Graviton, four dimensional Beltrami Parametrization, and BMS Symmetry and its Cocycles

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hepth2304.12369 : BRST BMS4 Symmetry and its Cocycles from Horizontality Conditions

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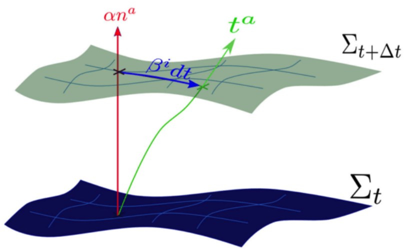


The standard ADM picture. It needs refinements to take care of gravitons.

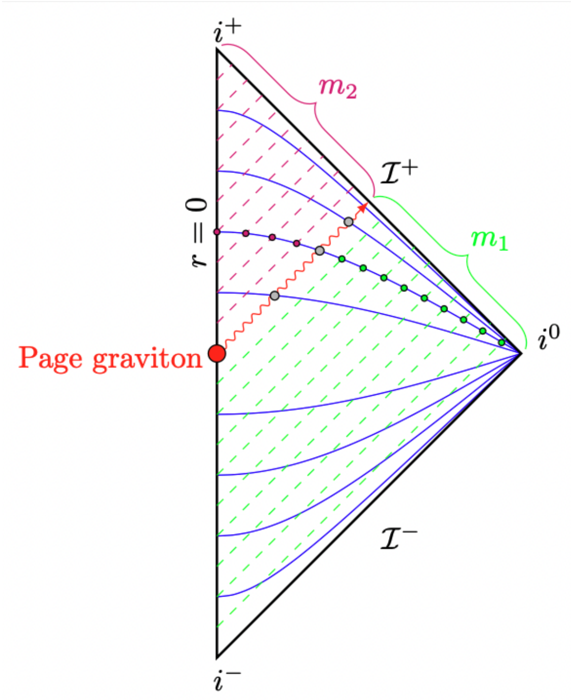
The 2d Beltrami metric $ds^2_{Beltrami} \equiv \exp\Phi ||dz + \mu_{\bar z}^z d\bar z||^2$ **generalizes for** $d = 4$ **as (LB arXiv:2109.06681) :**

$$ds^2_{d=4_{\text{Beltrami}}} \equiv -M^2\Big(d\tau - \mu^t_\tau dt\Big)^2 + N^2\Big(dt + \mu^t_\tau d\tau\Big)^2 + \exp\Phi\Big|\Big|dz + \boxed{\mu^z_{\bar z}}d\bar z + \mu^z_t dt + \mu^z_\tau d\tau\Big|\Big|^2$$

$$= -\mathcal{M}^2(d\tau^+ + \boxed{\mu^+_-}d\tau^-)(d\tau^- + \underline{\mu^-_+}d\tau^+) + \exp\Phi\Big|\Big|dz + \boxed{\mu^z_{\bar z}}d\bar z + \mu^z_- d\tau^- + \underline{\mu^z_+}d\tau^+\Big|\Big|^2$$

The basic idea is a further foliation of ADM leafs $^3\Sigma(\tau)$ **by spatial 2-manifolds** Σ_2**. The 10 Beltrami fields have different physical interpretations, in particular for the graviton excitations and Bondi mass effect.**

The $d = 4$ **Beltami parametrization is a Diff**$_4$$\subset$**Diff**$_4$$\times$**Lorentz covariant gauge fixing of the 16 components of generic vierbeins for** M_4**.**

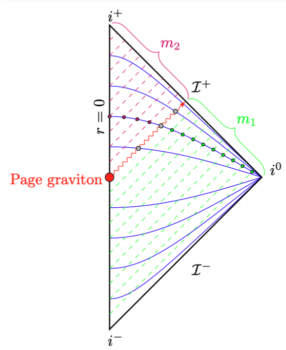


Penrose diagram

M_4 is asymptotically flat. The Penrose diagram Finkelstein-Eddington coordinates $z, \overline{z}, \tau^{\pm}$ better illustrates graviton creations/absorptions the boundaries $\mathcal{I}^{\pm}$ while ignoring the fundamentals of QGFT.

The 1963 ”physical” gauge Bondi gauge is best to describe gravitons near the boundaries. It has the virtue of clarifying the time problem of RG.

Penrose diagram



As a physical gauge analogous to a temporal gauge in YM, the Bondi gauge lets ungauged fixed part of Diff_4 **symmetry. It defines a physics that is by definition invariant under the BMS symmetry.**

It implies the existence of absorbed of emitted gravitons in the null boundaries $\mathcal{I}^{\pm}$ **as well as "would be " gravitational dof's, the soft gravitons, parametrizing the distortions of the celestial sphere when the latter absorbs/emits gravitons from/to the bulk,**

$$\rightarrow \textit{the so called memory effect.}$$

The paternity of the latter effect is perhaps Christodoulou. Effective BMS invrariant theories in $\mathcal{I}^{\pm}$**.**

$R^{classical}_{\mu\nu} =< T^{quantum}_{\mu\nu} >$ MUST be replaced by $\boxed{<?R^{quantum}_{\mu\nu}? >=< T^{quantum}_{\mu\nu} >}$

not to contradict the principles of quantum measurement, eg in the case of pendulum experiment (!!) Not yet appropriate glasses for a precise reading of the left-hand side of the boxed equation.

