcal{M}} \frac{1}{2e^2} (F \wedge *F) + A \wedge *j \, , \tag{7.18}$$

where $F = \mathrm{d}A$ is the field strength 2-form, and j is the matter current 1-form so that $*j$ yields a $(D-1)$-form on spacetime which is linearly coupled the the local gauge connection 1-form A in the coupling term D-form entering the Lagrangian top form. As can be easily checked by direct computation, the variation of the above Lagrangian leads to the following equation of motion and (pre)symplectic potential forms

$$E(A) = \frac{1}{e^2} (\mathrm{d} * F) - *j \qquad , \qquad \theta(A, \delta A) = \frac{1}{e^2} \delta A \wedge *F \, . \tag{7.19}$$

Let us now look at the quantities introduced in the previous (sub)section for the case of the Lagrangian gauge symmetry X_λ given in Eq. (7.17). The Lie derivative of Lagrangian along X_λ yields

$$\mathcal{L}_{X_\lambda} L = -\delta_\lambda A \wedge *j = -(\mathrm{d}\lambda) \wedge *j = -\mathrm{d}(\lambda * j) - \lambda \mathrm{d} * j$$

from which, comparing with the definition (7.2) of a Lagrangian symmetry, we see that gauge invariance of the action (7.18) leads us to identify

$$\ell_{X_\lambda} = -\lambda * j \ , \tag{7.20}$$

and the continuity equation for the matter current

$$\mathrm{d} * j = 0 \ . \tag{7.21}$$

Moreover, from the expression (7.19) of θ, it follows that the Noether current J_{X_λ} reads as

$$J_{X_\lambda} = I_{X_\lambda}\theta - \ell_{X_\lambda} = \frac{1}{e^2}\mathrm{d}\lambda \wedge *F + \lambda * j \ . \tag{7.22}$$

The two terms in Eq. (7.22), respectively linear in the electromagnetic field and the matter charge current, are sometimes called the "soft-current" and the "hard-current". Such names were introduced in the context of Maxwell electrodynamics in asymptotically flat spacetime and usually refer to the currents carrying no energy ("soft") and the energy-carrying matter fields ("hard") through future null infinity. Here in these notes we will not enter the details of the analysis of null boundaries and focus only on time-like boundaries and their codimension 2 spatial corners. The Noether potential (7.22) satisfies the equation (cfr. Eq. (7.12))

$$\mathrm{d}J_{X_\lambda} = \mathrm{d}(I_{X_\lambda}\theta - \ell_{X_\lambda}) = -I_{X_\lambda}(E \cdot \delta A) \ , \tag{7.23}$$

whose r.h.s., in the case of a gauge symmetry, can be further decomposed into

$$I_{X_\lambda}(E \cdot \delta A) = E \cdot \delta_\lambda A = E \cdot \mathrm{d}\lambda = \mathrm{d}(\underbrace{\lambda E}_{=:C_{X_\lambda}}) - \lambda \mathrm{d}E \ , \tag{7.24}$$

with C_{X_λ} denoting a "constraint" which vanishes when the equations of motion are imposed[13]. Plugging Eq. (7.24) back into Eq. (7.23), we find

$$\mathrm{d}(J_{X_\lambda} + C_{X_\lambda}) = \lambda \mathrm{d}E = 0 \ , \tag{7.25}$$

where the r.h.s. vanishes off-shell due to $d^2 = 0$ and the continuity equation (7.21). Therefore, for a gauge theory, Noether's conservation law reads as

$$\mathrm{d}(J_{X_\lambda} + C_{X_\lambda}) = \mathrm{d}(I_{X_\lambda}\theta - \ell_{X_\lambda} + C_{X_\lambda}) = 0 \ , \tag{7.26}$$

independently of the equations of motion being imposed. The fact that the l.h.s. is a closed form off-shell in turn means that there exist a $(D-2,0)$-form Q_{X_λ} called the charge aspect such that

$$J_{X_\lambda} + C_{X_\lambda} = \mathrm{d}Q_{X_\lambda} \ , \tag{7.27}$$

and on-shell we have

$$J_{X_\lambda} \stackrel{E=0}{=} \mathrm{d}Q_{X_\lambda} \tag{7.28}$$

[13] Recalling from Sec. 6 that, in the covariant formalism, canonical constraints are encoded as components of the equations of motion, one of the components of such a covariant constraint is in fact the familiar Gauß constraint associated with the gauge symmetry in canonical electrodynamics.

i.e., the Noether current is a pure boundary term on-shell. This is the hallmark of gauge theories. Explicitly, in the Maxwell case, we have

$$\begin{aligned}
\mathrm{d}Q_{X_\lambda} &= \frac{1}{e^2}\mathrm{d}\lambda \wedge *F + \lambda * j + \lambda E \\
&= \frac{1}{e^2}\mathrm{d}\lambda \wedge *F + \lambda * j + \lambda\left(\frac{1}{e^2}(\mathrm{d} * F) - *j\right) \\
&= \frac{1}{e^2}\mathrm{d}(\lambda * F)\ ,
\end{aligned} \tag{7.29}$$

i.e.

