GR lecture 12

Gravitational waves: polarizations and production

I. CARROLL'S BOOK: SECTIONS 7.4-7.5

II. A WORKED-OUT EXAMPLE OF DE DONDER AND TRANSVERSE-TRACELESS GAUGE

In the previous lecture, we considered small perturbations of the metric around flat spacetime:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \ , \tag{1}$$

and established a notation in which indices are raised and lowered using the flat metric $\eta_{\mu\nu}$. We found it convenient to define the trace-reversed version of the metric perturbation $h_{\mu\nu}$:

$$\tilde{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}h\eta_{\mu\nu} \ ; \quad \tilde{h} = -h \ . \tag{2}$$

In terms of this quantity, we defined de Donder gauge:

$$\partial^\mu \tilde{h}_{\mu\nu} = 0 \ . \tag{3}$$

In this gauge, to first order in $h_{\mu\nu}$, the Einstein tensor becomes simply:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = -\frac{1}{2}\Box\tilde{h}_{\mu\nu} \ , \tag{4}$$

where $\Box$ is the d'Alambertian:

$$\Box \equiv \partial_\mu\partial^\mu = -\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \ . \tag{5}$$

The linearized Einstein equation thus reads:

$$\Box\tilde{h}_{\mu\nu} = -16\pi G T_{\mu\nu} \ . \tag{6}$$

Let's now consider the source-free version:

$$\Box\tilde{h}_{\mu\nu} = 0 \ . \tag{7}$$

The non-trivial solutions to this equation are gravitational waves. By the usual Fourier transform techniques, these solutions are spanned by plane waves of the form:

$$\tilde{h}_{\mu\nu}(x) = \tilde{C}_{\mu\nu} e^{ik_\rho x^\rho} \ , \tag{8}$$

where the wavevector k_μ and the polarization tensor $\tilde{C}_{\mu\nu}$ are constants. The field equation (7) implies that k_μ is lightlike: $k_\mu k^\mu = 0$, while de Donder gauge (3) implies that $\tilde{C}_{\mu\nu}$ is transverse: $k^\mu \tilde{C}_{\mu\nu} = 0$. As an example, consider the null wavevector:

$$k^\mu = \omega(1,1,0,0) \ ; \quad k_\mu = \omega(-1,1,0,0) \ , \tag{9}$$

which represents a wave with frequency ω moving along the x direction. The most general symmetric polarization tensor $\tilde{C}_{\mu\nu}$ that satisfies $k^\mu \tilde{C}_{\mu\nu}$ reads:

$$\tilde{C}_{\mu\nu} = \begin{pmatrix} \alpha & -\alpha & \beta & \gamma \\ -\alpha & \alpha & -\beta & -\gamma \\ \beta & -\beta & \tilde{C}_{yy} & \tilde{C}_{yz} \\ \gamma & -\gamma & \tilde{C}_{yz} & \tilde{C}_{zz} \end{pmatrix} \ . \tag{10}$$

From $\tilde{h}_{\mu\nu}$, we can extract the actual metric perturbation $h_{\mu\nu}$, via:

$$h_{\mu\nu} = \tilde{h}_{\mu\nu} - \frac{1}{2}\tilde{h}\eta_{\mu\nu} = C_{\mu\nu} e^{ik_\rho x^\rho} \ , \tag{11}$$

where the polarization tensor $C_{\mu\nu}$ in our example (10) reads:

$$C_{\mu\nu} = \tilde{C}_{\mu\nu} - \frac{1}{2}\tilde{C}\eta_{\mu\nu} = \begin{pmatrix} \frac{2\alpha+\tilde{C}_{yy}+\tilde{C}_{zz}}{2} & -\alpha & \beta & \gamma \\ -\alpha & \frac{2\alpha-\tilde{C}_{yy}-\tilde{C}_{zz}}{2} & -\beta & -\gamma \\ \beta & -\beta & \frac{\tilde{C}_{yy}-\tilde{C}_{zz}}{2} & \tilde{C}_{yz} \\ \gamma & -\gamma & \tilde{C}_{yz} & \frac{\tilde{C}_{zz}-\tilde{C}_{yy}}{2} \end{pmatrix} \ . \tag{12}$$

Now, recall that de Donder gauge still leaves us with some of the gauge symmetry $h_{\mu\nu} \to h_{\mu\nu} - \partial_\mu\xi_\nu - \partial_\nu\xi_\mu$. Specifically, we can still apply gauge transformations that satisfy $\Box\xi_\mu = 0$. The parameter of this residual gauge symmetry can again be decomposed into plane waves of the form:

$$\xi_\mu(x) = \epsilon_\mu e^{ik_\rho x^\rho} \ . \tag{13}$$

In our example (12), we can now perform a gauge transformation with ϵ_μ of the form:

$$\epsilon_\mu = \frac{1}{\omega}\left(-\frac{2\alpha+\tilde{C}_{yy}+\tilde{C}_{zz}}{4} \, , \ \frac{2\alpha-\tilde{C}_{yy}-\tilde{C}_{zz}}{4} \, , \ -\beta \, , \ -\gamma\right) \ , \tag{14}$$

which leaves us with the traceless-transverse polarization tensor:

$$C_{\mu\nu} = \tilde{C}_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\tilde{C}_{yy}-\tilde{C}_{zz}}{2} & \tilde{C}_{yz} \\ 0 & 0 & \tilde{C}_{yz} & \frac{\tilde{C}_{zz}-\tilde{C}_{yy}}{2} \end{pmatrix} \equiv \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & C_{\oplus} & C_{\otimes} \\ 0 & 0 & C_{\otimes} & -C_{\oplus} \end{pmatrix} . \tag{15}$$

Here, $C_{\oplus}$ and $C_{\otimes}$ are the amplitudes of the "+" and "×" polarizations of the gravitational wave.