Postulating that gravitons do exist as massless particles allows one to go a long way:.

- possibly introducing coherent states made of gravitons;

- possibly detecting divergent classical gravitational waves issued from $\sim$ the celestial sphere.

- The asymptotic BMS symmetry is a superb and suggestive mathematical tool, that is physically obvious when one uses the Bondi gauge near the boundaries asymptotically flat spacetimes.

- It is the leftover of the reparametrization symmetry near $\mathcal{I}^{\pm}$

$$z \to z, \quad \bar{z} \to \bar{z}, \quad u \to u + \alpha(z, \bar{z}),$$

modulo z and $\bar{z}$ reparametrizations on the celestial sphere. BMS symmetry is called "supertranslation".

To work out the whole scheme, usefull to slightly broaden the null boundaries.

$$\mathcal{I}^{\pm} \to \delta M^4 = \mathcal{I}^{\pm} \times \delta r \quad \text{with } \delta r \propto \hbar$$

One may see the ingoing emission as a sort of Hawking effect for the spherical sphere as a sort of quantum broadened horizon. One then uses the Eddington Finkestein coodinates.

On must compute the asymptotic metric in lowest non trivial order in $1/r$ for asymptotically flat spaces and physically understand the roles of the various metric components in the Bondi gauge .

BMS pionnered all this in the 60's, doing brute force computarions, which had to be refined over the years, but the whole ideas were brilliant.

The Beltrami parametrization is quite useful for making them transparent.

Motivation for the Beltrami parametrization:

York (1970) defined gravitational observables as functionals of the classes of equivalences of the leaf metrics g_{ab}, defined modulo reparametrizations and Weyl transformations. It is technically not obvious to compute the correlation functions of such entities in QFT, and which gauge should be used to do so. Defining gravity in the class of unimodular gauges is possible but quite tedious. (ghost of ghost phenomena, because the unimodular gauge implies the use of divergentless diffeomorphisms ghosts, etc...)

But in the d-dimensional Beltrami formulation, the excitations of a geometrical well-defined and covariant subset of $\frac{d(d-3)}{2}$ local Weyl invaraint fields define from scratch the physical d-dimensional gravitons near the boundaries.

For $d = 4$ The "Beltrami" result is that both helicities ± 2 of four dimensional propagating gravitons (as well as the soft gravitons) are nothing but $\mu^{z}_{\bar{z}-1}(z, \bar{z}, \tau^{\pm})$ and $\mu^{\bar{z}}_{z-1}(z, \bar{z}, \tau^{\pm})$ near the boundaries. $\mu^{+}_{-1-}(z, \bar{z}, \tau^{-})$ is the Bondi mass effect, idebntiufied as gneralized Beltrami differential excitations.

- The soft gravitons remain confined in the celestial sphere and cannot absorb or emit energy in or from the bulk.

- The physical gravitons can travel in the bulk can serve as "Page" gravitons (they solve the Page request).

- Strominger names the soft gravitons : Goldstone bosons for the spontaneous breaking of the BMS symmetry.

- My own preference : creations and absorptions of soft gravitons in the celestial sphere at different values of $\tau^{\pm}$ in $\mathcal{I}^{\mp}$ imply a correlated vacuum change of the celestial sphere.

- Asymptotic physical gravitons are excitations of the Beltrami differential of celestial sphere of order 1/r.. Their existence is necessary for possibly getting a trivial gravitational dynamics in the bulk

- BMS opened the way and discover in the early 60' these properties by computing the fall-off conditions on the metric in function of $t \equiv r$ for asymptotically flat manifolds suitably combined with the Einstein e.o.m.'s in a perturbatively modified asymptotical vacuum.

The Beltrami fields that distort the flat metric are small for two reasons:

- One gets a factor $\propto \hbar$ for the emission of a single graviton that is a quantum object.

-One also has fall-off conditions of the asymptotically flat metric components, such that

$$X(z,\overline{z},u,r) \equiv \sum_{p\geq 0} r^{-p} X_{-p}(z,\overline{z},u)$$

and the graviton behaves asymptotically as $\frac{1}{r}\mu_{\overline{z}}^{z}(u,z,\overline{z})_{-1}$, as can be seen when solving the Einstein equation of motion near the boundaries.

- The one graviton fiels are multiplied by a gigantic number typically the Avogadro number. The associated metric linearly describes a gravitational wave propagating near the celestial sphere.