$$Q_{X_\lambda} = \frac{1}{e^2}(\lambda * F)\ . \tag{7.30}$$

This means that in gauge theories like Maxwell electrodynamics, there are infinitely many local boundary charges given by

$$Q_{\mathcal{S}}[\lambda] = \int_{\mathcal{S}} Q_{X_\lambda} = \frac{1}{e^2}\int_{\mathcal{S}=\partial\Sigma} \lambda * F\ , \tag{7.31}$$

with $\mathcal{S} = \partial\Sigma$ the codimension 2 spatial corner (recall that Q_{X_λ} is a $(D-2)$-form on spacetime). These are the generators of local physical symmetries on the boundary and not mere gauge redundancies. In fact, as discussed in Sec. 6 (cfr. Eqs. (6.10)-(6.13)), we have

$$\begin{aligned}
I_{X_\lambda}\Omega_\Sigma &= \frac{1}{e^2}\int_\Sigma I_{X_\lambda}(\delta A \wedge \delta * F) \\
&= \frac{1}{e^2}\int_\Sigma \mathrm{d}\lambda \wedge \delta * F \\
&= \frac{1}{e^2}\int_\Sigma \mathrm{d}(\lambda\delta * F) + \lambda\delta(d * F) \\
&\overset{E=0}{=} \frac{1}{e^2}\int_{\partial\Sigma} \lambda\delta * F\ ,
\end{aligned} \tag{7.32}$$

where the second term in the third line vanishes on-shell, as can be seen by using the equations of motion (7.19) and the fact that j does not depend on on Maxwell dynamical fields, and we used Stokes' theorem for the first term to get the last equality. Therefore, we see that

- when there is no boundary or λ vanishes on $\partial\Sigma$ ($\lambda|_{\partial\Sigma} = 0$), then the r.h.s. of (7.32) would vanish and X_λ would be a zero mode of the presymplectic structure (i.e. $I_{X_\lambda}\Omega_\Sigma = 0$). These are the pure gauge redundancies that we quotient out in constructing the physical phase space. In presence of boundaries, such gauge transformations vanishing at spatial boundaries are also called *trivial gauge symmetries* as they just correspond to redundancies of our description of the system with trivial action on the space of solutions.
- when instead λ does not vanish on $\partial\Sigma$, the corresponding vector field X_λ will generate a non-trivial flow on the physical phase space.

Schematically, the physical symmetries of a gauge theory with boundaries can be then thought of as being given by

$$\text{physical symmetries} = \frac{\text{allowed Lagrangian symmetries}}{\text{trivial gauge symmetries}}\ .$$

These can in principle include also isometries of background structures. For instance, in the case of generally covariant theories, these might include also Killing symmetries of the asymptotic metric, thus leading to the set of physical asymptotic symmetries. We will be looking at such asymptotic symmetries in the context of black holes, e.g. when looking at spatial infinity for black hole thermodynamics[14].

Finally, in the case of non-trivial gauge symmetries supported on the boundary ($\lambda|_{\partial\Sigma} \neq 0$), we can further distinguish the following two cases:

Case 1 - Field-independent boundary symmetries

In this case, Eq. (7.32) yields

$$I_{X_\lambda}\Omega_\Sigma = \delta\left(\frac{1}{e^2}\int_{\partial\Sigma}\lambda * F\right) = \delta Q_{\mathcal{S}}[\lambda] \ , \tag{7.33}$$

from which we see that on-shell X_λ are Hamiltonian vector fields with Hamiltonian $Q_{\mathcal{S}}[\lambda]$. Using then the symplectic structure, we can also compute the Poisson brackets of the boundary charges as

$$\begin{aligned}
\{Q_{\mathcal{S}}[X], Q_{\mathcal{S}}[Y]\} &= \Omega_\Sigma(X, Y) \\
&= I_Y I_X \Omega_\Sigma \\
&= I_Y \delta Q_{\mathcal{S}}[X] \\
&= \mathcal{L}_Y Q_{\mathcal{S}}[X] \ ,
\end{aligned}$$

but

$$\begin{aligned}
\delta\{Q_{\mathcal{S}}[X], Q_{\mathcal{S}}[Y]\} &= \delta\mathcal{L}_Y Q_{\mathcal{S}}[X] \\
&= \mathcal{L}_Y \delta Q_{\mathcal{S}}[X] \\
&= -[\mathcal{L}_Y, I_X]\Omega_\Sigma \\
&= I_{[X,Y]}\Omega_\Sigma \\
&= \delta Q_{\mathcal{S}}([X, Y]) \ ,
\end{aligned}$$

so that

$$\{Q_{\mathcal{S}}[X], Q_{\mathcal{S}}[Y]\} = Q_{\mathcal{S}}([X, Y]) \ , \tag{7.34}$$

i.e., the generators close an infinite-dimensional Lie algebra.

