GR lecture 8

Einstein-Hilbert action; Conical singularities; The Schwarzchild solution

I. CARROLL'S BOOK: SECTIONS 4.3, 5.1-5.4

II. $T^{\mu\nu}$ AS NOETHER CHARGE VS. $T^{\mu\nu}$ AS VARIATION WITH RESPECT TO THE METRIC

When deriving the Einstein equations from an action principle, we found ourselves identifying the stress-energy tensor as:

$$T^{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\partial(\sqrt{-g}L)}{\partial g_{\mu\nu}} \ , \tag{1}$$

or, equivalently:

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\partial(\sqrt{-g}L)}{\partial g^{\mu\nu}} = Lg_{\mu\nu} - 2\frac{\partial L}{\partial g^{\mu\nu}} \ , \tag{2}$$

where L is the matter Lagrangian. As we've seen for the electromagnetic field, this definition actually doesn't directly coincide with the one derived by considering $T^{\mu\nu}$ as the 4-current of 4-momentum, which is in turn the Noether charge associated with translations. Nevertheless, the claim is that (1) defines something very much like the Noether current for translations, such that e.g. the integrated total 4-momentum calculated from both definitions is the same (at least in flat spacetime, where such an integrated quantity makes sense). Once we believe that (1) defines something like the 4-current of 4-momentum, then it is clearly the superior definition, since it's automatically symmetric and gauge-invariant. However, why should we believe that? In this section, we attempt to answer that question.

Recall that a symmetric matrix such as $T_{\mu\nu}$ is fully determined by its products $T_{\mu\nu}u^\mu u^\nu$ with arbitrary timelike unit vectors. Thus, to understand $T_{\mu\nu}$, it's enough to consider T_{tt} in arbitrary Lorentz frames. From the Noether point of view, T_{tt} should be the energy density. To make this concrete, let's consider the action in flat spacetime with initial conditions somewhere and final conditions at $t = t_f$. Then the action's variation upon putting the same final conditions but at a slightly later time $t_f + \delta t$ reads:

$$\delta S = -\delta t \int_{t=t_f} d^3x \, T_{tt} \ . \tag{3}$$

Now, let us notice that there's another way to change the time duration of the spacetime region associated with S: instead of changing the final time coordinate t_f, we can just stretch the metric near $t = t_f$! In particular, to obtain the same shift δt_f of proper time, we can stretch a short time interval $(t_f - \Delta t, t_f)$ by a factor of $1 + \delta t/\Delta t$, where we take Δt small but much longer than δt. Thus, we must stretch $\sqrt{-g_{tt}}$ by a factor of $1 + \delta t/\Delta t$, which is equivalent to:

$$\delta g_{tt} = -2\sqrt{-g_{tt}}\,\delta(\sqrt{-g_{tt}}) = -2\frac{\delta t}{\Delta t} \ , \tag{4}$$

where we used the flat value $g_{tt} = -1$ before the variation. The resulting variation of the action $S = \int d^4x\sqrt{-g}\,L$ reads:

$$\delta S = \Delta t \int d^3x\, \frac{\delta(\sqrt{-g}L)}{\delta g_{tt}}\,\delta g_{tt} = -2\delta t \int d^3x\, \frac{\delta(\sqrt{-g}L)}{\delta g_{tt}} \ . \tag{5}$$

Comparing with (3), we conclude that it indeed makes sense to identify (1) as the stress-energy tensor.

III. GAUGE INVARIANCE VS. CONSERVATION

Another comment is that (1) has a close analog in electromagnetism. Indeed, the electric 4-current of a charged field can be defined by varying the action with respect to the electromagnetic potential:

$$j^\mu = \frac{\delta L}{\delta A_\mu} \ . \tag{6}$$

Charge conservation can then be beautifully derived as a consequence of gauge invariance. We simply consider a variation of the particular form $\delta A_\mu = \partial_\mu\theta$, which is a gauge transformation, and must leave the action invariant:

$$0 = \delta S = \int d^4x\, \frac{\delta L}{\delta A_\mu}\delta A_\mu = \int d^4x\, j^\mu\partial_\mu\theta = -\int d^4x\,\theta\,\partial_\mu j^\mu \ , \tag{7}$$

where we integrated by parts and disposed of the boundary term by choosing $\theta(x)$ that vanishes on the boundary. Since (7) must be true for otherwise arbitrary $\theta(x)$, we conclude that charge is locally conserved: $\partial_\mu j^\mu = 0$.