- $m_{Bondi}(u) = -\frac{1}{2}\frac{1}{2}\int_{\text{celestial sphere}} dz d\overline{z}\ \mu_{-}^{+}(z,\overline{z},u)_{-1}$ **is the so called Bondi mass .**

-the evolution of the Bondi mass M along $\mathcal{I}^{\pm}$ is what allows the ADM mass conservation when the graviton with field $\mu_{\overline{z}}^{z}(u,z,\overline{z})_{-1}$ emitted by the celestial sphere successively crosses ADM 3d Manifold during its time evolution from its creation in the bulk till its gets absorbed in the boundary or by a gravitational effect in the bulk that may change its dynamics.

1) fall-off conditions of the Beltrami fields for asymptotically flat manifolds By definition:

$$g_{uu} = -1 + \frac{2m_B}{t} + \mathcal{O}(t^{-2}), \quad g_{ut} = -1 + \mathcal{O}(t^{-2}), \quad g_{AB} = t^2 q_{AB} + t\ C_{AB} + D_{AB} + \mathcal{O}(t^{-1}), \quad g_{uA} = \mathcal{O}(1)$$

$$\mu_-^+ = \frac{\mu^+_{-\,-1}(z,\overline{z},u)}{t} + \mathcal{O}(t^{-2}),$$

$$\mu_{\overline{z}}^z = \frac{\mu^z_{\overline{z}-1}(z,\overline{z},u)}{t} + \mathcal{O}(t^{-2}), \quad \mu_z^{\overline{z}} = \frac{\mu^{\overline{z}}_{z-1}(z,\overline{z},u)}{t} + \mathcal{O}(t^{-2}),$$

$$\mu_-^A = \frac{\mu^A_{u\,-2}(z,\overline{z},u)}{t^2} + \mathcal{O}(t^{-3}),$$

$$\mathcal{M} = 1 + \frac{\mathcal{M}_{-2}(z,\overline{z},u)}{t^2} + \mathcal{O}(t^{-3}), \quad \Phi = 2\ln t + \Phi' = 2\ln t + \Phi'_0(z,\overline{z}) + \frac{\Phi'_{-2}(z,\overline{z},u)}{t^2} + \mathcal{O}(t^{-3})$$

and the analogous equations with $u \leftrightarrow v$. At the first non trivial order in $1/t$ the metrics reads as

$$\boxed{ds^2_{\infty-\delta t\leq t<\infty} = -(1+\frac{1}{t}\mu^+_{-\,-1})du^2 - 2dtdu + t^2 \exp\Phi'_0(dzd\overline{z} + \frac{\mu^{\overline{z}}_{z-1}}{t}dzdz + \frac{\mu^z_{\overline{z}-1}}{t}d\overline{z}d\overline{z}) + \text{lower order in } 1/\text{t}}$$

The so-called soft gravitons solve the Einstein eom and remain confined in δM^4. They cannot transfer energy in the bulk, they are as Goldstone particles for the broken BMS supertranslation symmetry of the vacuum in Strominger et al terminology.

The Beltrami parametrization simplifies getting the physical gravitons eqs of motion by suggesting solving the Mikowski self/antidual Spin connection equations.

Split $\omega^{ab^{\pm}} \equiv \omega^{ab} \pm i\epsilon^{abcd}\omega_{cd}$ **with** $\omega_{\mu}^{ab}(\omega_{\mu}^{ij}, \omega_{\mu}^{\tau i}) \sim (\omega_{\mu}^{i}, \omega_{\mu}^{\tau i}) \equiv (\omega_{\mu}^{z}, \omega_{\mu}^{\overline{z}}, \omega_{\mu}^{0}; \omega_{\mu}^{\tau z}, \omega_{\mu}^{\tau \overline{z}}, \omega_{\mu}^{\tau 0})$ **and then** $\boxed{\omega_{\mu}^{\pm\ i} = \omega_{\mu}^{i} \pm \omega_{\mu}^{\tau i}}$**.**

$$I_{Einstein} = \int \epsilon_{abcd} e^{a} \wedge e^{b} \wedge R^{bc} \sim \int \eta_{ab} (\omega^{ac^{-}} \wedge e_{c}) \wedge (\omega^{bd^{-}} \wedge e_{d})$$

$$\to I_{Einstein} = \int e^{i} \wedge \omega^{i-} \wedge e^{j} \wedge \omega^{j-} + \epsilon_{ijk} e^{\tau} \wedge e^{i} \wedge \omega^{j-} \wedge \omega^{k-} \text{with no second order derivatives of the metric.}$$

with Solving the first order equation $\omega^{\pm}(e^{\pm})$**=0 extremizes the action and solves the second order Einstein eom's.**

Not all solutions are selfdual or antiself/dual. Moreover, non linear equations, etc...Linearly, it's OK.

