SPECIAL RELATIVITY HOMEWORK – WEEK 1

For these exercises, we live in 3d Euclidean space $\mathbb{R}^3$.

Exercise 1. *A general matrix* A_{ij} *can be decomposed as:*

$$A_{ij} = a\delta_{ij} + B_{ij} + C_{ij} \ , \tag{1}$$

where B_{ij} *is symmetric and traceless, while* C_{ij} *is antisymmetric:*

$$B_{ij} = B_{ji} \ ; \quad B_{ii} = 0 \ ; \quad C_{ij} = -C_{ji} \ . \tag{2}$$

Find the explicit expressions for a, B_{ij} *and* C_{ij}.

Exercise 2. *Consider a totally symmetric rank-3 tensor* $A_{ijk} = A_{(ijk)}$ *(make sure you understand this equation). Denote its trace as* $A_i \equiv A_{ijj}$. *Find an expression for the traceless part of* A_{ijk}.

Exercise 3. *Denote the partial derivative with respect to the coordinates* x_i *as* $\partial_i \equiv \partial/\partial x_i$. *In this notation, the Laplacian is* $\nabla^2 \equiv \partial_i \partial_i$. *Consider a rank-n polynomial function:*

$$f(x_i) = T_{i_1 \dots i_n} x_{i_1} \dots x_{i_n} \ . \tag{3}$$

1. *What is the derivative* $\partial_i x_j$*?*
2. *Find the Laplacian* $\nabla^2 f$. *What happens in the special case where* $T_{i_1 \dots i_n}$ *is traceless?*
3. *Notice that in spherical coordinates* (r, θ, ϕ), *the function* f *factorizes as* $f(x_i) = g(r)h(\theta, \phi)$. *Find the radial dependence* $g(r)$.
4. *Recall that the Laplacian in spherical coordinates reads:*

$$\nabla^2 f = \frac{1}{r^2}\left(\frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right) + \hat{\nabla}^2 f\right) \ , \tag{4}$$

where $\hat{\nabla}^2$ *is the Laplacian on the unit sphere* (θ, ϕ). *Show that, when* $T_{i_1 \dots i_n}$ *is traceless,* f *is an eigenfunction of* $\hat{\nabla}^2$. *What is the eigenvalue?*