GENERAL RELATIVITY HOMEWORK – WEEK 2

Exercise 1. *Consider the following equations from electromagnetism, that involve vector products and curls:*

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A} \ ; \quad \boldsymbol{\nabla} \times \mathbf{E} = -\dot{\mathbf{B}} \ ; \quad \boldsymbol{\nabla} \times \mathbf{B} = \mathbf{j} + \dot{\mathbf{E}} \ ; \quad \mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \ . \tag{1}$$

1. *Write eqs.* (1) *in tensor notation, using the Levi-Civita tensor* ϵ_{ijk}.
2. *Define* $B_{ij} = \epsilon_{ijk} B_k$. *Invert this relation, i.e. express the original vector* B_i *in terms of* B_{ij}.
3. *Rewrite eqs.* (1), *using* B_{ij} *instead of* $\mathbf{B}$ *everywhere. Verify that* ϵ_{ijk} *no longer appears. This means that EM respects reflection symmetry.*

Exercise 2. *Recall our definitions vis. a general basis* $\mathbf{e}_i$ *and its dual basis* $\mathbf{e}^i$*:*

$$\mathbf{v} = v^i \mathbf{e}_i = v_i \mathbf{e}^i \ ; \tag{2}$$

$$g_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j \ ; \quad g^{ij} = \mathbf{e}^i \cdot \mathbf{e}^j \ ; \quad \mathbf{e}_i \cdot \mathbf{e}^j = \delta_i^j \ . \tag{3}$$

From these definitions, derive the following (closely related) statements:

1. *Index raising/lowering works as* $v_i = g_{ij} v^j$ *and* $v^i = g^{ij} v_j$.
2. g^{ij} *is the matrix inverse of* g_{ij}.
3. g_{ij}, δ_i^j *and* g^{ij} *are all related to each other by index raising/lowering.*

Exercise 3. *Look up the crystal structure of graphite, i.e. the inter-atomic distances and angles. In terms of the crystal's natural vector basis* $\mathbf{e}_i$ *(which we sketched on the board in class), compute the elements of the metric* g_{ij} *and the inverse metric* g^{ij}. *What are units of these matrix elements?*