Using light cone and then Eddington–Finkelstein Beltrami coordinates $(\tau^\pm, r, z, \overline{z})$:

$$\begin{pmatrix} e^z \\ e^{\overline{z}} \\ e^t \\ e^\tau \end{pmatrix} \equiv \begin{pmatrix} \exp\Phi & 0 & 0 & 0 \\ 0 & \exp\Phi & 0 & 0 \\ 0 & 0 & \mathcal{M} & 0 \\ 0 & 0 & 0 & \mathcal{M} \end{pmatrix} \begin{pmatrix} 1 & \mu_{\overline{z}}^z & \mu_+^z & \mu_-^z \\ \mu_z^{\overline{z}} & 1 & \mu_+^{\overline{z}} & \mu_-^{\overline{z}} \\ 0\,0 & & 1 & \mu_+^- \\ 0\,0 & & \mu_-^+ & 1 \end{pmatrix} \begin{pmatrix} dz \\ d\overline{z} \\ dt^+ \\ dt^- \end{pmatrix}$$

Bondi gauge is imposing the lapse $\boxed{\mu_+^z = \mu_+^{\overline{z}} = \mu_+^- = 0 \text{ and } \sqrt{g} \propto r^2.}$

$$\begin{aligned} ds^2 &= -\mathcal{M}^2(d\tau^+ + \mu_-^+ d\tau^-)(d\tau^- + \mu_+^- d\tau^+) + \exp\Phi||dz + \mu_{\overline{z}}^z d\overline{z} + \mu_+^z d\tau^+ + \mu_-^z d\tau^-||^2 \\ &= -\mathcal{M}^2\Big(d\tau^{-2} + 2dtd\tau^-(1 + \frac{1}{2}\mu_-^+)\Big) + \exp\Phi||dz + \boxed{\mu_{\overline{z}}^z} d\overline{z} + \mu_-^z d\tau^-||^2) \end{aligned}$$

$$\boxed{\rightarrow \textit{asymptotic metric}: ds^2 \sim -\mathcal{M}_0^2\Big(d\tau^{-2} + 2dtd\tau^-(1 + \frac{1}{2}\frac{\mu_{-}^{+}{}_{-1}}{t})\Big) + \exp\Phi_0||dz + \boxed{\frac{\mu_{\overline{z}}^{z}{}_{-1}}{t}} d\overline{z}||^2)}$$

Using eom's, one may check that this is physical gauge for describing gravitons flying in hyperplanes orthogonal directions $d\tau^\pm = 0$ (after or before being emitted and absorbed in the celestial sphere.)

This gauge mix of unimodular gauge and of vanishing shift vector along $\imath^\pm$. $\mu_\pm^i = 0$. It implies zero modes for the reparametrization ghosts. In principle one needs ghosts of ghosts.

→ **the BMS symmetry stems from the non trivial consistency of the 4 Bondi gauge conditions with the** Diff_4 **invariance.**

$$
\begin{aligned}
Lie_\xi g_{tt} &= -2\mathcal{M}^2\partial_t\xi^u = 0 \\
Lie_\xi g_{tz} &= -\mathcal{M}^2\partial_z\xi^u + \mu_z^{\bar z}\exp\Phi\;\partial_0 t\xi^z + \frac{1}{2}(1+\mu_{\bar z}^z\mu_z^{\bar z})\exp\Phi\;\partial_t\xi^{\bar z} = 0 \\
Lie_\xi g_{t\bar z} &= -\mathcal{M}^2\partial_{\bar z}\xi^u + \mu_{\bar z}^z\exp\Phi\;\partial_t\xi^{\bar z} + \frac{1}{2}(1+\mu_{\bar z}^z\mu_z^{\bar z})\exp\Phi\;\partial_0 t\xi^z = 0
\end{aligned}
$$

$$
g^{AB}Lie_\xi g_{AB} = t\left(\partial_t\Phi - \frac{\mu_{\bar z}^z\partial_t\mu_z^{\bar z}+\mu_z^{\bar z}\partial_t\mu_{\bar z}^z}{1-\mu_{\bar z}^z\mu_z^{\bar z}}\right) + \xi^i\left(\partial_i\Phi - \frac{\mu_{\bar z}^z\partial_i\mu_z^{\bar z}+\mu_z^{\bar z}\partial_i\mu_{\bar z}^z}{1-\mu_{\bar z}^z\mu_z^{\bar z}}\right)(\partial_z\xi^z + \mu_u^z\mathcal{D}_z\xi^u) + \left(\partial_{\bar z}\xi^{\bar z} + \mu_u^{\bar z}\mathcal{D}_{\bar z}\xi^u\right) = 0
$$

$$
Indeed: \int L^{Bondi\ \ gauge\ \ fixing} = \int s\Big(\bar\xi^{tt}g_{tt} + \bar\xi^{A}g_{tA} + \bar\xi^{g}(\sqrt{\det g_{AB}} - t^2\omega(x)\Big)
$$

$$
\Rightarrow\ \ non\ trivial\ zero\ modes\ \xi^\mu_{BMS}\ for\ the\ ghosts\ \xi^\mu.
$$

