SPECIAL RELATIVITY HOMEWORK – WEEK 2

Exercise 1. *In this exercise, we continue our discussion of dimensional analysis and the fine structure constant.*

1. *Let m be the mass of the electron. Using $\hbar$ and c, construct from it a quantity with units of length. This is known as the Compton wavelength.*

2. *Recall our definition of the fine structure constant:*

$$\alpha \equiv \frac{ke^2}{\hbar c} \approx \frac{1}{137} \ . \tag{1}$$

   Express the ratio between the radius of a hydrogen atom and the electron's Compton wavelength in terms of α.

3. *Express in terms of α the ratio between the radius of a hydrogen atom and the wavelength of a photon that is emitted as the atom relaxes from its first excited state $n = 2$ to the ground state $n = 1$.*

Exercise 2. *In this exercise, we explore the generators of the Lorentz group. Let us represent Lorentz group elements as 4×4 matrices Λ that linearly transform the coordinates (t, x, y, z):*

$$\begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} \longrightarrow \begin{pmatrix} \Lambda^t{}_t & \Lambda^t{}_x & \Lambda^t{}_y & \Lambda^t{}_z \\ \Lambda^x{}_t & \Lambda^x{}_x & \Lambda^x{}_y & \Lambda^x{}_z \\ \Lambda^y{}_t & \Lambda^y{}_x & \Lambda^y{}_y & \Lambda^y{}_z \\ \Lambda^z{}_t & \Lambda^z{}_x & \Lambda^z{}_y & \Lambda^z{}_z \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} \ . \tag{2}$$

1. *Write down the matrix that corresponds to a rotation in the xy plane by an angle θ. Then, consider an infinitesimal θ, and write down the corresponding generator.*

2. *Do the same for a Lorentz boost along the x axis.*

3. *Find the commutator $[K_x, K_y]$ of the Lorentz boost generators along the x and y axes. The generator of which transformation did you find?*

Exercise 3. *In this exercise, we discover $E = Mc^2$ from the commutation relations of spacetime symmetries. Let K_x, K_y, K_z denote the generators of Lorentz boosts, as in the previous exercise. Let H, P_x, P_y, P_z denote the energy and momentum, i.e. the generators of* *time and space translations. Find the commutators $[K_x, H]$ and $[K_x, P_x]$, using the method of Exercise 1 from Week 1, i.e. by performing the corresponding infinitesimal transformations followed by their inverses. Compare with the Poisson brackets $\{K_x, H\}$ and $\{K_x, P_x\}$ from Week 1. What do you see?*