To construct the analogous argument in gravity, recall that an infinitesimal coordinate transformation $x^\mu \to x^\mu + \xi^\mu$ acts on various fields through the Lie derivative $\mathcal{L}_\xi$. Indeed, in adapted coordinates for which $\xi^\mu = (\epsilon, 0, 0, 0)$, the coordinate transformation acts simply as the partial derivative $\epsilon(\partial/\partial x^0)$; the Lie derivative $\mathcal{L}_\xi$ is the coordinate-independent formulation of the same geometric concept. As we've seen in Lecture 5, the Lie derivative of the metric can be written in terms of covariant derivatives as:

$$\delta g_{\mu\nu} = \mathcal{L}_\xi g_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 2\nabla_{(\mu}\xi_{\nu)} \ . \tag{8}$$

Since the action should be invariant under this coordinate transformation, we conclude:

$$0 = \delta S = \int d^4x \, \frac{\delta(\sqrt{-g}L)}{\delta g_{\mu\nu}} \delta g_{\mu\nu} = \int d^4x \sqrt{-g}\, T^{\mu\nu} \nabla_\mu \xi_\nu = -\int d^4x \, \xi_\nu \nabla_\mu T^{\mu\nu} \ . \tag{9}$$

Again, for this to be true for arbitrary infinitesimal $\xi^\mu(x)$, we must have the conservation law $\nabla_\mu T^{\mu\nu} = 0$.

IV. THE CONICAL SINGULARITY SOLUTION IN 2+1D GR

As a warmup towards the Schwarzschild solution in 3+1d, let's consider time-independent, rotationally symmetric, non-rotating vacuum solutions in 2+1d. In other words, let's find the gravitational field of a stationary point mass in 2+1d GR. We begin by writing the following ansatz for the metric:

$$ds^2 = g_{tt}(r)dt^2 + g_{rr}(r)dr^2 + r^2 d\phi^2 \ . \tag{10}$$

This is the most general metric that satisfies the following assumptions:

- Rotationally invariant, i.e. invariant under $\phi \to \phi + \text{const}$. This implies $\partial_\phi g_{\mu\nu}$.
- Treats the clockwise and anticlockwise directions equally, i.e. invariant under $\phi \to -\phi$. This implies $g_{t\phi} = g_{r\phi} = 0$.
- Static, i.e. $\partial_t g_{\mu\nu} = 0$ and $g_{tr} = g_{t\phi} = 0$.

As we will see in an exercise, the last assumption isn't actually necessary. Note that we don't need to consider a more general $g_{\phi\phi}(r)$ in (10), since we can always use the tangential length element $\sqrt{g_{\phi\phi}}\, d\phi \equiv r d\phi$ as a definition of the r coordinate. The nonzero elements of

$g_{\mu\nu}$, $g^{\mu\nu}$ and $\partial_\mu g_{\nu\rho}$ read:

$$g_{tt} \ ; \quad g_{rr} \ ; \quad g_{\phi\phi} = r^2 \ ; \tag{11}$$

$$g^{tt} = \frac{1}{g_{tt}} \ ; \quad g^{rr} = \frac{1}{g_{rr}} \ ; \quad g^{\phi\phi} = \frac{1}{r^2} \ ; \tag{12}$$

$$\partial_r g_{tt} \equiv g'_{tt} \ ; \quad \partial_r g_{rr} \equiv g'_{rr} \ ; \quad \partial_r g_{\phi\phi} = 2r \ . \tag{13}$$