→ **Some computations willl need ghosts of ghosts.**

Solving the consistency equations at all order in the $1/r$ expansion provides the following metric dependent restrictions for $\xi^\mu(u,r,z,\overline{z})$:

$$\begin{aligned}
\xi^u(u,r,z,\overline{z}) &\equiv \xi^u_0(u,z,\overline{z}),\\
\xi^z(u,r,z,\overline{z}) &\equiv \xi^z_0(u,z,\overline{z}) - 2\partial_{\overline{z}}\xi^u_0 \int_r^\infty dr'\ \frac{\mathcal{M}^2(1+\mu^z_{\overline{z}}\mu^{\overline{z}}_z)\exp -\Phi}{(1-\mu^z_{\overline{z}}\mu^{\overline{z}}_z)^2} + 4\partial_z\xi^u_0 \int_r^\infty dr'\ \frac{\mathcal{M}^2\mu^z_{\overline{z}}\exp -\Phi}{(1-\mu^z_{\overline{z}}\mu^{\overline{z}}_z)^2},\\
\xi^{\overline{z}}(u,r,z,\overline{z}) &\equiv \xi^{\overline{z}}_0(u,z,\overline{z}) - 2\partial_z\xi^u_0 \int_r^\infty dr'\ \frac{\mathcal{M}^2(1+\mu^z_{\overline{z}}\mu^{\overline{z}}_z)\exp -\Phi}{(1-\mu^z_{\overline{z}}\mu^{\overline{z}}_z)^2} + 4\partial_{\overline{z}}\xi^u_0 \int_r^\infty dr'\ \frac{\mathcal{M}^2\mu^{\overline{z}}_z\exp -\Phi}{(1-\mu^z_{\overline{z}}\mu^{\overline{z}}_z)^2},\\
\xi^r(u,r,z,\overline{z}) &= -\frac{r}{2}\left[(\partial_z\xi^z + \mu^z_u\mathcal{D}_z\xi^u) + \left(\partial_{\overline{z}}\xi^{\overline{z}} + \mu^{\overline{z}}_u\mathcal{D}_{\overline{z}}\xi^u\right) + \xi^i\left(\partial_i\Phi - \frac{\partial_i(\mu^z_{\overline{z}}\mu^{\overline{z}}_z)}{1-\mu^z_{\overline{z}}\mu^{\overline{z}}_z}\right)\right].
\end{aligned} \tag{1}$$

At first non trivial order in $1/r$ they are

$$\begin{aligned}
\xi^z(u,r,z,\overline{z}) &= \xi^z_0(u,z,\overline{z}) - \frac{1}{r}\nabla^z\xi^u + \mathcal{O}(r^{-2}),\\
\xi^{\overline{z}}(u,r,z,\overline{z}) &= \xi^{\overline{z}}_0(u,z,\overline{z}) - \frac{1}{r}\nabla^{\overline{z}}\xi^u + \mathcal{O}(r^{-2}),\\
\xi^u(u,r,z,\overline{z}) &= \xi^u_0(u,z,\overline{z}),\\
\xi^r(u,r,z,\overline{z}) &= -\frac{r}{2}\nabla_A\xi^A_0 + \frac{1}{2}\nabla_A\nabla^A\xi^u + \mathcal{O}(r^{-1}).
\end{aligned} \tag{2}$$

∇_A is the covariant derivative with respect to the unit 2-sphere.

The subset of the Diff_4 **symmetry with infinitesimal parameters solutions of the previous equation defines the leftover gauge invariance of the gravitational action in the Bondi gauge.**

This proves that it doesn't provide a full gauge fixing, as announced.

The restrictions on the ghosts ξ^μ and the Beltrami fields falloffs can then be combined.

One gets

$$0 = (s\mu_{\bar{z}}^{z})_0 = \partial_{\bar{z}}\xi_0^z \quad \text{and} \quad 0 = (s\mu_u^z)_0 = \partial_u \xi_0^z \quad \Rightarrow \quad \boxed{\xi_0^z = \xi_0^z(z)}$$

$$0 = 2(s\mathcal{M})_0 = \partial_u \xi^u + \xi_1^r \quad \Rightarrow \quad \boxed{\xi^u = \alpha(z,\bar{z}) + \frac{u}{2}\nabla_A \xi_0^A}$$

$\alpha(z,\bar{z})$ **is an arbitrary given function of** $(z,\bar{z})$ **obtained by a trivial quadrature over the** u **coordinate.**

Analogously as for $\xi_0^z = \xi_0^z(z)$**, one gets** $\xi_0^{\bar{z}} = \xi_0^{\bar{z}}(\bar{z})$**.**

At this stage, one can check that all fall-off constraints are satisfied.

