SPECIAL RELATIVITY HOMEWORK – WEEK 6

Exercise 1. *We've seen that 4d spacetime vectors can be converted into* 2×2 *spinor matrices via* $X^{\alpha\dot{\alpha}} = x^\mu \sigma_\mu^{\alpha\dot{\alpha}}$, *where* $\sigma_\mu^{\alpha\dot{\alpha}}$ *are the Pauli matrices:*

$$\sigma_t^{\alpha\dot{\alpha}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \;;\quad \sigma_x^{\alpha\dot{\alpha}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \;;\quad \sigma_y^{\alpha\dot{\alpha}} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \;;\quad \sigma_z^{\alpha\dot{\alpha}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \,. \tag{1}$$

Surely the opposite operation should also work, i.e. $\sigma^\mu_{\alpha\dot{\alpha}} X^{\alpha\dot{\alpha}}$ *should reproduce the vector* x^μ *up to some numerical factor. Verify this, and find the factor.*

Exercise 2. *Consider the following basis of traceless* $SL(2,\mathbb{C})$ *generators* $M^\alpha{}_\beta$ *(where we treat real and imaginary components as independent):*

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ i & 0 \end{pmatrix} \right\} \tag{2}$$

How does each of these generators act on the basis spinors $\psi^\alpha = (1,0)$ *and* $\psi^\alpha = (0,1)$*? What transformation do they induce on the complex sky coordinate* $\xi = \psi^1/\psi^0$*? Which generators of the Lorentz group are these, i.e. what is the spacetime plane in which each generator is rotating?*

Remember that we care about correct answers, but not about how clean your hands are. If you prefer to crunch components, go ahead. If you can see the answer based on the geometric intuition we've accumulated so far – cheers to you. Just get at the truth, please.

Exercise 3. *Let* x^μ *be a spacetime vector (take your pick between 3d and 4d), and* $X^{\alpha\beta}$ *or* $X^{\alpha\dot{\alpha}}$ *the corresponding spinor matrix. Let us consider this matrix as a linear operation on spinors, i.e.* $\psi^\alpha \to \tilde{\psi}^\alpha = X^\alpha{}_\beta \psi^\beta$ *or* $\psi^{\dot{\alpha}} \to \tilde{\psi}^\alpha = X^\alpha{}_{\dot{\alpha}} \psi^{\dot{\alpha}}$. *What is the geometric meaning of this operation when* x^μ *is a unit (spacelike or timelike) vector? How about when* x^μ *is a lightlike vector? When approaching this question, remember that theoretical physics is an experimental science. Try out different vectors and spinors, and see what happens! There is no need to prove the general answer. Just figure out what it is!*