The Christoffel symbols then read:

$$\Gamma^r_{tt} = -\frac{g'_{tt}}{2g_{rr}} \ ; \quad \Gamma^t_{tr} = \frac{g'_{tt}}{2g_{tt}} \ ; \quad \Gamma^r_{rr} = \frac{g'_{rr}}{2g_{rr}} \ ; \quad \Gamma^r_{\phi\phi} = -\frac{r}{g_{rr}} \ ; \quad \Gamma^\phi_{\phi r} = \frac{1}{r} \ , \tag{14}$$

where all other components are either related to the above by symmetries (e.g. $\Gamma^t_{rt} = \Gamma^t_{tr}$) or vanishing. We see that a lot of Christoffel components have a form similar to $g'_{rr}/(2g_{rr})$. This is not a coincidence: the Christoffel is really about curvature angles, which are related not to the absolute size of the metric, but to its relative rate of change; finally, the factor of 1/2 in the Christoffel's definition tells us that it's directly sensitive not to the metric – which gives lengths squared – but to lengths themselves. Thus, it's a better idea to reparameterize the original metric (10) as:

$$ds^2 = -e^{2\alpha(r)}dt^2 + e^{2\beta(r)}dr^2 + r^2 d\phi^2 \ , \tag{15}$$

which gives us:

$$\Gamma^r_{tt} = \alpha' e^{2(\alpha-\beta)} \ ; \quad \Gamma^t_{tr} = \alpha' \ ; \quad \Gamma^r_{rr} = \beta' \ ; \quad \Gamma^r_{\phi\phi} = -re^{-2\beta} \ ; \quad \Gamma^\phi_{\phi r} = \frac{1}{r} \ . \tag{16}$$

We can now directly compute the Ricci tensor as:

$$R_{\mu\nu} = \partial_\rho \Gamma^\rho_{\mu\nu} - \partial_\mu \Gamma^\rho_{\nu\rho} + \Gamma^\rho_{\rho\sigma}\Gamma^\sigma_{\mu\nu} - \Gamma^\rho_{\mu\sigma}\Gamma^\sigma_{\nu\rho} \ , \tag{17}$$

which yields:

$$\begin{aligned} &R_{tt} = e^{2(\alpha-\beta)}\left(\alpha'' + \alpha'^2 - \alpha'\beta' + \frac{\alpha'}{r}\right) \ ; \quad R_{rr} = -\alpha'' - \alpha'^2 + \alpha'\beta' + \frac{\beta'}{r} \ ; \\ &R_{\phi\phi} = re^{-2\beta}(\beta' - \alpha') \ . \end{aligned} \tag{18}$$

Let us now apply the vacuum Einstein equations $R_{\mu\nu} = 0$. From $e^{2(\beta-\alpha)}R_{tt} + R_{rr}$, we find $\alpha' + \beta' = 0$. On the other hand, from $R_{\phi\phi}$, we find $\beta' - \alpha' = 0$. It follows that α' and β' both vanish, i.e. that α and β are both constants! We can get rid of these constants by rescaling the coordinates as:

$$t \to e^\alpha t \ ; \quad r \to e^\beta r \ ; \quad \phi \to e^{-\beta}\phi \ , \tag{19}$$

which brings the metric to the flat form:

$$ds^2 = -dt^2 + dr^2 + r^2 d\phi^2 \ . \tag{20}$$

Note that rescaling ϕ as in (19) may affect its 2π periodicity, which so far we've been taking for granted. As we'll now see, the non-trivial part of the geometry (20) is precisely encoded in this periodicity.