This defines the BMS symmetry, $z \to z$ **and** $u \to u + \alpha(z,\bar{z})$**.**

The residual symmetry of Diff_4 is thus carried by the restricted four ghosts $\xi^\mu_{\rm BMS}$, that are actually well defined functionals of a "fundamental geometrical ghost representation" made of

$$\alpha(z,\overline{z})\ ,\quad \xi_0^z(z)\ ,\quad \xi_0^{\overline{z}}(\overline{z}). \tag{3}$$

One can in fact use the following notation

$$\xi_{\rm BMS} = \xi(\alpha, \xi_0^A, g_{mn}^{Bondi}) \tag{4}$$

and one has the final form:

$$\begin{aligned}
\xi^u_{\rm BMS} &= \alpha(z,\overline{z}) + \frac{u}{2}\nabla_A\xi_0^A,\\
\xi^z_{\rm BMS} &= \xi_0^z(z) - \frac{1}{r}\nabla^z\xi^u_{\rm BMS} + \mathcal{O}(r^{-2}),\\
\xi^{\overline{z}}_{\rm BMS} &= \xi_0^{\overline{z}}(\overline{z}) - \frac{1}{r}\nabla^{\overline{z}}\xi^u_{\rm BMS} + \mathcal{O}(r^{-2}),\\
\xi^r_{\rm BMS} &= -\frac{r}{2}\nabla_A\xi_0^A + \frac{1}{2}\nabla_A\nabla^A\xi^u_{\rm BMS} + \mathcal{O}(r^{-1}).
\end{aligned} \tag{5}$$

The BMS beautiful result is of course recovered by expanding in spherical harmonics the arbitrary function $\alpha(z,\overline{z})$.

Moreover, the geometrical equations that express the $d = 4$ Lorentz$\times$Diff symmetry in the Beltrami formulation determinerne the following action of the nilpotent BRST operator on the ghosts

$$
\begin{aligned}
s\alpha(z,\bar{z}) &= \xi_0^A \partial_A \alpha + \frac{\alpha}{2} \nabla_A \xi_0^A \\
s\xi_0^z(z) &= \xi_0^z \partial_z \xi_0^z \\
s\xi_0^{\bar{z}}(\bar{z}) &= \xi_0^{\bar{z}} \partial_{\bar{z}} \xi_0^{\bar{z}}.
\end{aligned} \tag{6}
$$

and, among other formula

$$
\begin{aligned}
s\mu_{\bar{z}-1}^{z} &= \partial_{\bar{z}}\xi_{-1}^{z} + (\partial_{\bar{z}}\mu_{\bar{z}-1}^{z})\xi_0^{\bar{z}} + \xi_0^z\partial_z\mu_{\bar{z}-1}^{z} - \mu_{\bar{z}-1}^{z}\partial_z\xi_0^z + \xi^u\partial_u\mu_{\bar{z}-1}^{z} - \xi_1^r\mu_{\bar{z}-1}^{z} \\
&= \Big(\xi^u\partial_u + \xi_0^A\partial_A + \tfrac{3}{2}\partial_{\bar{z}}\xi_0^{\bar{z}} - \tfrac{1}{2}\partial_z\xi_0^z\Big)\mu_{\bar{z}-1}^{z} - \partial_{\bar{z}}^2\xi^u
\end{aligned}
$$

The last term $-\partial_{\bar{z}}^2\xi^u$ term that expresses the $d = 4$ subtlety of $s\mu_{\bar{z}-1}^{z}$.

The above computed values of the BMS transformation of the Beltrami differential $s\mu_{\bar{z}-1}^{z}$ and $s\mu_{z-1}^{\bar{z}}$ of the primary ghosts $\xi_0^z, \xi_0^{\bar{z}}$ and α do coincide with the transformations of the BMS4 algebroid computed by Barnich in Hamiltonian formalism.

The Lagrangian formalism is better suited for classifying the possible anmalies of the BMS4 symmetry.

BMS4 cocycles and possible anomalies for three dimensional theories in $\mathcal{I}^{\pm}$

To gain intuition for computing possible non trivial cocycles for the BRST BMS4 symmetry, consider first the quite simpler Diff_2 BRST operation s for the ghosts $\xi_0^z(z)$ and $\xi_0^{\bar z}(\bar z)$.

Defines the unified 1-forms $\tilde M_0^z \equiv dz + \mu_{\bar z 0}^z d\bar z + \xi_0^z(z)$ and $\tilde M_0^{\bar z} \equiv d\bar z + \mu_{z 0}^{\bar z} dz + \xi_0^{\bar z}(\bar z)$. $\mu_{\bar z 0}^z(z)$ is the background Beltrami differential of the celestial sphere (chosen equal to zero when constructing the BMS4 symmetry). Computing the action of the Diff_2 BRST operation s amounts to impose the " horizontality conditions" $(s+d)\tilde M_0^z - \frac{1}{2}\{\tilde M_0^z, \tilde M_0^z\}_z = 0$ and $(s+d)\tilde M_0^{\bar z} - \frac{1}{2}\{\tilde M_0^{\bar z}, \tilde M_0^{\bar z}\}_{\bar z} = 0$.

