SPECIAL RELATIVITY HOMEWORK – WEEK 4

Exercise 1. *Let $J_{\mu\nu}$ denote the generator of (Lorentz) rotations in the plane specified by the indices $\mu\nu$. When acting on vectors, this generator takes the form of a matrix. Write down the elements $(J_{\mu\nu})^{\rho}{}_{\sigma}$ of this matrix. Your final answer should be a single compact tensorial expression, containing all the matrix elements of all the generators, all at once. Hint: it may be easier to do the raised-index version $(J_{\mu\nu})^{\rho\sigma}$ first.*

Exercise 2. *Consider 3d Minkowski spacetime with coordinates (t, x, y) and distance element $ds^2 = -dt^2 + dx^2 + dy^2$. Let us switch to the more lightlike coordinates $x^\mu = (u, v, y)$, where $u = t + x$ and $v = t - x$. All the exercises below are in this coordinate system, with indices taking the values $\mu = (u, v, y)$. Write down the metric $\eta_{\mu\nu}$ in the new coordinates, i.e. the symmetric matrix that defines the distance element $ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu$, for $x^\mu = (u, v, y)$. Write down the inverse metric $\eta^{\mu\nu}$ as well. Write down the components of x_μ in terms of the components of x^μ.*

Exercise 3. *Staying in the (u, v, y) coordinates of the previous exercise, write down the generators J_{uv}, J_{uy} and J_{vy} as matrices, i.e. write down their matrix elements $(J_{uv})^{\mu}{}_{\nu}$ etc. Raise each of these matrices to the n'th power (this is easier than it sounds – try the first few powers and you'll see). Write down the matrices for the finite group elements $e^{\chi J_{uv}}$, $e^{\chi J_{uy}}$ and $e^{\chi J_{vy}}$, where χ is the "angle" specifying how far we "rotate" with our generators.*

Exercise 4. *Now, consider a null vector $\frac{\lambda}{2}(1, \cos\theta, \sin\theta)$ in (t, x, y) coordinates, i.e. $x^\mu = \lambda(\cos^2\frac{\theta}{2}, \sin^2\frac{\theta}{2}, \cos\frac{\theta}{2}\sin\frac{\theta}{2})$ in (u, v, y) coordinates. Let us encode the direction of this null vector using the parameter $\xi \equiv \tan\frac{\theta}{2}$, which takes values on the entire real line. How does ξ transform under:*

1. *A boost $e^{\chi J_{uv}}$ by an angle χ?*
2. *A "lightlike boost" $e^{\chi J_{uy}}$ by an angle χ?*
3. *A reflection $x \to -x$ along the x axis, which interchanges $u \leftrightarrow v$?*

Exercise 5. *Combine what you've seen in the previous exercise, and write down the most general transformation of the sky coordinate ξ under the Lorentz group $SO(2, 1)$.*