First, let's recall that the flat answer (20) should have been expected: we know that in 3d spacetime, $R_{\mu\nu} = 0$ implies that the entire Riemann curvature vanishes. However, now we must be careful. For a point mass, $T_{\mu\nu}$ and thus $R_{\mu\nu}$ vanishes everywhere except at $r = 0$. Thus, we may have some curvature that's concentrated, like a delta function, just at the origin. To see what this curvature should look like, let's "zoom in" on the point mass so it isn't look pointlike anymore. Recall the form of $T_{\mu\nu}$ for a mass density at rest, in locally inertial coordinates:

$$T_{\mu\nu} = \begin{pmatrix} \varepsilon & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \ , \tag{21}$$

By the 3d Einstein equation, the Ricci tensor then takes the form:

$$R_{\mu\nu} = 8\pi G(T_{\mu\nu} - T g_{\mu\nu}) = 8\pi G \begin{pmatrix} 0 & 0 & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & \varepsilon \end{pmatrix} \ . \tag{22}$$

Thus, we expect a purely spatial 2d curvature $R_{xx} = R_{yy}$. Recall that in 2d, the Riemann tensor has just one independent component. The same curvature can be expressed equivalently as:

$$R_{xyxy} = R_{xx} = R_{yy} = \frac{1}{2} R \ . \tag{23}$$

Thus, returning to the pointlike mass case, we are dealing with a distributional curvature of the form:

$$R_{xyxy} = 8\pi G M \delta^2(\mathbf{x}) \ , \tag{24}$$

where M is the mass at the origin, and $\delta^2(\mathbf{x})$ is a spatial delta function that integrates to 1. What do we call a 2d flat manifold with distributional curvature at the origin? We call this a cone. Indeed, a 2d cone is constructed by simply "cutting out" some angle χ from a flat plane, and gluing the two sides of the cut together. The geometry throughout the cone is the same as that of the plane, i.e. flat, with the exception of the apex. To see that there is curvature at the apex, we recall our definition of the Riemann in terms of parallel transport along a closed loop. If we parallel-transport a vector around the apex of the cone, it ends up at angle χ to its original orientation. Taking χ to be small for simplicity and taking care with the signs (better to make a drawing for this purpose), we find that the rotation matrix upon traversing a counterclockwise loop is:

$$M_i{}^j = \begin{pmatrix} 1 & -\chi \\ \chi & 1 \end{pmatrix} . \tag{25}$$

Recalling that the Riemann tensor element $R^x{}_{yxy}$ is $M_y{}^x$ per unit area of a counterclockwise loop, we read off:

$$R_{xyxy} = \chi\delta^2(\mathbf{x}) . \tag{26}$$

Comparing with (24), we see that in 2+1d GR, the geometry around a (small) mass M is conical, with deficit angle $\chi = 8\pi GM$. Returning to the polar coordinates (20), we note that the deficit angle can be encoded by simply changing the period of ϕ from 2π to $2\pi - \chi$, without any change to the ds^2 formula.

EXERCISES

Exercise 1. *Prove by direct calculation that the variation of the Ricci tensor is:*

$$\delta R_{\mu\nu} = \nabla_\rho \delta\Gamma^\rho_{\nu\mu} - \nabla_\nu \delta\Gamma^\rho_{\rho\mu} . \tag{27}$$

Exercise 2. *Prove the 2+1d version of Birkhoff's theorem. Starting from an ansatz that doesn't assume time independence:*

$$ds^2 = -e^{2\alpha(t,r)}dt^2 + e^{2\beta(t,r)}dr^2 + r^2 d\phi^2 , \tag{28}$$

show that the vacuum Einstein equations $R_{\mu\nu} = 0$ *imply* $\partial_r\alpha = \partial_r\beta = \partial_t\beta = 0$*, which brings the metric to the form:*

$$ds^2 = -e^{2\alpha(t)}dt^2 + e^{2\beta}dr^2 + r^2 d\phi^2 . \tag{29}$$

Finally, find a coordinate transformation which brings this metric to the flat form (20)*.*

Exercise 3. *Consider the Schwarzschild metric:*

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \frac{dr^2}{1 - 2GM/r} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \ . \tag{30}$$

Show that this metric satisfies the vacuum Einstein equations $R_{\mu\nu} = 0$*. On the other hand, show that* R_{trtr} *is nonzero, and compare it to the Newtonian prediction at* $r \gg GM$*.*