GENERAL RELATIVITY HOMEWORK – WEEK 1

Exercise 1. *Write the product $\epsilon_{ijk}\epsilon_{lmn}$ in terms of Kronecker deltas only. Contract successive pairs of indices to obtain $\epsilon_{ijm}\epsilon_{klm}$, $\epsilon_{ikl}\epsilon_{jkl}$, and $\epsilon_{ijk}\epsilon_{ijk}$.*

Exercise 2. *Reproduce the derivation of the inverse-matrix formula for 3×3 matrices (note the index positions):*

$$(A^{-1})_{ij} = \frac{\epsilon_{jkl}\epsilon_{imn}A_{km}A_{ln}}{2\det A} \ . \tag{1}$$

For $n\times n$ matrices, the 2 in the denominator should become $(n-1)!$.

Exercise 3. *In Gibbs' unholy notation, the Maxwell equations read:*

$$\boldsymbol{\nabla}\cdot\mathbf{E} = \rho/\epsilon_0 \ ; \quad \boldsymbol{\nabla}\times\mathbf{E} = -\dot{\mathbf{B}} \ ; \quad \boldsymbol{\nabla}\cdot\mathbf{B} = 0; \quad \boldsymbol{\nabla}\times\mathbf{B} = \mu_0(\mathbf{j}+\epsilon_0\dot{\mathbf{E}}) \ . \tag{2}$$

Rewrite these using tensor-index notation. Use $B_{ij} = \epsilon_{ijk}B_k$ instead of B_i for the magnetic field, and show that all factors of ϵ_{ijk} disappear.

Exercise 4. *Consider the motion $\mathbf{r}(t) = (x(t), y(t), z(t))$ of a non-relativistic object of mass m in the Earth's gravitational field, governed by the action:*

$$S = m\int dt\left(\frac{1}{2}(\dot{x}^2+\dot{y}^2+\dot{z}^2) - gz\right) \ , \tag{3}$$

assuming a constant free-fall acceleration g.

Find the object's trajectory, as a function of its initial position $\mathbf{r_1}$ at time t_1 and its final position $\mathbf{r_2}$ at time t_2. Find the value of the action $S(t_2,\mathbf{r_2};t_1,\mathbf{r_1})$ for this trajectory. Verify that the derivatives $\partial S/\partial\mathbf{r_2}$ coincide with the momentum $\mathbf{p} = m\dot{\mathbf{r}}$ at time t_2. Show also that $-\partial S/\partial t_2$ is the energy $E = m\left(\frac{1}{2}\dot{\mathbf{r}}^2 + gz\right)$ at t_2.