$\{,\}_A$ is the Poisson bracket along x^A. $s^2 = 0$ expresses the Jacobi identity of the Poisson bracket. Expansion in ghost number implies

$$s\xi^z = \xi^z\partial_z\xi^z, \quad s\xi^{\bar z} = \xi^{\bar z}\partial_{\bar z}\xi^{\bar z}, \quad s\mu_{\bar z 0}^z = \{\mu_{\bar z 0}^z, m_{\bar z}^z\}_z, \quad s\mu_{z 0}^{\bar z} = \{\mu_{z 0}^{\bar z}, m_z^{\bar z}\}_{\bar z}$$

.

$$\text{This definition of s implies} \qquad (s+d)[du\tilde M^z \tilde\partial_z^2 \tilde M^z] = 0 \quad \tilde d[du\tilde M^z\partial_z\tilde M^z\partial_z^2\tilde M^z] = 0$$

$$\text{Thus,} \quad \int_{\mathcal{I}^+} du\wedge dz\wedge d\bar z \ (\partial_z\xi_0^z)\partial_z^2\mu_{\bar z 0}^z \ \text{is a consistent d} = 3 \text{ ghost number 1 cocycle.}$$

But what truly matters are possible anomaly when computing entities relative to the production/absorption of gravitons in the celestial sphere from a BMS_4 invariant action in a given $d = 3$ QFT in $\mathcal{I}^{\pm}$. Then field representation is $\mu^{z}_{\bar{z}-1}(z, \bar{z}, u)$ and not $\mu^{\bar{z}}_{z0}$).

If such an anomaly had a vanishing coefficient, conservation laws such as the Bondi mass loss formula are to be possibly violated.

Anomalies may show up in a model dependent way if non trivial cocycle (defined modulo d and s exact terms) $\Delta^{g}_{4-g}(\xi^{z}, \xi^{\bar{z}}, \alpha, \mu^{z}_{\bar{z}-1}, \mu^{\bar{z}}_{z-1}), g = 1, 2, 3, 4$ exist, with

$$\boxed{s\Delta^{g}_{4-g} + d\Delta^{g+1}_{3-g} = 0 \quad ??}$$

Their form degrees on $\mathcal{I}^{+}$ is through a dependence on exterior products of dz, $d\bar{z}$ and du for emitted graviton (or on dz, $d\bar{z}$ and dv when $\mathcal{I}^{+} \rightarrow \mathcal{I}^{-}$ for absorbed gravitons).

Solving $s\Delta^{g}_{4-g} + d\Delta^{g+1}_{3-g} = 0.$ is non trivial due to the complication of the BRST BMS4 transformation law of $\mu^{z}_{\bar{z}-1}$ and $\mu^{\bar{z}}_{z-1}$.

However, if one manages to define $\Delta_3^1 \propto du dz d\bar{z}$ such that $ds\Delta_3^1 = -sd\Delta_3^1 = 0$, the algebraic Poincaré lemma implies the existence of a 2-form Δ_2^2 such that

$$ds\Delta_3^1 = 0 \Rightarrow s\Delta_3^1 + d\Delta_2^2 = 0$$

After that, using the then obvious equation $ds\Delta_2^2 = 0$ and so on, one gets successively the obvious existence of non trivial cocycles satisfying $s\Delta_2^2 + d\Delta_1^3 = 0, \ s\Delta_1^3 + d\Delta_0^4 = 0, \ s\Delta_0^4 = 0.$

Therefore everything boils down to guessing a possible form for Δ_3^1 such that

$$\boxed{ds\Delta_3^1 = 0 \quad ???}$$

Going back to the above $\mu^z_{\bar z 0}$ and $\mu^{\bar z}_{z0}$ dependent s-cocycle $\Delta^1_{3(0)} = 2du{\wedge}dz{\wedge}d\bar z\ (\mu^z_{\bar z 0}\partial^3_z\xi^z_0 + \mu^{\bar z}_{z0}\partial^3_{\bar z}\xi^{\bar z}_0)$ one can observe that, not only its computation was suggestive, but the way the conformal indices get contracted to get a 3-form is quite unique.

This makes it quite natural to check if the following equation is valid:

$$\boxed{sd(\Delta^1_3) = 0 \text{ with } \quad \Delta^1_3 = du{\wedge}dz{\wedge}d\bar z\ (\mu^z_{\bar z -1}\partial^3_z\xi^z_0 + \mu^{\bar z}_{z-1}\partial^3_{\bar z}\xi^{\bar z}_0).}$$

Amazingly. , the last equation is correct !

The simplest quadratic field dependence of Δ^1_3 makes it rather obvious to check that it is neither d- or s-exact.

So, Δ^1_3 is a non trivial s cocycle that can stand on the right hand side of the BMS4 Ward identity and possibly break it in a model dependent way.

Hence, its descendent cocycles Δ^g_{4-g} are also non trivial.