Case 2 - Field-dependent boundary symmetries

In this case $\delta\lambda \neq 0$ and Eq. (7.32) can be rewritten as

$$I_{X_\lambda}\Omega_\Sigma = \delta\left(\frac{1}{e^2}\int_{\partial\Sigma}\lambda * F\right) - \frac{1}{e^2}\int_{\partial\Sigma}(\delta\lambda) * F = \delta Q_{\mathcal{S}}[\lambda] - \frac{1}{e^2}\int_{\partial\Sigma}(\delta\lambda) * F \ . \tag{7.35}$$

According to Eq. (7.15), the r.h.s. of (7.35) is δH_{X_λ} so that, recalling the expressions (7.9), (7.10) and (7.28), the Lagrangian symmetry under consideration must satisfy boundary conditions on $\mathcal{S} = \partial\Sigma$ such that $\delta\alpha_{X_\lambda} = -\frac{1}{e^2}(\delta\lambda) * F$, that is $(\delta\lambda)\delta * F|_{\partial\Sigma} = 0$. There might be however

[14] Similar story holds also at null infinity and additional symmetries arise as demanded by the requirement of preserving boundary conditions at the past/future bounday of future/past null infinity. Such symmetries are known as large gauge transformations and are generated by the so-called soft charges. As anticipated before, for more details, we refer the interested students to Strominger's lecture notes on the infrared structure of gravity.

situations in which we need to relax our boundary conditions. This is for instance the case of finite boundaries where one considers $\mathcal{M}$ to be a finite region in spacetime rather than the whole spacetime. The behaviour of fields at its finite boundaries are in general different from the corresponding behaviour at infinity. In such situations, different boundary conditions are required, which in turn leads to the inclusion of boundary Lagrangians in the action of the theory, and Eq. (7.35) tells us that the symplectic structure is not preserved anymore (there is a symplectic anomaly), namely

$$\mathcal{L}_{X_\lambda}\Omega_\Sigma = \delta I_{X_\lambda}\Omega_\Sigma = \int_{\mathcal{S}} \delta\vartheta_{X_\lambda} \qquad , \qquad \vartheta_{X_\lambda} = \frac{1}{e^2}(\delta\lambda) * F = Q_{\delta\lambda}(x) \qquad x \in \mathcal{S}$$

or (recalling that $\Omega_\Sigma = \int_\Sigma \omega$ and using Stokes' theorem on the $\int_{\mathcal{S}=\partial\Sigma}$ term)

$$\mathcal{L}_{X_\lambda}\omega = \mathrm{d}\delta\vartheta_{X_\lambda} \; . \tag{7.36}$$

The fact that the symplectic structure is not preserved means that the number of degrees of freedom is not conserved. In other words, there are some missing degrees of freedom associated with the corner $\mathcal{S} = \partial\Sigma$ (the "edge" of our splacelike region). This is the initial idea behind the introduction of the so-called edge modes in gauge theories where one seek for an extended phase space description in which there is an additional term in the symplectic potential $\theta^{\text{ext}} := \theta + \mathrm{d}\vartheta$, with ϑ such that $\delta(\vartheta_X + \mathcal{L}_X\vartheta) = 0$ on-shell and the corresponding extended symplectic structure is preserved (cfr. Eq. (7.36)). This goes however beyond the purposes of present course and is part of ongoing research in recent years. We refer to the list of further readings below for more details.

In the next lecture (Sec. 8), we will move the discussion to covariant Lagrangians and diffeomorphism charges in which case the vector field X will be the lift X_ξ to field configuration space of a vector field $\xi \in \mathfrak{X}(\mathcal{M})$ generating a spacetime diffeomorphism. We will see that, in the case of diffeomorphism symmetry, the Noether charge H_ξ in Eq. (7.9) will read the same just with $\ell_{X_\xi} = \iota_\xi L$ and $\alpha_{X_\xi} = \iota_\xi \ell$ with ξ parallel to $\mathcal{I}$.

Further reading

[1] D. Harlow and J. Wu, *Covariant phase space with boundaries*, JHEP 10 (2020) 146, `hep-th/1906.08616`

[2] G. Compère and A. Fiorucci, *Advanced Lectures on General Relativity*, (2019) `hep-th/1801.07064`

[3] J. Lee and R. M. Wald, *Local symmetries and constraints*, Journal of Mathematical Physics 31 3 (1990), pp.725-743.

[4] F. Gieres, *Covariant canonical formulations of classical field theories*, `hep-th/2109.07330`

[5] A. Strominger, *Lectures on the Infrared Structure of Gravity and Gauge Theory*, (2018), `hep-th/1703.05448`

[6] W. Donnelly and L. Freidel, *Local subsystems in gauge theory and gravity*, JHEP 09 (2016) 102, `hep-th/1601.04744`

[7] M. Geiller and P. Jai-akson, *Extended actions, dynamics of edge modes, and entanglement entropy*, JHEP 2020, 134 (2020), `hep-th/1912.06025`

[8] S. Carrozza and P. H. Hoehn, *Edge modes as reference frames and boundary actions from post-selection*, (2021), `hep-th/2109.06184`