We actually computed, using $\Delta^1_3 = du{\wedge}dz{\wedge}d\overline{z}\ (\mu^{z}_{\overline{z}-1}\partial^3_z\xi^z_0 + \mu^{\overline{z}}_{z-1}\partial^3_{\overline{z}}\xi^{\overline{z}}_0$, that the ghost number 2, 3 and 4 are

$$
\begin{aligned}
\Delta^2_2 &= dz{\wedge}d\overline{z}\ \xi^u(\mu\partial^3\xi + \overline{\mu}\overline{\partial}^3\overline{\xi}) - du{\wedge}d\overline{z}\ \big(\xi(\mu\partial^3\xi + \overline{\mu}\overline{\partial}^3\overline{\xi}) + \partial\xi^u\overline{\partial}^3\overline{\xi}\big) + du{\wedge}dz\ \big(c.c.\big),\\
\Delta^3_1 &= \ -du\ \big(\xi\overline{\xi}(\mu\partial^3\xi + \overline{\mu}\overline{\partial}^3\overline{\xi}) + \xi(\partial^3\xi)(\overline{\partial}\xi^u) - \overline{\xi}(\overline{\partial}^3\overline{\xi})(\partial\xi^u)\big)\\
&\quad - d\overline{z}\ \xi^u\big(\xi(\mu\partial^3\xi + \overline{\mu}\overline{\partial}^3\overline{\xi}) + \partial\xi^u\overline{\partial}^3\overline{\xi}\big) + dz\ \xi^u(c.c.)),\\
\Delta^4_0 &= \xi\partial^3\xi(\overline{\xi}\xi^u\mu + \xi^u\overline{\partial}\xi^u) - \overline{\xi}\ \overline{\partial}^3\overline{\xi}(\xi\xi^u\overline{\mu} + \xi^u\partial\xi^u)
\end{aligned}
$$

where $\mu = \mu^{z}_{\overline{z}-1}, \overline{\mu} = \mu^{\overline{z}}_{z-1}, \partial = \partial_z, \overline{\partial} = \partial_{\overline{z}}, \xi = \xi^z_0, \overline{\xi} = \xi^{\overline{z}}_0$**..**

One can eg check the last cocycle is not s exact and satisfies $s\Delta^4_0 = 0$.

The complicated expressions of the lower descendant cocycles Δ^2_2, Δ^3_1 and Δ^4_0 contrast with the simplicity of their generating ghost number one up ascendant $\boxed{\Delta^1_3 = du{\wedge}dz{\wedge}d\overline{z}\ (\mu\partial^3\xi + \overline{\mu}\overline{\partial}^3\overline{\mu})}$, which is the signal that one may have an anomaly in the context of three dimensional Lagrangian quantum field theories in $\mathcal{I}^+$ relying on the principles of the BRST BMS4 invariance.

Having weakened the Diff_4 symmetry in the asymptotic Bondi gauge to its BMS_4 subgroup, although anomalies that cannot occur for the full Diff_4 symmetry.

In practice, if a theory generates a Ward identity breaking term

$$\mathbf{a}\int_{\mathcal{I}^+}\Delta_3^1(\xi,\overline{\xi},\mu,\overline{\mu})$$

with a $\neq 0$, one encounters an obstruction for imposing the Ward identity of the BRST BMS4 symmetry of the chosen three-dimensional quantum field theories in $\mathcal{I}^+$.

The study of this possibly broken Ward identity should permit one to identify which correlation functions must be investigated to compute the value of the model dependent anomaly coefficient a.

Physically, having a $\neq 0$ may complicate the construction of coinformal theories describing graviton absorptions and creations in the null boundaries $\mathcal{I}^\pm$ of asymptotically flat manifolds, and my restrict their choices.

Alternatively, if one uses a Hamiltonian formalism, the Lagrangian anomaly coefficient a is to be interpreted as a central charge. In fact the breaking of the Hamiltonian BRST BMS4 Ward identity implies that of the nilpotency of its BRST charge Q, as was shown long time ago by Kato and Ogawa [?] for the Polyakov covariant bosonic string theory where $a = D - 26$

One actually expects $Q^2 = \mathbf{a} \int_{\mathcal{S}^2} \Delta_2^2$ **in operational form.**

It is noteworthy that the Δ^g_{4-g}**'s for** $g \geq 2$ **coincide to the BMS4 algebroid cocycles that Barnich computed. However, the Hamiltonian approach leaves undetermined the generating Lagrangian top cocycle** Δ^1_3**.**

Conclusion :

One therefore understands that not only the BMS symmetry that is is the leftover local symmetry of the Diff_4 symmetry has an interesting geometrical formulation, but it can become anomalous, meaning that the subtle gravitational Bondi mass formula equilibrium formula may possibly go wrong due to perturbative graviton loop effects.

Getting a Lagrangian formulation was important in order to reach this conclusion.

Thank you!